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The convergence theorem for fourth-order super-Halley method in weaker conditions

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Abstract

In this paper, we establish the Newton-Kantorovich convergence theorem of a fourth-order super-Halley method under weaker conditions in Banach space, which is used to solve the nonlinear equations. Finally, some examples are provided to show the application of our theorem.

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Keywords: nonlinear equations in Banach spaces; super-Halley method; semilocal convergence; Newton-Kantorovich theorem; weaker conditions

1 Introduction

For a number of problems arising in scientific and engineering areas one often needs to find the solution of nonlinear equations in Banach spaces

$$F(x) = 0, \tag{1}$$

where F is a third-order Fréchet-differentiable operator defined on a convex subset Ω of a Banach space X with values in a Banach space Y.

There are kinds of methods to find a solution of equation (1). Generally, iterative methods are often used to solve this problem [1]. The best-known iterative method is Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{2}$$

which has quadratic convergence. Recently a lot of research has been carried out to provide improvements. Third-order iterative methods such as Halley's method, Chebyshev's method, super-Halley's method, Chebyshev-like's method *etc.* [2–12] are used to solve equation (1). To improve the convergence order, fourth-order iterative methods are also discussed in [13–19].

Kou *et al.* [20] presented a variant of the super-Halley method which improves the order of the super-Halley method from three to four by using the values of the second derivative at $(x_n - \frac{1}{3}f(x_n)/f'(x_n))$ instead of x_n. Wang *et al.* [15] established the semilocal convergence of the fourth-order super-Halley method in Banach spaces by using recurrence relations. This method in Banach spaces can be given by

$$x_{n+1} = x_n - \left[I + \frac{1}{2}K_F(x_n)\big[I - K_F(x_n)\big]^{-1}\right]\Gamma_n F(x_n), \tag{3}$$

where $\Gamma_n = [F'(x_n)]^{-1}$, $K_F(x_n) = \Gamma_n F''(u_n)\Gamma_n F(x_n)$, and $u_n = x_n - \frac{1}{3}\Gamma_n F(x_n)$.

Let $x_0 \in \Omega$ and the nonlinear operator $F : \Omega \subset X \to Y$ be continuously third-order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. Assume that

(C1) $\|\Gamma_0 F(x_0)\| \leq \eta$,
(C2) $\|\Gamma_0\| \leq \beta$,
(C3) $\|F''(x)\| \leq M, x \in \Omega$,
(C4) $\|F'''(x)\| \leq N, x \in \Omega$,
(C5) there exists a positive real number L such that

$$\big\|F'''(x) - F'''(y)\big\| \leq L\|x - y\|, \quad \forall x, y \in \Omega.$$

Under the above assumptions, we apply majorizing functions to prove the semilocal convergence of the method (3) to solve nonlinear equations in Banach spaces and establish its convergence theorems in [21]. The main results is as follows.

Theorem 1 ([21]) *Let X and Y be two Banach spaces and $F : \Omega \subseteq X \to Y$ be a third-order Fréchet differentiable on a non-empty open convex subset Ω. Assume that all conditions* (C1)-(C5) *hold and $x_0 \in \Omega$, $h = K\beta\eta \leq 1/2$, $\overline{B(x_0, t^*)} \subset \Omega$, then the sequence $\{x_n\}$ generated by the method* (3) *is well defined, $x_n \in \overline{B(x_0, t^*)}$ and converges to the unique solution $x^* \in B(x_0, t^{**})$ of $F(x)$, and $\|x_n - x^*\| \leq t^* - t_n$, where*

$$t^* = \frac{1 - \sqrt{1 - 2h}}{h}\eta, \qquad t^{**} = \frac{1 + \sqrt{1 - 2h}}{h}\eta,$$
$$K \geq M\left[1 + \frac{N}{M^2\beta} + \frac{35L}{36M^3\beta^2}\right]. \tag{4}$$

We know the conditions of Theorem 1 cannot be satisfied by some general nonlinear operator equations. For example,

$$F(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3} = 0. \tag{5}$$

Let the initial point $x_0 = 0$, $\Omega = [-1, 1]$. Then we know

$$\beta = \big\|F'(x_0)^{-1}\big\| = \frac{6}{5}, \qquad \eta = \big\|F'(x_0)^{-1}F(x_0)\big\| = \frac{2}{5},$$
$$M = \frac{4}{3}, \qquad N = 1, \qquad L = 0.$$

From (4), we can get $K \geq M$, so

$$h = K\beta\eta \geq M\beta\eta = \frac{16}{25} > \frac{1}{2}.$$

The conditions of Theorem 1 cannot be satisfied. Hence, we cannot know whether the sequence $\{x_n\}$ generated by the method (3) converges to the solution x^*.

In this paper, we consider weaker conditions and establish a new Newton-Kantorovich convergence theorem. The paper is organized as follows: in Section 2 the convergence analysis based on weaker conditions is given and in Section 3, a new Newton-Kantorovich convergence theorem is established. In Section 4, some numerical examples are worked out. We finish the work with some conclusions and references.

2 Analysis of convergence

Let $x_0 \in \Omega$ and nonlinear operator $F : \Omega \subset X \to Y$ be continuously third-order Fréchet differentiable where Ω is an open set and X and Y are Banach spaces. We assume that:

(C6) $\|\Gamma_0 F(x_0)\| \leq \eta$,
(C7) $\|F'(x_0)^{-1}F''(x_0)\| \leq \gamma$,
(C8) $\|F'(x_0)^{-1}F'''(x)\| \leq N, x \in \Omega$,
(C9) there exists a positive real number L such that

$$\left\|F'(x_0)^{-1}\left[F'''(x) - F'''(y)\right]\right\| \leq L\|x - y\|, \quad \forall x, y \in \Omega. \tag{6}$$

Denote

$$g(t) = \frac{1}{6}Kt^3 + \frac{1}{2}\gamma t^2 - t + \eta, \tag{7}$$

where K, γ, η are positive real numbers and

$$\frac{5L}{12N\eta + 36\gamma} + N \leq K. \tag{8}$$

Lemma 1 ([19]) *Let* $\alpha = \frac{2}{\gamma+\sqrt{\gamma^2+2K}}$, $\beta = \alpha - \frac{1}{6}K\alpha^3 - \frac{1}{2}\gamma\alpha^2 = \frac{2(\gamma+2\sqrt{\gamma^2+2K})}{3(\gamma+\sqrt{\gamma^2+2K})}$. *If* $\eta \leq \beta$, *then the polynomial equation* $g(t)$ *has two positive real roots* r_1, r_2 $(r_1 \leq r_2)$ *and a negative root* $-r_0$ $(r_0 > 0)$.

Lemma 2 *Let* $r_1, r_2, -r_0$ *be three roots of* $g(t)$ *and* $r_1 \leq r_2, r_0 > 0$. *Write* $u = r_0 + t$, $a = r_1 - t$, $b = r_2 - t$, *and*

$$q(t) = \frac{b-u}{a-u} \cdot \frac{(a-u)b^3 + u(u-a)b^2 + u^2(a-u)b - au^3}{(b-u)a^3 + u(u-b)a^2 + u^2(b-u)a - bu^3}. \tag{9}$$

Then as $0 \leq t \leq r_1$, *we have*

$$q(0) \leq q(t) \leq q(r_1) \leq 1. \tag{10}$$

Proof Since $g(t) = \frac{K}{6}abu$ and $g''(t) \geq 0$ $(t \geq 0)$, we have

$$u - a - b \geq 0. \tag{11}$$

Differentiating q and noticing $q'(t) \geq 0$ $(0 \leq t \leq r_1)$, we obtain

$$q(0) \leq q(t) \leq q(r_1). \tag{12}$$

On the other hand, since

$$q(t) - 1 \leq 0,$$

the lemma is proved.

Now we consider the majorizing sequences $\{t_n\}$, $\{s_n\}$ $(n \geq 0)$, $t_0 = 0$,

$$\begin{cases} s_n = t_n - \frac{g(t_n)}{g'(t_n)}, \\ h_n = -g'(t_n)^{-1} g''(r_n)(s_n - t_n), \\ t_{n+1} = t_n - [1 + \frac{1}{2}\frac{h_n}{1-h_n}]\frac{g(t_n)}{g'(t_n)}, \end{cases} \tag{13}$$

where $r_n = t_n + 1/3(s_n - t_n)$. □

Lemma 3 *Let $g(t)$ be defined by* (7) *and satisfy the condition $\eta \leq \beta$, then we have*

$$\frac{(\sqrt[3]{\lambda_2}\theta)^{4^n}}{\sqrt[3]{\lambda_2} - (\sqrt[3]{\lambda_2}\theta)^{4^n}}(r_2 - r_1) \leq r_1 - t_n \leq \frac{(\sqrt[3]{\lambda_1}\theta)^{4^n}}{\sqrt[3]{\lambda_1} - (\sqrt[3]{\lambda_1}\theta)^{4^n}}(r_2 - r_1), \quad n = 0, 1, \ldots, \tag{14}$$

where $\theta = \frac{r_1}{r_2}, \lambda_1 = q(r_1), \lambda_2 = q(0)$.

Proof Let $a_n = r_1 - t_n, b_n = r_2 - t_n, u_n = r_0 + t_n$, then

$$g(t_n) = \frac{K}{6} a_n b_n u_n, \tag{15}$$

$$g'(t_n) = -\frac{K}{6}[a_n u_n + b_n u_n - a_n b_n], \tag{16}$$

$$g''(t_n) = \frac{K}{3}[u_n - b_n - a_n]. \tag{17}$$

Write $\varphi(t_n) = a_n u_n + b_n u_n - a_n b_n$, then we have

$$\begin{aligned} a_{n+1} &= a_n + \left[1 + \frac{1}{2}\frac{h_n}{1-h_n}\right]\frac{g(t_n)}{g'(t_n)} \\ &= \frac{a_n^4(b_n - u_n)[(a_n - u_n)b_n^3 + u_n(u_n - a_n)b_n^2 + u_n^2(a_n - u_n)b_n - a_n u_n^3]}{\varphi^2(t_n)(a_n^2 u_n^2 + b_n^2 u_n^2 + a_n^2 b_n^2) - 2a_n^2 b_n^2 \varphi(t_n)}, \end{aligned} \tag{18}$$

$$\begin{aligned} b_{n+1} &= b_n + \left[1 + \frac{1}{2}\frac{h_n}{1-h_n}\right]\frac{g(t_n)}{g'(t_n)} \\ &= \frac{b_n^4(a_n - u_n)[(b_n - u_n)a_n^3 + u_n(u_n - b_n)b_n^2 + u_n^2(b_n - u_n)a_n - b_n u_n^3]}{\varphi^2(t_n)(a_n^2 u_n^2 + b_n^2 u_n^2 + a_n^2 b_n^2) - 2a_n^2 b_n^2 \varphi(t_n)}. \end{aligned} \tag{19}$$

We can obtain

$$\frac{a_{n+1}}{b_{n+1}} = \frac{b_n - u_n}{a_n - u_n} \cdot \frac{(a_n - u_n)b_n^3 + u_n(u_n - a_n)b_n^2 + u_n^2(a_n - u_n)b_n - a_n u_n^3}{(b_n - u_n)a_n^3 + u_n(u_n - b_n)a_n^2 + u_n^2(b_n - u_n)a_n - b_n u_n^3} \cdot \left(\frac{a_n}{b_n}\right)^4. \tag{20}$$

From Lemma 2, we have $\lambda_2 \leq q(t) \leq \lambda_1$. Thus

$$\frac{a_n}{b_n} \leq \lambda_1 \left(\frac{a_{n-1}}{b_{n-1}}\right)^4 \leq \cdots \leq (\lambda_1)^{1+4+\cdots+4^{n-1}} \left(\frac{a_0}{b_0}\right)^{4^n} = \frac{1}{\sqrt[3]{\lambda_1}} \left(\sqrt[3]{\lambda_1}\theta\right)^{4^n}. \tag{21}$$

In a similar way,

$$\frac{a_n}{b_n} \geq \frac{1}{\sqrt[3]{\lambda_2}} \left(\sqrt[3]{\lambda_2}\theta\right)^{4^n}. \tag{22}$$

That completes the proof of the lemma. □

Lemma 4 *Suppose t_n, s_n are generated by* (13). *If $\eta < \beta$, then the sequences $\{t_n\}, \{s_n\}$ increase and converge to r_1, and*

$$0 \leq t_n \leq s_n \leq t_{n+1} < r_1. \tag{23}$$

Proof Let

$$\begin{aligned} &U(t) = t - g'(t)^{-1} g(t), \\ &H(t) = \left(g'(t)^{-1}\right)^2 g''(T) g(t), \\ &V(t) = t + \left[1 + \frac{1}{2} \frac{H(t)}{1 - H(t)}\right] \left(U(t) - t\right), \end{aligned} \tag{24}$$

where $T = (2t + U(t))/3$.

When $0 \leq t \leq r_1$, we can obtain $g(t) \geq 0$, $g'(t) < 0$, $g''(t) > 0$. Hence

$$U(t) = t - \frac{g(t)}{g'(t)} \geq t \geq 0. \tag{25}$$

So $\forall t \in [0, r_1]$, we always have $U(t) \geq t$.

Since $T = \frac{2t+U(t)}{3} \geq t \geq 0$, we have

$$H(t) = \frac{g''(T) g(t)}{g'(t)^2} \geq 0. \tag{26}$$

On the other hand $g''(T)g(t) - g'(t)^2 > 0$, then

$$0 \leq H(t) = \frac{g''(T) g(t)}{g'(t)^2} < 1. \tag{27}$$

Thus

$$V(t) = U(t) + \frac{1}{2} \frac{H(t)}{1 - H(t)} \left(U(t) - t\right) \geq 0,$$

and $\forall t \in [0, r_1]$, we always have $V(t) \geq U(t)$.

Since

$$\begin{aligned} V'(t) &= \frac{g(t)[3g'(t)^2(g''(T) - g''(t))(g''(T)g(t) - 2g'(t)^2) - Kg(t)^2 g'(t) g''(t)}{6g'(t)^2[g''(T)g(t) - g'(t)^2]^2} \\ &\quad + \frac{-Kg(t)^2 g''(t) g'(t) + 3g(t)^2 g''(t) g''(T)]}{6g'(t)^2[g''(T)g(t) - g'(t)^2]^2} \\ &= \frac{g(t)[-Kg(t)^2 g'(t) g''(t) - Kg(t)^2 g''(t) g'(t) + 3g(t)^2 g''(t) g''(T)]}{6g'(t)^2[g''(T)g(t) - g'(t)^2]^2}, \end{aligned} \tag{28}$$

we know $V'(t) > 0$ for $0 \leq t \leq r_1$. That is to say that $V(t)$ is monotonically increasing. By this we will inductively prove that

$$0 \leq t_n \leq s_n \leq t_{n+1} < V(r_1) = r_1. \tag{29}$$

In fact, (29) is obviously true for $n = 0$. Assume (29) holds until some n. Since $t_{n+1} < r_1$, s_{n+1}, t_{n+2} are well defined and $t_{n+1} \leq s_{n+1} \leq t_{n+2}$. On the other hand, by the monotonicity of $V(t)$, we also have

$$t_{n+2} = V(t_{n+1}) < V(r_1) = r_1.$$

Thus, (29) also holds for $n + 1$.

From Lemma 3, we can see that $\{t_n\}$ converges to r_1. That completes the proof of the lemma. □

Lemma 5 *Assume F satisfies the conditions* (C6)-(C9), *then* $\forall x \in \overline{B(x_0, r_1)}$, $F'(x)^{-1}$ *exists and satisfies the inequality*

(I) $\|F'(x_0)^{-1}F''(x)\| \leq g''(\|x - x_0\|)$,

(II) $\|F'(x)^{-1}F'(x_0)\| \leq -g'(\|x - x_0\|)^{-1}$.

Proof (I) From the above assumptions, we have

$$\begin{aligned}
\bigl\|F'(x_0)^{-1}F''(x)\bigr\| &= \bigl\|F'(x_0)^{-1}F''(x_0) + F'(x_0)^{-1}\bigl[F''(x) - F''(x_0)\bigr]\bigr\| \\
&\leq \gamma + N\|x - x_0\| \leq \gamma + K\|x - x_0\| \\
&= g''\bigl(\|x - x_0\|\bigr).
\end{aligned}$$

(II) When $t \in [0, r_1)$, we know $g'(t) < 0$. Hence when $x \in \overline{B(x_0, r_1)}$,

$$\begin{aligned}
&\bigl\|F'(x_0)^{-1}F'(x) - I\bigr\| \\
&\quad= \bigl\|F'(x_0)^{-1}\bigl[F'(x) - F'(x_0) - F''(x_0)(x - x_0) + F''(x_0)(x - x_0)\bigr]\bigr\| \\
&\quad\leq \left\|\int_0^1 F'(x_0)^{-1}\bigl[F''\bigl(x_0 + t(x - x_0)\bigr) - F''(x_0)\bigr]\,dt(x - x_0)\right\| + \gamma\|x - x_0\| \\
&\quad\leq \left\|\int_0^1 Nt\,dt(x - x_0)^2\right\| + \gamma\|x - x_0\| \leq \frac{1}{2}K\|x - x_0\|^2 + \gamma\|x - x_0\| \\
&\quad= 1 + g'\bigl(\|x - x_0\|\bigr) < 1.
\end{aligned}$$

By the Banach lemma, we know $(F'(x_0)^{-1}F'(x))^{-1} = F'(x)^{-1}F'(x_0)$ exists and

$$\bigl\|F'(x)^{-1}F'(x_0)\bigr\| \leq \frac{1}{1 - \|I - F'(x_0)^{-1}F'(x)\|} \leq -g'\bigl(\|x - x_0\|\bigr)^{-1}.$$

That completes the proof of the lemma. □

Lemma 6 ([21]) *Assume that the nonlinear operator* $F : \Omega \subset X \to Y$ *is continuously third-order Fréchet differentiable where* Ω *is an open set and X and Y are Banach spaces. The* *sequences* $\{x_n\}$, $\{y_n\}$ *are generated by* (3). *Then we have*

$$\begin{aligned} F(x_{n+1}) &= \frac{1}{2}F''(u_n)(x_{n+1}-y_n)^2 + \frac{1}{6}F'''(x_n)(x_{n+1}-y_n)(x_{n+1}-x_n)^2 \\ &\quad - \frac{1}{6}\int_0^1 \left[F'''\left(x_n + \frac{1}{3}t(y_n-x_n)\right) - F'''(x_n)\right] dt(y_n-x_n)(x_{n+1}-x_n)^2 \\ &\quad + \frac{1}{2}\int_0^1 \left[F'''\big(x_n + t(x_{n+1}-x_n)\big) - F'''(x_n)\right](1-t)^2\, dt(x_{n+1}-x_n)^3, \end{aligned}$$

where $y_n = x_n - \Gamma_n F(x_n)$ *and* $u_n = x_n + \frac{1}{3}(y_n - x_n)$.

3 Newton-Kantorovich convergence theorem

Now we give a theorem to establish the semilocal convergence of the method (3) in weaker conditions, the existence and uniqueness of the solution and the domain in which it is located, along with a priori error bounds, which lead to the R-order of convergence of at least four of the iterations (3).

Theorem 2 *Let* X *and* Y *be two Banach spaces, and* $F : \Omega \subseteq X \to Y$ *be a third-order Fréchet differentiable on a non-empty open convex subset* Ω. *Assume that all conditions* (C6)-(C9) *hold true and* $x_0 \in \Omega$. *If* $\eta < \beta$, $\overline{B(x_0, r_1)} \subset \Omega$, *then the sequence* $\{x_n\}$ *generated by* (3) *is well defined*, $\{x_n\} \in \overline{B(x_0, r_1)}$ *and converges to the unique solution* $x^* \in B(x_0, \alpha)$, *and* $\|x_n - x^*\| \le r_1 - t_n$. *Further, we have*

$$\|x_n - x^*\| \le \frac{(\sqrt[3]{\lambda_1}\theta)^{4^n}}{\sqrt[3]{\lambda_1} - (\sqrt[3]{\lambda_1}\theta)^{4^n}}(r_2 - r_1), \quad n = 0, 1, \ldots, \tag{30}$$

where $\theta = \frac{r_1}{r_2}$, $\lambda_1 = q(r_1)$, $\alpha = \frac{2}{\gamma + \sqrt{\gamma^2 + 2K}}$.

Proof We will prove the following formula by induction:

(I_n) $x_n \in \overline{B(x_0, t_n)}$,
(II_n) $\|F'(x_n)^{-1}F'(x_0)\| \le -g'(t_n)^{-1}$,
(III_n) $\|F'(x_0)^{-1}F''(x_n)\| \le g''(\|x_n - x_0\|) \le g''(t_n)$,
(IV_n) $\|y_n - x_n\| \le s_n - t_n$,
(V_n) $y_n \in \overline{B(x_0, s_n)}$,
(VI_n) $\|x_{n+1} - y_n\| \le t_{n+1} - s_n$.

Estimate that (I_n)-(VI_n) are true for $n = 0$ by the initial conditions. Now, assume that (I_n)-(VI_n) are true for all integers $k \le n$.

(I_{n+1}) From the above assumptions, we have

$$\begin{aligned} \|x_{n+1} - x_0\| &\le \|x_{n+1} - y_n\| + \|y_n - x_n\| + \|x_n - x_0\| \\ &\le (t_{n+1} - s_n) + (s_n - t_n) + (t_n - t_0) = t_{n+1}. \end{aligned} \tag{31}$$

(II_{n+1}) From (II) of Lemma 5, we can obtain

$$\|F'(x_{n+1})^{-1}F'(x_0)\| \le -g'\big(\|x_{n+1} - x_0\|\big)^{-1} \le -g'(t_{n+1})^{-1}. \tag{32}$$

(III_{n+1}) From (I) of Lemma 5, we can obtain

$$\left\|F'(x_0)^{-1}F''(x_{n+1})\right\| \le g''\bigl(\|x_{n+1}-x_0\|\bigr) \le g''(t_{n+1}). \tag{33}$$

(IV_{n+1}) From Lemma 5, we have

$$\begin{aligned}\left\|F'(x_0)^{-1}F(x_{n+1})\right\| &\le \frac{1}{2}g''(r_n)(t_{n+1}-s_n)^2 + \frac{1}{6}N(t_{n+1}-s_n)(t_{n+1}-t_n)^2 \\ &\quad + \frac{L}{36}(s_n-t_n)^2(t_{n+1}-t_n)^2 + \frac{L}{24}(t_{n+1}-t_n)^4 \\ &\le \frac{1}{2}g''(r_n)(t_{n+1}-s_n)^2 \\ &\quad + \frac{1}{6}\left(N + \frac{L}{36}\cdot\frac{(s_n-t_n)^2}{t_{n+1}-s_n} + \frac{L}{24}\cdot\frac{(t_{n+1}-t_n)^2}{t_{n+1}-s_n}\right)(t_{n+1}-s_n)(t_{n+1}-t_n)^2 \\ &\le \frac{1}{2}g''(r_n)(t_{n+1}-s_n)^2 + \frac{1}{6}K(t_{n+1}-s_n)(t_{n+1}-t_n)^2 \\ &\le g(t_{n+1}). \end{aligned} \tag{34}$$

Thus, we have

$$\begin{aligned}\|y_{n+1}-x_{n+1}\| &\le \left\|-F'(x_{n+1})F'(x_0)\right\|\left\|F'(x_0)^{-1}F(x_{n+1})\right\| \\ &\le -\frac{g(t_{n+1})}{g'(t_{n+1})} = s_{n+1}-t_{n+1}. \end{aligned} \tag{35}$$

(V_{n+1}) From the above assumptions and (35), we obtain

$$\begin{aligned}\|y_{n+1}-x_0\| &\le \|y_{n+1}-x_{n+1}\| + \|x_{n+1}-y_n\| + \|y_n-x_n\| + \|x_n-x_0\| \\ &\le (s_{n+1}-t_{n+1}) + (t_{n+1}-s_n) + (s_n-t_n) + (t_n-t_0) = s_{n+1}, \end{aligned} \tag{36}$$

so $y_{n+1} \in \overline{B(x_0, s_{n+1})}$.

(VI_{n+1}) Since

$$\begin{aligned}\left\|I-K_F(x_{n+1})\right\| &\ge 1-\left\|K_F(x_{n+1})\right\| \ge 1-\bigl(-g'(t_{n+1})\bigr)^{-1}g''(r_{n+1})(s_{n+1}-t_{n+1}) \\ &= 1+g'(t_{n+1})^{-1}g''(r_{n+1})(s_{n+1}-t_{n+1}) = 1-h_{n+1}, \end{aligned}$$

we have

$$\begin{aligned}\|x_{n+2}-y_{n+1}\| &= \frac{1}{2}\left\|K_F(x_{n+1})\bigl[I-K_F(x_{n+1})\bigr]^{-1}\right\|\left\|F'(x_{n+1})^{-1}F(x_{n+1})\right\| \\ &\le \frac{1}{2}\frac{h_{n+1}}{1-h_{n+1}}\cdot\frac{g(t_{n+1})}{-g'(t_{n+1})} = t_{n+2}-s_{n+1}. \end{aligned} \tag{37}$$

Further, we have

$$\|x_{n+2}-x_{n+1}\| \le \|x_{n+2}-y_{n+1}\| + \|y_{n+1}-x_{n+1}\| \le t_{n+2}-t_{n+1}, \tag{38}$$

and when $m > n$

$$\|x_m - x_n\| \leq \|x_m - x_{m-1}\| + \cdots + \|x_{n+1} - x_n\| \leq t_m - t_n. \tag{39}$$

It then follows that the sequence $\{x_n\}$ is convergent to a limit x^*. Take $n \to \infty$ in (34), we deduce $F(x^*) = 0$. From (39), we also get

$$\left\|x^* - x_n\right\| \leq r_1 - t_n. \tag{40}$$

Now, we prove the uniqueness. Suppose x^{**} is also the solution of $F(x)$ on $B(x_0, \alpha)$. By Taylor expansion, we have

$$0 = F\left(x^{**}\right) - F\left(x^*\right) = \int_0^1 F'\left((1-t)x^* + tx^{**}\right) dt\left(x^{**} - x^*\right). \tag{41}$$

Since

$$\begin{aligned}
&\left\|F'(x_0)^{-1} \int_0^1 \left[F'\left((1-t)x^* + tx^{**}\right) - F'(x_0)\right] dt\right\| \\
&\quad \leq \left\|F'(x_0)^{-1} \int_0^1 \int_0^1 F''\left[x_0 + t\left(x^* - x_0\right) + t\left(x^{**} - t^*\right)\right] ds\, dt\left[x^* - x_0 + t\left(x^{**} - x^*\right)\right]\right\| \\
&\quad \leq \int_0^1 \int_0^1 g''\left[s\left\|x^* - x_0 + t\left(x^{**} - x^*\right)\right\|\right] ds\, dt \left\|x^* - x_0 + t\left(x^{**} - x^*\right)\right\| \\
&\quad = \int_0^1 g'\left(\left\|\left(x^* - x_0\right) + t\left(x^{**} - x^*\right)\right\|\right) dt - g'(0) \\
&\quad = \int_0^1 g'\left(\left\|(1-t)\left(x^* - x_0\right) + t\left(x^{**} - x_0\right)\right\|\right) + 1 \\
&\quad < \frac{g'(r_1) + g'(\alpha)}{2} + 1 \leq 1,
\end{aligned} \tag{42}$$

we can find that the inverse of $\int_0^1 F'((1-t)x^* + tx^{**})\, dt$ exists, so $x^{**} = x^*$.

From Lemma 3, we get

$$\left\|x_n - x^*\right\| \leq \frac{(\sqrt[3]{\lambda_1}\theta)^{4^n}}{\sqrt[3]{\lambda_1} - (\sqrt[3]{\lambda_1}\theta)^{4^n}}(r_2 - r_1), \quad n = 0, 1, \ldots. \tag{43}$$

This completes the proof of the theorem. □

4 Numerical examples

In this section, we illustrate the previous study with an application to the following nonlinear equations.

Example 1 Let $X = Y = R$, and

$$F(x) = \frac{1}{6}x^3 + \frac{1}{6}x^2 - \frac{5}{6}x + \frac{1}{3} = 0. \tag{44}$$

We consider the initial point $x_0 = 0$, $\Omega = [-1,1]$, we can get

$$\eta = \gamma = \frac{2}{5}, \qquad N = \frac{6}{5}, \qquad L = 0.$$

Hence, from (8), we have $K = N = \frac{6}{5}$ and

$$\beta = \frac{2(\gamma + 2\sqrt{\gamma^2 + 2K})}{3(\gamma + \sqrt{\gamma^2 + 2K})^2} = \frac{3}{5}, \quad \eta < \beta.$$

This means that the hypotheses of Theorem 2 are satisfied, we can get the sequence $\{x_n\}_{(n\geq 0)}$ generated by the method (3) is well defined and converges.

Example 2 Consider an interesting case as follows:

$$x(s) = 1 + \frac{1}{4}x(s)\int_0^1 \frac{s}{s+t}x(t)\,dt, \tag{45}$$

where we have the space $X = C[0,1]$ with norm

$$\|x\| = \max_{0\leq s\leq 1}\bigl|x(s)\bigr|.$$

This equation arises in the theory of the radiative transfer, neutron transport and the kinetic theory of gases.

Let us define the operator F on X by

$$F(x) = \frac{1}{4}x(s)\int_0^1 \frac{s}{s+t}x(t)\,dt - x(s) + 1. \tag{46}$$

Then for $x_0 = 1$ we can obtain

$$\begin{aligned}
&N = 0, \qquad L = 0, \qquad K = 0, \qquad \bigl\|F'(x_0)^{-1}\bigr\| = 1.5304, \qquad \eta = 0.2652,\\
&\gamma = \bigl\|F'(x_0)^{-1}F''(x_0)\bigr\| = 1.5304 \times 2\cdot\frac{1}{4}\max_{0\leq s\leq 1}\left|\int_0^1 \frac{s}{s+t}\,dt\right| = 1.5304 \times \frac{\ln 2}{2} = 0.5303,\\
&\frac{2(\gamma + 2\sqrt{\gamma^2 + 2K})}{3(\gamma + \sqrt{\gamma^2 + 2K})^2} = 0.9429 > \eta.
\end{aligned}$$

That means that the hypotheses of Theorem 2 are satisfied.

Example 3 Consider the problem of finding the minimizer of the chained Rosenbrock function [22]:

$$g(\mathbf{x}) = \sum_{i=1}^{m}\bigl[4\bigl(x_i - x_{i+1}^2\bigr)^2 + \bigl(1 - x_{i+1}^2\bigr)\bigr], \quad \mathbf{x}\in\mathbf{R}^m. \tag{47}$$

For finding the minimum of g one needs to solve the nonlinear system $F(\mathbf{x}) = 0$, where $F(\mathbf{x}) = \nabla g(\mathbf{x})$. Here, we apply the method (3), and compare it with Chebyshev method (CM), the Halley method (HM), and the super-Halley method (SHM).

Table 1 The iterative errors ($\|x_n - x^*\|_2$) of various methods

n	CM	HM	SHM	Method (3)
1	3.086657e–001	2.169261e–001	5.962251e–002	3.119924e–002
2	1.198959e–003	4.506048e–003	1.913033e–005	1.989913e–006
3	1.099899e–007	2.048709e–008	1.435454e–014	7.771561e–016
4	3.140185e–016	5.438960e–016	8.599751e–016	

In a numerical tests, the stopping criterion of each method is $\|\mathbf{x}_k - \mathbf{x}^*\|_2 \le 1e-15$, where $\mathbf{x}^* = (1,1,\dots,1)^T$ is the exact solution. We choose $m = 30$ and $x_0 = 1.2\mathbf{x}^*$. Listed in Table 1 are the iterative errors ($\|\mathbf{x}_k - \mathbf{x}^*\|_2$) of various methods. From Table 1, we know that, as tested here, the performance of the method (3) is better.

5 Conclusions

In this paper, a new Newton-Kantorovich convergence theorem of a fourth-order super-Halley method is established. As compared with the method in [21], the differentiability conditions of the method in the paper are mild. Finally, some examples are provided to show the application of the convergence theorem.

Competing interests
The author declares to have no competing interests.

Author's contributions
Only the author contributed in writing this paper.

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