Bnouhachem et al. *Journal of Inequalities and Applications* (2016) 2016:297
DOI 10.1186/s13660-016-1226-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH

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On the $O(1/t)$ convergence rate of the alternating direction method with LQP regularization for solving structured variational inequality problems

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Abstract

In this paper, we propose a parallel descent LQP alternating direction method for solving structured variational inequality with three separable operators. The $O(1/t)$ convergence rate for this method is studied. We also present some numerical examples to illustrate the efficiency of the proposed method. The results presented in this paper extend and improve some well-known results in the literature.

MSC: Primary 90C33; 49J40; secondary 65N30

Keywords: structured variational inequalities; logarithmic-quadratic proximal method; convergence rate; projection method; alternating direction method

1 Introduction

Let $\mathbb{R}^n_+ = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \ \forall i = 1, 2, \ldots, n\}$ and $\mathbb{R}^n_{++} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > 0 \ \forall i = 1, 2, \ldots, n\}$. The variational inequality problem is to find

$$x \in \Omega := \left\{(u, v) : u \in \mathbb{R}^n_+, v \in \mathbb{R}^m_+, A_1 u + A_2 v = b\right\}$$

such that

$$\left(x' - x\right)^T F(x) \geq 0, \quad \forall x' \in \Omega, \tag{1.1}$$

with

$$x = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad F(x) = \begin{pmatrix} f_1(u) \\ f_2(v) \end{pmatrix}, \tag{1.2}$$

where $A_1 \in \mathbb{R}^{l \times n}$, $A_2 \in \mathbb{R}^{l \times m}$ are given matrices, $b \in \mathbb{R}^l$ is a given vector, and $f_1 : \mathbb{R}^n_+ \to \mathbb{R}^n$, $f_2 : \mathbb{R}^m_+ \to \mathbb{R}^m$ are given monotone operators. For further study and applications of such problems, we refer to [1–11] and the references therein. By attaching a Lagrange multiplier vector $\lambda \in \mathbb{R}^l$ to the linear constraints $A_1 u + A_2 v = b$, the problem (1.1)-(1.2) can be explained in terms of finding $z \in \mathcal{Z}' := \mathbb{R}^n_+ \times \mathbb{R}^m_+ \times \mathbb{R}^l$ such that

$$\left(z' - z\right)^\top Q(z) \geq 0, \quad \forall z' \in \mathcal{Z}', \tag{1.3}$$

where

$$z = \begin{pmatrix} u \\ v \\ \lambda \end{pmatrix}, \qquad Q(z) = \begin{pmatrix} f_1(u) - A_1^\top \lambda \\ f_2(v) - A_2^\top \lambda \\ A_1 u + A_2 v - b \end{pmatrix}, \tag{1.4}$$

and $A_1^\top$ denotes the transpose of the matrix A_1. The problem (1.3)-(1.4) is referred as a *structured variational inequality problem* (in short, SVI).

Yuan and Li [12] developed the following logarithmic-quadratic proximal (LQP)-based decomposition method by applying the LQP terms to regularize the ADM subproblems: For a given $z^k = (u^k, v^k, \lambda^k) \in \mathbb{R}^n_{++} \times \mathbb{R}^m_{++} \times \mathbb{R}^l$, and $\mu \in (0,1)$, the new iterative $(u^{k+1}, v^{k+1}, \lambda^{k+1})$ is obtained via solving the following system:

$$f_1(u) - A_1^\top \left[\lambda^k - H\left(A_1 u + A_2 v^k - b\right)\right] + R\left[\left(u - u^k\right) + \mu\left(u^k - U_k^2 u^{-1}\right)\right] = 0, \tag{1.5}$$

$$f_2(v) - A_2^\top \left[\lambda^k - H(A_1 u + A_2 v - b)\right] + S\left[\left(v - v^k\right) + \mu\left(v^k - V_k^2 v^{-1}\right)\right] = 0, \tag{1.6}$$

$$\lambda^{k+1} = \lambda^k - H\left(A_1 u^k + A_2 v^k - b\right), \tag{1.7}$$

where $H \in \mathbb{R}^{l\times l}$, $R \in \mathbb{R}^{n\times n}$, and $S \in \mathbb{R}^{m\times m}$ are symmetric positive definite.

Later, some LQP alternating direction methods have been proposed to make the LQP alternating direction method more practical, see, for example, [13–17] and the references therein. Each iteration of these methods contains a prediction and a correction, the predictor is obtained via solving (1.5)-(1.7) and the new iterate is obtained by a convex combination of the previous point and the one generated by a projection-type method along a descent direction. The main disadvantage of the methods proposed in [12–17] is that solving equation (1.6) requires the solution of equation (1.5). To overcome with this difficulty, Bnouhachem and Hamdi [18] proposed a parallel descent LQP alternating direction method for solving SVI.

In this paper, we propose a parallel descent LQP alternating direction method for solving the following structured variational inequality with three separable operators: Find $y \in \Omega := \{(u, v, w) : u \in \mathbb{R}^{n_1}_+, v \in \mathbb{R}^{n_2}_+, w \in \mathbb{R}^{n_2}_+, A_1 u + A_2 v + A_3 w = b\}$ such that

$$\left(y' - y\right)^\top F(y) \geq 0, \quad \forall y' \in \Omega, \tag{1.8}$$

with

$$y = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \qquad F(y) = \begin{pmatrix} f_1(u) \\ f_2(v) \\ f_3(w) \end{pmatrix}, \tag{1.9}$$

where $A_1 \in \mathbb{R}^{m\times n_1}$, $A_2 \in \mathbb{R}^{m\times n_2}$, $A_3 \in \mathbb{R}^{m\times n_3}$ are given matrices, $b \in \mathbb{R}^m$ is a given vector, and $f_1 : \mathbb{R}^{n_1}_+ \to \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}^{n_2}_+ \to \mathbb{R}^{n_2}$, $f_3 : \mathbb{R}^{n_3}_+ \to \mathbb{R}^{n_3}$ are given monotone operators. By attaching a Lagrange multiplier vector $\lambda \in \mathbb{R}^m$ to the linear constraints $A_1 u + A_2 v + A_3 w = b$, the problem (1.8)-(1.9) can be explained in terms of finding $z \in \mathcal{Z} := \mathbb{R}^{n_1}_+ \times \mathbb{R}^{n_2}_+ \times \mathbb{R}^{n_3}_+ \times \mathbb{R}^m$ such that

$$\bigl(z' - z\bigr)^\top Q(z) \geq 0, \quad \forall z' \in \mathcal{Z}, \tag{1.10}$$

where

$$z = \begin{pmatrix} u \\ v \\ w \\ \lambda \end{pmatrix} \quad \text{and} \quad Q(z) = \begin{pmatrix} f_1(u) - A_1^\top \lambda \\ f_2(v) - A_2^\top \lambda \\ f_3(w) - A_3^\top \lambda \\ A_1 u + A_2 v + A_3 w - b \end{pmatrix}. \tag{1.11}$$

The problem (1.10)-(1.11) is referred as SVI_3.

The main aim of this paper is to present the parallel descent LQP alternating direction method for solving SVI_3 and to investigate the convergence rate of this method. We show that the proposed method has the $O(1/t)$ convergence rate. The iterative algorithm and results presented in this paper generalize, unify, and improve the previously known results in this area.

2 The proposed method

For any vector $u \in \mathbb{R}^n$, $\|u\|_\infty = \max\{|u_1|, \ldots, |u_n|\}$. Let $D \in \mathbb{R}^{n\times n}$ be a symmetry positive definite matrix, we denote the D-norm of u by $\|u\|_D^2 = u^T D u$.

The following lemma provides a basic property of projection operator onto a closed convex subset Ω of $\mathbb{R}^l$. We denote by $P_{\Omega,D}(\cdot)$ the projection operator under the D-norm, that is,

$$P_{\Omega,D}(v) = \operatorname{argmin}\bigl\{\|v - u\|_D : u \in \Omega\bigr\}.$$

Lemma 2.1 *Let D be a symmetry positive definite matrix and Ω be a nonempty closed convex subset of $\mathbb{R}^l$. Then*

$$\bigl(z - P_{\Omega,D}[z]\bigr)^\top D\bigl(P_{\Omega,D}[z] - v\bigr) \geq 0, \quad \forall z \in \mathbb{R}^l, v \in \Omega. \tag{2.1}$$

We make the following standard assumptions.

Assumption 2.1 f_1 is monotone with respect to $\mathbb{R}^{n_1}_+$, that is, $(f_1(x) - f_1(y))^T(x - y) \geq 0$, $\forall x, y \in \mathbb{R}^{n_1}_+$, f_2 is monotone with respect to $\mathbb{R}^{n_2}_+$, and f_3 is monotone with respect to $\mathbb{R}^{n_3}_+$.

Assumption 2.2 The solution set of SVI_3, denoted by $\mathcal{Z}^*$, is nonempty.

We propose the following parallel LQP alternating direction method for solving SVI_3:

Algorithm 2.1

Step 0. Given $\varepsilon > 0$, $\mu \in (0,1)$, $\beta \in (\frac{\sqrt{3}}{2}, 1)$, $\gamma \in (0,2)$ and $z^0 = (u^0, v^0, w^0, \lambda^0) \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$. Set $k = 0$.

Step 1. Compute $\tilde{z}^k = (\tilde{u}^k, \tilde{v}^k, \tilde{w}^k, \tilde{\lambda}^k) \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$ by solving the following system:

$$\begin{aligned} & f_1(u) - A_1^\top\bigl[\lambda^k - H\bigl(A_1u + A_2v^k + A_3w^k - b\bigr)\bigr] \\ & \quad + R_1\bigl[\bigl(u - u^k\bigr) + \mu\bigl(u^k - U_k^2u^{-1}\bigr)\bigr] = 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} & f_2(v) - A_2^\top\bigl[\lambda^k - H\bigl(A_1u^k + A_2v + A_3w^k - b\bigr)\bigr] \\ & \quad + R_2\bigl[\bigl(v - v^k\bigr) + \mu\bigl(v^k - V_k^2v^{-1}\bigr)\bigr] = 0, \end{aligned} \tag{2.3}$$

$$\begin{aligned} & f_3(w) - A_3^\top\bigl[\lambda^k - H\bigl(A_1u^k + A_2v^k + A_3w - b\bigr)\bigr] \\ & \quad + R_3\bigl[\bigl(w - w^k\bigr) + \mu\bigl(w^k - W_k^2w^{-1}\bigr)\bigr] = 0, \end{aligned} \tag{2.4}$$

$$\tilde{\lambda}^k = \lambda^k - \beta H\bigl(A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b\bigr), \tag{2.5}$$

where $H \in \mathbb{R}^{m\times m}$, $R_1 \in \mathbb{R}^{n_1\times n_1}$, $R_2 \in \mathbb{R}^{n_2\times n_2}$ and $R_3 \in \mathbb{R}^{n_3\times n_3}$ are symmetric positive definite matrices. U_k, V_k, and W_k are positive definite diagonal matrices defined by $U_k = \operatorname{diag}(u_1^k, \dots, u_n^k)$, $V_k = \operatorname{diag}(v_1^k, \dots, v_n^k)$, $W_k = \operatorname{diag}(w_1^k, \dots, w_n^k)$.

Step 2. If $\max\{\|u^k - \tilde{u}^k\|_\infty, \|v^k - \tilde{v}^k\|_\infty, \|w^k - \tilde{w}^k\|_\infty, \|\lambda^k - \tilde{\lambda}^k\|_\infty\} < \epsilon$, then stop.

Step 3. The new iterate $z^{k+1}(\tau_k) = (u^{k+1}, v^{k+1}, w^{k+1}, \lambda^{k+1})$ is given by

$$z^{k+1}(\tau_k) = (1-\sigma)z^k + \sigma P_{\mathcal{Z},G}\bigl[z^k - \gamma\tau_k G^{-1}g\bigl(z^k, \tilde{z}^k\bigr)\bigr], \quad \sigma \in (0,1), \tag{2.6}$$

where

$$\tau_k = \frac{\varphi(z^k, \tilde{z}^k)}{\|z^k - \tilde{z}^k\|_G^2}, \tag{2.7}$$

$$\begin{aligned} \varphi\bigl(z^k, \tilde{z}^k\bigr) &= \bigl\|z^k - \tilde{z}^k\bigr\|_M^2 \\ &\quad + \frac{1}{\beta}\bigl(\lambda^k - \tilde{\lambda}^k\bigr)^T\bigl(A_1\bigl(u^k - \tilde{u}^k\bigr) + A_2\bigl(v^k - \tilde{v}^k\bigr) + A_3\bigl(w^k - \tilde{w}^k\bigr)\bigr), \end{aligned} \tag{2.8}$$

$$\begin{aligned} & g(z^k, \tilde{z}^k) \\ & \quad = \begin{pmatrix} f_1(\tilde{u}^k) - A_1^\top\tilde{\lambda}^k + A_1^\top H[A_1(u^k - \tilde{u}^k) + A_2(v^k - \tilde{v}^k) + A_3(w^k - \tilde{w}^k) + \frac{1-\beta}{\beta}H^{-1}(\lambda^k - \tilde{\lambda}^k)] \\ f_2(\tilde{v}^k) - A_2^\top\tilde{\lambda}^k + A_2^\top H[A_1(u^k - \tilde{u}^k) + A_2(v^k - \tilde{v}^k) + A_3(w^k - \tilde{w}^k) + \frac{1-\beta}{\beta}H^{-1}(\lambda^k - \tilde{\lambda}^k)] \\ f_3(\tilde{w}^k) - A_3^\top\tilde{\lambda}^k + A_3^\top H[A_1(u^k - \tilde{u}^k) + A_2(v^k - \tilde{v}^k) + A_3(w^k - \tilde{w}^k) + \frac{1-\beta}{\beta}H^{-1}(\lambda^k - \tilde{\lambda}^k)] \\ A_1\tilde{u}^k + A_2\tilde{v}^k + A_3\tilde{w}^k - b \end{pmatrix}, \end{aligned} \tag{2.9}$$

$$G = \begin{pmatrix} (1+\mu)R_1 + A_1^\top HA_1 & 0 & 0 & 0 \\ 0 & (1+\mu)R_2 + A_2^\top HA_2 & 0 & 0 \\ 0 & 0 & (1+\mu)R_3 + A_3^\top HA_3 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta}H^{-1} \end{pmatrix},$$

and

$$M = \begin{pmatrix} R_1 + A_1^\top HA_1 & 0 & 0 & 0 \\ 0 & R_2 + A_2^\top HA_2 & 0 & 0 \\ 0 & 0 & R_3 + A_3^\top HA_3 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta}H^{-1} \end{pmatrix}.$$

Set $k := k + 1$ and go to Step 1.

Remark 2.1 As special cases, we can obtain some new LQP alternating methods as follows:

(a) If $u^{k+1} = \tilde{u}^k$, $v^{k+1} = \tilde{v}^k$, $w^{k+1} = \tilde{w}^k$ and $\lambda^{k+1} = \tilde{\lambda}^k$ in (2.2), (2.3), (2.4) and (2.5), respectively, we obtain a new method which can be viewed as an extension of that proposed in [10] for solving structured variational inequality with three separable operators in a parallel way.

(b) If $u^{k+1} = \tilde{u}^k$, $v^{k+1} = \tilde{v}^k$, $w^{k+1} = \tilde{w}^k$, $\lambda^{k+1} = \tilde{\lambda}^k$, and $\beta = 1$ in (2.2), (2.3), (2.4) and (2.5), respectively, we obtain a new method which can be viewed as an extension of that proposed in [12] for solving structured variational inequality with three separable operators in a parallel wise.

(c) If $\beta = 1$, the proposed method can be viewed as an extension of that proposed in [18] for solving structured variational inequality with three separable operators.

We need the following result in the convergence analysis of the proposed method.

Lemma 2.2 ([12]) *Let $q(u) \in \mathbb{R}^n$ be a monotone mapping of u with respect to $\mathbb{R}^n_+$ and $R \in \mathbb{R}^{n\times n}$ be a positive definite diagonal matrix. For a given $u^k > 0$, if $U_k := \operatorname{diag}(u^k_1, u^k_2, \dots, u^k_n)$ (the diagonal matrix with elements $u^k_1, u^k_2, \dots, u^k_n$) and u^{-1} be an n-vector whose jth element is $1/u_j$, then the equation*

$$q(u) + R\bigl[\bigl(u - u^k\bigr) + \mu\bigl(u^k - U_k^2 u^{-1}\bigr)\bigr] = 0 \tag{2.10}$$

has a unique positive solution u. Moreover, for any $v \geq 0$, we have

$$(v - u)^\top q(u) \geq \frac{1+\mu}{2}\bigl(\|u - v\|_R^2 - \bigl\|u^k - v\bigr\|_R^2\bigr) + \frac{1-\mu}{2}\bigl\|u^k - u\bigr\|_R^2. \tag{2.11}$$

The next theorem is useful for the convergence analysis.

Theorem 2.1 *For given $z^k \in \mathbb{R}^{n_1}_{++} \times \mathbb{R}^{n_2}_{++} \times \mathbb{R}^{n_3}_{++} \times \mathbb{R}^m$, let $\tilde{z}^k$ be generated by* (2.2)-(2.5). *Then*

$$\varphi\bigl(z^k, \tilde{z}^k\bigr) \geq \frac{2\beta - \sqrt{3}}{2\beta}\bigl\|z^k - \tilde{z}^k\bigr\|_G^2 \tag{2.12}$$

and

$$\tau_k \geq \frac{2\beta - \sqrt{3}}{2\beta}. \tag{2.13}$$

Proof It follows from (2.8) that

$$\begin{aligned}
\varphi\bigl(z^k, \tilde{z}^k\bigr) &= \bigl\|z^k - \tilde{z}^k\bigr\|_M^2 + \frac{1}{\beta}\bigl(\lambda^k - \tilde{\lambda}^k\bigr)^\top\bigl(A_1\bigl(u^k - \tilde{u}^k\bigr) + A_2\bigl(v^k - \tilde{v}^k\bigr) + A_3\bigl(w^k - \tilde{w}^k\bigr)\bigr) \\
&= \bigl\|u^k - \tilde{u}^k\bigr\|_{R_1}^2 + \bigl\|A_1u^k - A_1\tilde{u}^k\bigr\|_H^2 + \bigl\|v^k - \tilde{v}^k\bigr\|_{R_2}^2 + \bigl\|A_2v^k - A_2\tilde{v}^k\bigr\|_H^2 \\
&\quad + \bigl\|w^k - \tilde{w}^k\bigr\|_{R_3}^2 + \bigl\|A_3w^k - A_3\tilde{w}^k\bigr\|_H^2 + \frac{1}{\beta}\bigl\|\lambda^k - \tilde{\lambda}^k\bigr\|_{H^{-1}}^2 \\
&\quad + \frac{1}{\beta}\bigl(\lambda^k - \tilde{\lambda}^k\bigr)^\top\bigl(A_1\bigl(u^k - \tilde{u}^k\bigr) + A_2\bigl(v^k - \tilde{v}^k\bigr) + A_3\bigl(w^k - \tilde{w}^k\bigr)\bigr).
\end{aligned} \tag{2.14}$$

By using the Cauchy-Schwarz inequality, we have

$$\big(\lambda^k-\tilde\lambda^k\big)^\top\big(A_1\big(u^k-\tilde u^k\big)\big)\ge -\frac{1}{2}\bigg(\sqrt{3}\big\|A_1\big(u^k-\tilde u^k\big)\big\|_H^2+\frac{1}{\sqrt{3}}\big\|\lambda^k-\tilde\lambda^k\big\|_{H^{-1}}^2\bigg), \tag{2.15}$$

$$\big(\lambda^k-\tilde\lambda^k\big)^\top\big(A_2\big(v^k-\tilde v^k\big)\big)\ge -\frac{1}{2}\bigg(\sqrt{3}\big\|A_2\big(v^k-\tilde v^k\big)\big\|_H^2+\frac{1}{\sqrt{3}}\big\|\lambda^k-\tilde\lambda^k\big\|_{H^{-1}}^2\bigg), \tag{2.16}$$

and

$$\big(\lambda^k-\tilde\lambda^k\big)^\top\big(A_3\big(w^k-\tilde w^k\big)\big)\ge -\frac{1}{2}\bigg(\sqrt{3}\big\|A_3\big(w^k-\tilde w^k\big)\big\|_H^2+\frac{1}{\sqrt{3}}\big\|\lambda^k-\tilde\lambda^k\big\|_{H^{-1}}^2\bigg). \tag{2.17}$$

Substituting (2.15), (2.16), and (2.17) into (2.14), we get

$$\begin{aligned}
\varphi\big(z^k,\tilde z^k\big) &\ge \frac{2\beta-\sqrt{3}}{2\beta}\big(\big\|A_1u^k-A_1\tilde u^k\big\|_H^2+\big\|A_2v^k-A_2\tilde v^k\big\|_H^2+\big\|A_3w^k-A_3\tilde w^k\big\|_H^2\big)\\
&\quad+\frac{2-\sqrt{3}}{2\beta}\big\|\lambda^k-\tilde\lambda^k\big\|_{H^{-1}}^2+\big\|u^k-\tilde u^k\big\|_{R_1}^2+\big\|v^k-\tilde v^k\big\|_{R_2}^2+\big\|w^k-\tilde w^k\big\|_{R_3}^2\\
&\ge \frac{2\beta-\sqrt{3}}{2\beta}\bigg(\big\|A_1u^k-A_1\tilde u^k\big\|_H^2+\big\|A_2v^k-A_2\tilde v^k\big\|_H^2\\
&\quad+\big\|A_3w^k-A_3\tilde w^k\big\|_H^2+\frac{1}{\beta}\big\|\lambda^k-\tilde\lambda^k\big\|_{H^{-1}}^2\bigg)\\
&\quad+\frac{2\beta-\sqrt{3}}{2\beta}\big(\big\|u^k-\tilde u^k\big\|_{R_1}^2+\big\|v^k-\tilde v^k\big\|_{R_2}^2+\big\|w^k-\tilde w^k\big\|_{R_3}^2\big)\\
&= \frac{2\beta-\sqrt{3}}{2\beta}\big(\big\|z^k-\tilde z^k\big\|_G^2+(1-\mu)\big\|u^k-\tilde u^k\big\|_{R_1}^2\\
&\quad+(1-\mu)\big\|v^k-\tilde v^k\big\|_{R_2}^2+(1-\mu)\big\|w^k-\tilde w^k\big\|_{R_3}^2\big)\\
&\ge \frac{2\beta-\sqrt{3}}{2\beta}\big\|z^k-\tilde z^k\big\|_G^2.
\end{aligned}$$

Therefore, it follows from (2.7) and (2.12) that

$$\tau_k\ge\frac{2\beta-\sqrt{3}}{2\beta}, \tag{2.18}$$

and this completes the proof. □

3 Convergence rate

Recall that $\mathcal{Z}^*$ can be characterized as (see (2.3.2) in p.159 of [19])

$$\mathcal{Z}^*=\bigcap_{z\in\mathcal{Z}}\big\{\hat z\in\mathcal{Z}:(z-\hat z)^\top Q(z)\ge 0\big\}.$$

This implies that $\hat z$ is an approximate solution of SVI_3 with the accuracy $\epsilon>0$ if it satisfies

$$\hat z\in\mathcal{Z}\quad\text{and}\quad\sup_{z\in\mathcal{Z}}\big\{(z-\hat z)^\top Q(z)\big\}\le\epsilon. \tag{3.1}$$

Now we show that after t iterations of the proposed method, we can find a $\hat{z} \in \mathcal{Z}$ such that (3.1) is satisfied with $\epsilon = O(1/t)$.

We introduce the following matrices,

$$N = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ -\beta HA_1 & -\beta HA_2 & -\beta HA_3 & \beta I \end{pmatrix} \tag{3.2}$$

and

$$J = \begin{pmatrix} (1+\mu)R_1 + A_1^\top HA_1 & 0 & 0 & 0 \\ 0 & (1+\mu)R_2 + A_2^\top HA_2 & 0 & 0 \\ 0 & 0 & (1+\mu)R_3 + A_3^\top HA_3 & 0 \\ -A_1 & -A_2 & -A_3 & H^{-1} \end{pmatrix}. \tag{3.3}$$

By simple manipulations, we can find that $J = GN$.

Our analysis needs a new sequence defined by

$$\hat{z}^k = \begin{pmatrix} \hat{u}^k \\ \hat{v}^k \\ \hat{w}^k \\ \hat{\lambda}^k \end{pmatrix} = \begin{pmatrix} \tilde{u}^k \\ \tilde{v}^k \\ \tilde{w}^k \\ \lambda^k - H(A_1u^k + A_2v^k + A_3w^k - b) \end{pmatrix}. \tag{3.4}$$

Based on (3.2) and (3.4), we can easily have

$$z^k - \tilde{z}^k = N(z^k - \hat{z}^k). \tag{3.5}$$

Using (1.11), (2.9), and (3.4), we obtain

$$g(z^k, \tilde{z}^k) = Q(\hat{z}^k). \tag{3.6}$$

Lemma 3.1 *Let $\hat{z}^k$ be defined by* (3.4), $z \in \mathcal{Z}$, *and the matrix J be given by* (3.3). *Then*

$$(z - \hat{z}^k)^\top (Q(\hat{z}^k) - J(z^k - \hat{z}^k)) \geq -\mu \|u^k - \hat{u}^k\|_{R_1}^2 - \mu \|v^k - \hat{v}^k\|_{R_2}^2 - \mu \|w^k - \hat{w}^k\|_{R_3}^2. \tag{3.7}$$

Proof Applying Lemma 2.2 to (2.2), we get

$$\begin{aligned} &(u - \tilde{u}^k)^\top \{f_1(\tilde{u}^k) - A_1^\top [\lambda^k - H(A_1\tilde{u}^k + A_2v^k + A_3w^k - b)]\} \\ &\quad \geq \frac{1+\mu}{2}(\|\tilde{u}^k - u\|_{R_1}^2 - \|u^k - u\|_{R_1}^2) + \frac{1-\mu}{2}\|u^k - \tilde{u}^k\|_{R_1}^2. \end{aligned} \tag{3.8}$$

Since

$$\|u^k - u\|_{R_1}^2 = \|u^k - \tilde{u}^k\|_{R_1}^2 + \|\tilde{u}^k - u\|_{R_1}^2 + 2(\tilde{u}^k - u)^T R_1(u^k - \tilde{u}^k),$$

we have

$$(u - \tilde{u}^k)^\top R_1(u^k - \tilde{u}^k) = \frac{1}{2}(\|\tilde{u}^k - u\|_{R_1}^2 - \|u^k - u\|_{R_1}^2) + \frac{1}{2}\|u^k - \tilde{u}^k\|_{R_1}^2. \tag{3.9}$$

Adding (3.8) and (3.9), we obtain

$$\begin{aligned}&(u-\tilde{u}^k)^\top\{(1+\mu)R_1(u^k-\tilde{u}^k)-f_1(\tilde{u}^k)+A_1^\top[\lambda^k-H(A_1u^k+A_2v^k+A_3w^k-b)]\\&\quad+A_1^\top HA_1(u^k-\tilde{u}^k)\}\le\mu\|u^k-\tilde{u}^k\|_{R_1}^2.\end{aligned}\tag{3.10}$$

Similarly, applying Lemma 2.2 to (2.3), we get

$$\begin{aligned}&(v-\tilde{v}^k)^\top\{f_2(\tilde{v}^k)-A_2^\top[\lambda^k-H(A_1u^k+A_2\tilde{v}^k+A_3w^k-b)]\}\\&\quad\ge\frac{1+\mu}{2}(\|\tilde{v}^k-v\|_{R_2}^2-\|v^k-v\|_{R_2}^2)+\frac{1-\mu}{2}\|v^k-\tilde{v}^k\|_{R_2}^2.\end{aligned}\tag{3.11}$$

Similar to (3.9), we have

$$(v-\tilde{v}^k)^\top R_2(v^k-\tilde{v}^k)=\frac{1}{2}(\|\tilde{v}^k-v\|_{R_2}^2-\|v^k-v\|_{R_2}^2)+\frac{1}{2}\|v^k-\tilde{v}^k\|_{R_2}^2.\tag{3.12}$$

Adding (3.11) and (3.12), we have

$$\begin{aligned}&(v-\tilde{v}^k)^\top\{(1+\mu)R_2(v^k-\tilde{v}^k)-f_2(\tilde{v}^k)+A_2^\top[\lambda^k-H(A_1u^k+A_2v^k+A_3w^k-b)]\\&\quad+A_2^\top HA_2(v^k-\tilde{v}^k)\}\le\mu\|v^k-\tilde{v}^k\|_{R_2}^2.\end{aligned}\tag{3.13}$$

Similarly, we have

$$\begin{aligned}&(w-\tilde{w}^k)^\top\{(1+\mu)R_3(w^k-\tilde{w}^k)-f_3(\tilde{w}^k)+A_3^\top[\lambda^k-H(A_1u^k+A_2v^k+A_3w^k-b)]\\&\quad+A_3^\top HA_3(w^k-\tilde{w}^k)\}\le\mu\|w^k-\tilde{w}^k\|_{R_3}^2.\end{aligned}\tag{3.14}$$

Then, by using the notation of $\hat{z}^k$ in (3.4), (3.10), (3.13), and (3.14) can be written as

$$\begin{aligned}&(u-\hat{u}^k)^\top\{(1+\mu)R_1(u^k-\hat{u}^k)-f_1(\hat{u}^k)+A_1^\top\hat{\lambda}^k+A_1^\top HA_1(u^k-\hat{u}^k)\}\\&\quad\le\mu\|u^k-\hat{u}^k\|_{R_1}^2,\end{aligned}\tag{3.15}$$

$$\begin{aligned}&(v-\hat{v}^k)^\top\{(1+\mu)R_2(v^k-\hat{v}^k)-f_2(\hat{v}^k)+A_2^\top\hat{\lambda}^k+A_2^\top HA_2(v^k-\hat{v}^k)\}\\&\quad\le\mu\|v^k-\hat{v}^k\|_{R_2}^2,\end{aligned}\tag{3.16}$$

and

$$\begin{aligned}&(w-\hat{w}^k)^\top\{(1+\mu)R_3(w^k-\hat{w}^k)-f_3(\hat{w}^k)+A_3^\top\hat{\lambda}^k+A_3^\top HA_3(w^k-\hat{w}^k)\}\\&\quad\le\mu\|w^k-\hat{w}^k\|_{R_3}^2.\end{aligned}\tag{3.17}$$

In addition, it follows from (3.4) that

$$\begin{aligned}&A_1\hat{u}^k+A_2\hat{v}^k+A_3\hat{w}^k-b+H^{-1}(\hat{\lambda}^k-\lambda^k)\\&\quad-A_1(\hat{u}^k-u^k)-A_2(\hat{v}^k-v^k)-A_3(\hat{w}^k-w^k)=0.\end{aligned}\tag{3.18}$$

Combining (3.15)-(3.18), we get

$$\begin{pmatrix} u-\hat{u}^k \\ v-\hat{v}^k \\ w-\hat{w}^k \\ \lambda-\hat{\lambda}^k \end{pmatrix}^\top \begin{pmatrix} f_1(\hat{u}^k)-A_1^\top\hat{\lambda}^k-((1+\mu)R_1+A^THA_1)(u^k-\hat{u}^k) \\ f_2(\hat{v}^k)-A_2^\top\hat{\lambda}^k-((1+\mu)R_2+A_2^\top HA_2)(v^k-\hat{v}^k) \\ f_3(\hat{w}^k)-A_3^\top\hat{\lambda}^k-((1+\mu)R_3+A_3^\top HA_3)(w^k-\hat{w}^k) \\ A_1\hat{u}^k+A_2\hat{v}^k+A_3\hat{w}^k-b+A_1(u^k-\hat{u}^k)+A_2(v^k-\hat{v}^k)+A_3(w^k-\hat{w}^k)-H^{-1}(\lambda^k-\hat{\lambda}^k) \end{pmatrix}$$
$$\geq -\mu\|u^k-\hat{u}^k\|_{R_1}^2-\mu\|v^k-\hat{v}^k\|_{R_2}^2-\mu\|w^k-\hat{w}^k\|_{R_3}^2. \tag{3.19}$$

Recall the definition of J in (3.3), we obtain the assertion (3.7). The proof is completed. □

Lemma 3.2 *For given $z^k\in\mathbb{R}_{++}^{n_1}\times\mathbb{R}_{++}^{n_2}\times\mathbb{R}_{++}^{n_3}\times\mathbb{R}^m$ and $z_*^k:=P_{\mathcal{Z},G}[z^k-\tau_kG^{-1}g(z^k,\tilde{z}^k)]$, we have*

$$\gamma\tau_k\bigl(z-\hat{z}^k\bigr)^\top Q(z)+\frac{1}{2}\bigl(\|z-z^k\|_G^2-\|z-z_*^k\|_G^2\bigr)\geq\frac{1}{2}\gamma(2-\gamma)\tau_k^2\|z^k-\tilde{z}^k\|_G^2. \tag{3.20}$$

Proof Since $z_*^k\in\mathcal{Z}$, substituting $z=z_*^k$ in (3.7), we get

$$\begin{aligned}
&\gamma\tau_k\bigl(z_*^k-\hat{z}^k\bigr)^\top Q\bigl(\hat{z}^k\bigr) &(3.21)\\
&\quad\geq\gamma\tau_k\bigl(z_*^k-\hat{z}^k\bigr)^\top J\bigl(z^k-\hat{z}^k\bigr)-\mu\gamma\tau_k\|u^k-\hat{u}^k\|_{R_1}^2-\mu\gamma\tau_k\|v^k-\hat{v}^k\|_{R_2}^2\\
&\qquad-\mu\gamma\tau_k\|w^k-\hat{w}^k\|_{R_3}^2\\
&\quad=\gamma\tau_k\bigl(z^k-\hat{z}^k\bigr)^\top J\bigl(z^k-\hat{z}^k\bigr)+\gamma\tau_k\bigl(z_*^k-z^k\bigr)^\top J\bigl(z^k-\hat{z}^k\bigr)\\
&\qquad-\mu\gamma\tau_k\|u^k-\hat{u}^k\|_{R_1}^2-\mu\gamma\tau_k\|v^k-\hat{v}^k\|_{R_2}^2-\mu\gamma\tau_k\|w^k-\hat{w}^k\|_{R_3}^2\\
&\quad=\gamma\tau_k\bigl(z^k-\tilde{z}^k\bigr)^\top\bigl(N^{-1}\bigr)^\top JN^{-1}\bigl(z^k-\tilde{z}^k\bigr)+\gamma\tau_k\bigl(z_*^k-z^k\bigr)^\top JN^{-1}\bigl(z^k-\tilde{z}^k\bigr)\\
&\qquad-\gamma\tau_k\mu\|u^k-\hat{u}^k\|_{R_1}^2-\gamma\tau_k\mu\|v^k-\hat{v}^k\|_{R_2}^2-\mu\gamma\tau_k\|w^k-\hat{w}^k\|_{R_3}^2\\
&\quad=\gamma\tau_k\bigl(z^k-\tilde{z}^k\bigr)^\top\bigl(N^{-1}\bigr)^\top G\bigl(z^k-\tilde{z}^k\bigr)-\gamma\tau_k\mu\|u^k-\hat{u}^k\|_{R_1}^2-\gamma\tau_k\mu\|v^k-\hat{v}^k\|_{R_2}^2\\
&\qquad-\mu\gamma\tau_k\|w^k-\hat{w}^k\|_{R_3}^2+\gamma\tau_k\bigl(z_*^k-z^k\bigr)^\top G\bigl(z^k-\tilde{z}^k\bigr)\\
&\quad=\gamma\tau_k\bigl(z^k-\tilde{z}^k\bigr)^\top\bigl(N^{-1}\bigr)^\top M\bigl(z^k-\tilde{z}^k\bigr)+\gamma\tau_k\bigl(z_*^k-z^k\bigr)^\top G\bigl(z^k-\tilde{z}^k\bigr)\\
&\quad\geq\gamma\tau_k\varphi\bigl(z^k,\tilde{z}^k\bigr)+\gamma\tau_k\bigl(z_*^k-z^k\bigr)^\top G\bigl(z^k-\tilde{z}^k\bigr)\\
&\quad\geq\gamma\tau_k\varphi\bigl(z^k,\tilde{z}^k\bigr)-\frac{1}{2}\|z^k-z_*^k\|_G^2-\frac{1}{2}\gamma^2\tau_k^2\|z^k-\tilde{z}^k\|_G^2\\
&\quad=\frac{1}{2}\gamma(2-\gamma)\tau_k^2\|z^k-\tilde{z}^k\|_G^2-\frac{1}{2}\|z^k-z_*^k\|_G^2. &(3.22)
\end{aligned}$$

Using (3.6), z_*^k is the projection of $z^k-\gamma\tau_kG^{-1}Q(\hat{z}^k)$ on $\mathcal{Z}$, it follows from (2.1) that

$$\bigl(z^k-\gamma\tau_kG^{-1}Q\bigl(\hat{z}^k\bigr)-z_*^k\bigr)^\top G\bigl(z-z_*^k\bigr)\leq 0,\quad\forall z\in\mathcal{Z},$$

and consequently, we have

$$\gamma\tau_k\bigl(z-z_*^k\bigr)^\top Q\bigl(\hat{z}^k\bigr)\geq\bigl(z^k-z_*^k\bigr)^\top G\bigl(z-z_*^k\bigr).$$

Using the identity $x^\top Gy = \frac{1}{2}(\|x\|_G^2 - \|x-y\|_G^2 + \|y\|_G^2)$ to the right hand side of the last inequality, we obtain

$$\gamma\tau_k\big(z-z_*^k\big)^\top Q\big(\hat{z}^k\big) \geq \frac{1}{2}\big(\big\|z-z_*^k\big\|_G^2 - \big\|z-z^k\big\|_G^2\big) + \frac{1}{2}\big\|z^k - z_*^k\big\|_G^2. \tag{3.23}$$

Adding (3.21) and (3.23), we get

$$\gamma\tau_k\big(z-\hat{z}^k\big)^\top Q\big(\hat{z}^k\big) + \frac{1}{2}\big(\big\|z-z^k\big\|_G^2 - \big\|z-z_*^k\big\|_G^2\big) \geq \frac{1}{2}\gamma(2-\gamma)\tau_k^2\big\|z^k-\tilde{z}^k\big\|_G^2,$$

and by using the monotonicity of Q, we obtain (3.20) and the proof is completed. □

Lemma 3.3 *Let $z^k \in \mathbb{R}_{++}^{n_1} \times \mathbb{R}_{++}^{n_2} \times \mathbb{R}_{++}^{n_3} \times \mathbb{R}^m$ and $z^{k+1}(\tau_k)$ be generated by* (2.6). *Then*

$$\gamma\sigma\tau_k\big(z-\hat{z}^k\big)^\top Q(z) + \frac{1}{2}\big(\big\|z-z^k\big\|_G^2 - \big\|z-z^{k+1}(\tau_k)\big\|_G^2\big) \geq \frac{1}{2}\sigma\gamma(2-\gamma)\tau_k^2\big\|z-\tilde{z}^k\big\|_G^2. \tag{3.24}$$

Proof We have

$$\begin{aligned}
&\big\|z-z^k\big\|_G^2 - \big\|z-z^{k+1}(\tau_k)\big\|_G^2 &(3.25)\\
&\quad= \big\|z^k-z\big\|_G^2 - \big\|z^k - \sigma\big(z^k-z_*^k\big) - z\big\|_G^2\\
&\quad= 2\sigma\big(z^k-z\big)^\top G\big(z^k-z_*^k\big) - \sigma^2\big\|z^k-z_*^k\big\|_G^2\\
&\quad= 2\sigma\big(\big\|z^k-z_*^k\big\|_G^2 - \big(z-z_*^k\big)^\top G\big(z^k-z_*^k\big)\big) - \sigma^2\big\|z^k-z_*^k\big\|_G^2. &(3.26)
\end{aligned}$$

Using the identity

$$\big(z-z_*^k\big)^\top G\big(z^k-z_*^k\big) = \frac{1}{2}\big(\big\|z_*^k-z\big\|_G^2 - \big\|z^k-z\big\|_G^2\big) + \frac{1}{2}\big\|z^k-z_*^k\big\|_G^2,$$

we get

$$\big\|z^k-z_*^k\big\|_G^2 - 2\big(z-z_*^k\big)^\top G\big(z^k-z_*^k\big) = \big\|z^k-z\big\|_G^2 - \big\|z_*^k-z\big\|_G^2. \tag{3.27}$$

Substituting (3.27) into (3.25), we obtain

$$\begin{aligned}
\big\|z-z^k\big\|_G^2 - \big\|z-z^{k+1}(\tau_k)\big\|_G^2 &= \sigma\big(\big\|z-z^k\big\|_G^2 - \big\|z-z_*^k\big\|_G^2\big) + \sigma(1-\sigma)\big\|z^k-z_*^k\big\|_G^2\\
&\geq \sigma\big(\big\|z-z^k\big\|_G^2 - \big\|z-z_*^k\big\|_G^2\big). &(3.28)
\end{aligned}$$

Substituting (3.28) into (3.20), we obtain (3.24), the required result. □

Theorem 3.1 *Let z^* be a solution of* SVI_3 *and $z^{k+1}(\tau_k)$ be generated by* (2.6). *Then z^k and $\tilde{z}^k$ are bounded, and*

$$\big\|z^{k+1}(\tau_k) - z^*\big\|_G^2 \leq \big\|z^k-z^*\big\|_G^2 - c\big\|z^k-\tilde{z}^k\big\|_G^2, \tag{3.29}$$

where

$$c := \frac{\sigma\gamma(2-\gamma)(2\beta-\sqrt{3})^2}{4\beta^2} > 0.$$

Proof Setting $z = z^*$ in (3.24), we obtain

$$\begin{aligned}\|z^{k+1}(\tau_k) - z^*\|_G^2 &\le \|z^k - z^*\|_G^2 - \sigma\gamma(2-\gamma)\tau_k^2\|z^k - \tilde{z}^k\|_G^2 + 2\gamma\sigma\tau_k(z^* - \hat{z}^k)^\top Q(z^*) \\ &\le \|z^k - z^*\|_G^2 - \sigma\gamma(2-\gamma)\tau_k^2\|z^k - \tilde{z}^k\|_G^2 \\ &\le \|z^k - z^*\|_G^2 - \frac{\sigma\gamma(2-\gamma)(2\beta-\sqrt{3})^2}{4\beta^2}\|z^k - \tilde{z}^k\|_G^2.\end{aligned}$$

Then we have

$$\|z^{k+1}(\tau_k) - z^*\|_G \le \|z^k - z^*\|_G \le \cdots \le \|z^0 - z^*\|_G,$$

and thus, $\{z^k\}$ is a bounded sequence.

It follows from (3.29) that

$$\sum_{k=0}^{\infty} c\|z^k - \tilde{z}^k\|_G^2 < +\infty,$$

which means that

$$\lim_{k\to\infty}\|z^k - \tilde{z}^k\|_G = 0. \tag{3.30}$$

Since $\{z^k\}$ is a bounded sequence, we conclude that $\{\tilde{z}^k\}$ is also bounded. □

The global convergence of the proposed method can be proved by similar arguments as in [18]. Hence the proof is omitted.

Theorem 3.2 *The sequence $\{z^k\}$ generated by the proposed method converges to some z^∞ which is a solution of* SVI_3.

Now, we are ready to present the $O(1/t)$ convergence rate of the proposed method.

Theorem 3.3 *For any integer $t > 0$, we have a $\hat{z}_t \in \mathcal{Z}$ which satisfies*

$$(\hat{z}_t - z)^\top Q(z) \le \frac{1}{2\gamma\sigma\Upsilon_t}\|z - z^0\|_G^2, \quad \forall z \in \mathcal{Z},$$

where

$$\hat{z}_t = \frac{1}{\Upsilon_t}\sum_{k=0}^{t}\tau_k\hat{z}^k \quad \textit{and} \quad \Upsilon_t = \sum_{k=0}^{t}\tau_k.$$

Proof Summing the inequality (3.24) over $k = 0, \ldots, t$, we obtain

$$\left(\left(\sum_{k=0}^{t}\gamma\sigma\tau_k\right)z - \sum_{k=0}^{t}\gamma\sigma\tau_k\hat{z}^k\right)^\top Q(z) + \frac{1}{2}\|z - z^0\|_G^2 \ge 0.$$

Using the notations of Υ_t and $\hat{z}_t$ in the above inequality, we derive

$$(\hat{z}_t - z)^\top Q(z) \le \frac{1}{2\gamma\sigma\Upsilon_t}\left\|z - z^0\right\|_G^2, \quad \forall z \in \mathcal{Z}.$$

Indeed, $\hat{z}_t \in \mathcal{Z}$ because it is a convex combination of $\hat{z}^0, \hat{z}^1, \ldots, \hat{z}^t$. The proof is completed. □

Remark 3.1 It follows from (2.13) that

$$\Upsilon_t \ge \frac{2\beta - \sqrt{3}}{2\beta}(t+1).$$

Suppose that, for any compact set $\mathcal{D} \subset \mathcal{Z}$, let $d = \sup\{\|z - z^0\|_G | z \in \mathcal{D}\}$. For any given $\epsilon > 0$, after at most

$$t = \left\lceil \frac{\beta d^2}{(2\beta - \sqrt{3})\gamma\sigma\epsilon} \right\rceil$$

iterations, we have

$$(\hat{z}_t - z)^T Q(z) \le \epsilon, \quad \forall z \in \mathcal{D}.$$

That is, the $O(1/t)$ convergence rate is established in an ergodic sense.

4 Preliminary computational results

In this section, we present some numerical examples to illustrate the proposed method. We consider the following optimization problem with matrix variables, which is studied in [18]:

$$\min\left\{\frac{1}{2}\|U - C\|_F^2 \middle| U \in S_+^n\right\}, \tag{4.1}$$

where $\|\cdot\|_F$ is the matrix Fröbenius norm, *i.e.*, $\|C\|_F = (\sum_{i=1}^n \sum_{j=1}^n |C_{ij}|^2)^{1/2}$ and

$$S_+^n = \left\{M \in \mathbb{R}^{n\times n} : M^\top = M, M \succeq 0\right\}.$$

It has been shown in [18] that solving problem (4.1) is equivalent to the following variational inequality problem: Find $X^* = (U^*, V^*, Z^*) \in \Omega = S_+^n \times S_+^n \times \mathbb{R}^{n\times n}$ such that

$$\begin{cases} \langle U - U^*, (U^* - C) - Z^*\rangle \ge 0, \\ \langle V - V^*, (V^* - C) + Z^*\rangle \ge 0, \quad \forall X = (U, V, Z) \in \Omega, \\ U^* - V^* = 0. \end{cases} \tag{4.2}$$

The problem (4.2) is a special case of (1.3)-(1.4) with matrix variables where $A_1 = I_{n\times n}$, $A_2 = -I_{n\times n}$, $b = 0$, $f_1(U) = U - C$, $f_2(V) = V - C$, and $\mathcal{W} = S_+^n \times S_+^n \times \mathbb{R}^{n\times n}$. For simplification, we take $R_1 = r_1 I_{n\times n}$, $R_2 = r_2 I_{n\times n}$, and $H = I_{n\times n}$ where $r_1 > 0$ and $r_2 > 0$ are scalars. In all tests

Table 1 Numerical results for problem (4.1) with $r_1 = 0.5$, $r_2 = 5$

Dimension of the problem	The proposed method		The method in [18]		The method in [17]	
	k	**CPU (Sec.)**	***k***	**CPU (Sec.)**	***k***	**CPU (Sec.)**
100	43	0.83	49	0.96	71	2.47
300	48	3.98	53	4.85	79	6.33
500	50	11.54	56	13.27	82	20.2
700	52	29.91	57	34.33	85	44.06

Table 2 Numerical results for problem (4.1) with $r_1 = 1$, $r_2 = 10$

Dimension of the problem	The proposed method		The method in [18]		The method in [17]	
	k	**CPU (Sec.)**	***k***	**CPU (Sec.)**	***k***	**CPU (Sec.)**
100	106	0.87	109	1.18	124	2.61
300	119	6.85	123	7.54	140	9.06
500	125	25.85	128	29.71	147	37.25
700	129	53.19	132	58.06	152	64.35

we take $\mu = 0.5$, $\beta = 0.88$, $C = \mathrm{rand}(n)$, and $(U^0, V^0, Z^0) = (I_{n\times n}, I_{n\times n}, 0_{n\times n})$ as the initial point in the test. The iteration is stopped as soon as

$$\max\{\|U^k - \tilde{U}^k\|, \|V^k - \tilde{V}^k\|, \|Z^k - \tilde{Z}^k\|\} \le 10^{-6}.$$

All codes were written in Matlab, we compare the proposed method with those in [18] and [17]. The iteration numbers, denoted by k, and the computational time for the problem (4.1) with different dimensions are given in Tables 1-2.

Tables 1-2 show that the proposed method is more flexible and efficient for the problem tested.

5 Conclusions

In this paper, we proposed a new modified logarithmic-quadratic proximal alternating direction method for solving structured variational inequalities with three separable operators. The prediction point is obtained by solving series of related systems of nonlinear equations in a parallel way. Global convergence of the proposed method is proved under mild assumptions. We have proved the $O(1/t)$ convergence rate of the parallel LQP alternating direction method. Some preliminary numerical examples are reported to verify the effectiveness of the proposed method in practice.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
In this research, the second author was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. Therefore, he acknowledges with thanks DSR for the technical and financial support. The research part of the

third author was done during his visit to King Fahd University of Petroleum & Minerals (KFUPM), Dhahran, Saudi Arabia. He is grateful to KFUPM for providing excellent research facilities to carry out his part of this research.

Received: 3 October 2016 Accepted: 8 November 2016 Published online: 22 November 2016

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