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# Character sums over generalized Lehmer numbers

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**Abstract**

Let $q > 2$ be an integer, $n \geq 2$ be a fixed integer with $(n, q) = 1$, $\psi$ be a non-principal Dirichlet character $\bmod q$. An upper bound estimate for character sums of the form

$$\sum_{a \in \mathcal{C}(1,q)} \psi(a)$$

is given, where $\mathcal{C}(1, q) = \{a \mid 1 \leq a \leq q - 1, a\overline{a} \equiv 1(\bmod q), n \nmid (a + \overline{a})\}$.

**MSC:** 11L05; 11L40; 11N37

**Keywords:** Lehmer number; character sums; Kloosterman sums; upper bound estimate

## 1 Introduction

Let $q$ be an odd integer, $c$ be a fixed positive integer with $(c, q) = 1$. For each integer $a$ with $1 \leq a \leq q - 1$ and $(a, q) = 1$, it is clear that there exists one and only one integer $b$ with $1 \leq b \leq q - 1$ such that $ab \equiv c(\bmod q)$. If $a$ and $b$ are of opposite parity, then $a$ is called a Lehmer number. Let $\mathcal{A}(c, q)$ denote the set of all Lehmer numbers, and $r(c, q)$ the number of $\mathcal{A}(c, q)$. Lehmer [1] posed the problem of finding $r(1, q)$.

Before proceeding we need to recall that the notations $U = O(V)$ and $U \ll V$ are equivalent to $|U| \leq cV$ for some constant $c > 0$. We write $\ll_\rho$ and $O_\rho$ to indicate that this constant may depend on the parameter $\rho$. $\sum'$ means summing over reduced residue classes, $\overline{a}$ denotes the multiplicative inverse of $a$ modulo $q$ and for a real $x$ we denote $e(x) = e^{2\pi i x}$, $\{x\}$ the fractal part of $x$, and $\langle x \rangle = \min\{\{x\}, 1 - \{x\}\}$.

In 1993, Zhang [2] proved that

$$r(1, p^\alpha) = \frac{\phi(p^\alpha)}{2} + O\big(p^{\alpha/2} \ln^3(p^\alpha)\big),$$
$$r(1, pl) = \frac{\phi(pl)}{2} + O\big((pl)^{1/2} \ln^2(pl)\big),$$

where $p$, $l$ are two distinct odd primes, $\alpha$ is a positive integer, and $\phi(q)$ is the Euler function. For arbitrary odd integer $q \geq 3$, he [3] soon obtained

$$r(1, q) = \frac{\phi(q)}{2} + O\big(q^{1/2} d^2(q) \ln^2 q\big),$$

where $d(q)$ is the classical divisor function.

Later, Lu and Yi [4] generalized this problem to incomplete intervals. In fact, let $q \geq 3$ be an integer, $n \geq 2$ and $c$ be two fixed integers with $(n,q) = (c,q) = 1$, $0 < \delta_1, \delta_2 \leq 1$, they defined

$$r_n(\delta_1,\delta_2,c;q) = \sideset{}{'}\sum_{\substack{a\leq\delta_1 q \\ ab\equiv c(\bmod q) \\ n\nmid(a+b)}} \sideset{}{'}\sum_{b\leq\delta_2 q} 1,$$

and got an asymptotic formula as follows:

$$r_n(\delta_1,\delta_2,c;q) = \left(1-\frac{1}{n}\right)\delta_1\delta_2\phi(q) + O_n\left(q^{1/2}d^6(q)\log^2 q\right).$$

Recently, interesting connections between Lehmer numbers and character sums were investigated by some scholars. For example, for an odd prime $p$, and a fixed prime $w$ less than $p$, let

$$\mathcal{B}(w,p) = \left\{a \mid 1 \leq a \leq p-1, a\overline{a} \equiv 1(\bmod p), a \equiv \overline{a}(\bmod w)\right\}.$$

Then, for any non-principal Dirichlet character $\chi$ mod $w$, Ma, Zhang and Zhang [5] got an upper bound estimate of character sums over $\mathcal{B}(w,p)$ as

$$\sum_{\substack{a=1 \\ a\in\mathcal{B}(w,p)}}^{p-1} \chi(a) \ll_w p^{1/2+\epsilon}.$$

At almost the same time, Han and Zhang [6] obtained an upper bound estimate of the character sums over Lehmer numbers as

$$\sum_{a\in\mathcal{A}(1,p)} \chi(a) = \sum_{\substack{a=1 \\ 2\nmid(a+\overline{a})}}^{p-1} \chi(a) \ll p^{1/2}\ln^2 p, \tag{1.1}$$

where $\chi$ is an arbitrary non-principal character modulo an odd prime $p$.

The results of character sums over other special numbers or polynomials can also be found in [7] and [8]. For more properties of character sums and their various applications, see [9, 10] and the references therein.

It seems that (1.1) cannot be extended to arbitrary integer $q$ by their methods in [6]. However, relying on the methods in [4], we can overcome the obstacles.

Let $q \geq 3$ be an integer, $n \geq 2$ be a fixed integer with $(n,q) = 1$, $\psi$ be a non-principal Dirichlet character modulo $q$. If $n \nmid (a+\overline{a})$, then $a$ is called a generalized Lehmer number. Denote the set of all generalized Lehmer numbers by

$$\mathcal{C}(1,q) = \left\{a \mid 1 \leq a \leq q-1, a\overline{a} \equiv 1(\bmod q), n \nmid (a+\overline{a})\right\}.$$

Following the same technique as in [4], we obtain the following.

**Theorem** *Let $q \geq 3$ be an integer, $n \geq 2$ be a fixed integer with $(n,q) = 1$, $\psi$ be a non-principal Dirichlet character* mod $q$. *Then we have the upper bound estimate*

$$\sum_{a \in \mathcal{C}(1,q)} \psi(a) = \sum_{\substack{a=1 \\ n \nmid (a+\overline{a})}}^{q}{}' \psi(a) \ll_n q^{1/2} d^5(q) \log^2 q.$$

Let $q \geq 3$ be an odd integer, $n = 2$ in the theorem, we may immediately obtain the following.

**Corollary 1** *Let $\psi$ be a non-principal Dirichlet character modulo $q$. Then we have*

$$\sum_{a \in \mathcal{A}(1,q)} \psi(a) = \sum_{\substack{a=1 \\ 2 \nmid (a+\overline{a})}}^{q}{}' \psi(a) \ll q^{1/2} d^5(q) \log^2 q.$$

Let $q$ be an odd prime $p$, $n = 2$ in Corollary 1, then (1.1) can be deduced directly as follows.

**Corollary 2** *Let $\psi$ be a non-principal Dirichlet character modulo $p$. Then we have*

$$\sum_{a \in \mathcal{A}(1,p)} \psi(a) \ll p^{1/2} \log^2 p.$$

## 2 Some lemmas

To prove the theorem, we need the following several lemmas. First we need an upper bound estimate of the general Kloosterman sum $S(m,n,\chi;q)$ as follows.

**Lemma 1** *Let $q$ be a positive integer and $\chi$ a Dirichlet character* mod $q$. *Then for any integers $m$ and $n$, we have*

$$S(m,n,\chi;q) \ll q^{1/2}(m,n,q)^{1/2} d(q),$$

*where $S(m,n,\chi;q)$ is defined by*

$$S(m,n,\chi;q) = \sum_{a \bmod q} \chi(a) e\left(\frac{ma + n\overline{a}}{q}\right).$$

*Proof* See Lemma 1 of [7]. □

**Lemma 2** *Let $q$ be a positive integer, $\chi_0$ be the principal Dirichlet character* mod $q$, *$\psi$ be a non-principal character* mod$q$, *$r_1$, $r_2$ be integers with $1 \leq r_1, r_2 \leq q-1$. Then we have*

$$\left|G(r_1,\psi)G(r_2,\chi_0)\right| \leq q^{1/2}(r_1,q)(r_2,q).$$

*Proof* By Lemma 2 of Chapter 1.2 in [11], we have

$$G(r_2,\chi_0) = \mu\left(\frac{q}{(r_2,q)}\right)\phi(q)\phi^{-1}\left(\frac{q}{(r_2,q)}\right) \leq (r_2,q),$$

where we have used the fact $\phi(q)/\phi(t) \leq q/t$ if $t \mid q$.

Note that $\psi$ is a non-principal character $\bmod q$, we only need to consider the following cases.

If $(r_1, q) = 1$, we have

$$\left|G(r_1,\psi)\right| = \left|\overline{\psi}(r_1)G(1,\psi)\right| = \left|G(1,\psi)\right| = q^{1/2}.$$

If $(r_1, q) > 1$, and $\psi$ is a primitive character $\bmod q$, we have

$$\left|G(r_1,\psi)\right| = \left|\overline{\psi}(r_1)G(1,\psi)\right| \le q^{1/2}.$$

If $(r_1, q) > 1$, and $\psi$ is a non-primitive character $\bmod q$, then Lemma 5 of Chapter 1.2 in [11] indicates that there exists one and only one $q^*$ such that $q^* \mid q$, with $\chi^*$ the primitive character $\bmod q^*$ corresponding $\chi$. Thus

$$\begin{aligned}\left|G(r_1,\psi)\right| &\le \left|\overline{\chi}^*\left(\frac{r_1}{(r_1,q)}\right)\chi^*\left(\frac{q}{q^*(r_1,q)}\right)\mu\left(\frac{q}{q^*(r_1,q)}\right)\phi(q)\phi^{-1}\left(\frac{q}{(r_1,q)}\right)\tau\left(\chi^*\right)\right| \\ &\le q^{1/2}(r_1,q).\end{aligned}$$

Combining the above, we have

$$\left|G(r_1,\psi)G(r_2,\chi_0)\right| \le q^{1/2}(r_1,q)(r_2,q). \qquad \square$$

**Lemma 3** *Let $q \ge 3$ be an integer, $\chi$, $\psi$ be Dirichlet characters $\bmod q$ such that $\psi \ne \chi_0$ and $\psi\overline{\psi} = \chi_0$. Then we have the estimate*

$$\sum_{\substack{\chi \bmod q \\ \chi \ne \chi_0 \\ \chi \ne \overline{\psi}}} G(r_1,\chi\psi)G(r_2,\chi) \ll \phi(q)q^{1/2}(r_1,q)^{1/2}(r_2,q)^{1/2}d(q).$$

*Proof* Combining Lemmas 1 and 2, we have

$$\begin{aligned}&\sum_{\substack{\chi \bmod q \\ \chi \ne \chi_0 \\ \chi \ne \overline{\psi}}} G(r_1,\chi\psi)G(r_2,\chi) \\ &\quad= \sum_{\chi \bmod q} G(r_1,\chi\psi)G(r_2,\chi) - G(r_1,\psi)G(r_2,\chi_0) - G(r_1,\chi_0)G(r_2,\overline{\psi}) \\ &\quad= \sum_{\chi \bmod q}\sum_{a=1}^{q}\chi\psi(a)e\left(\frac{ar_1}{q}\right)\sum_{b=1}^{q}\chi(b)e\left(\frac{br_2}{q}\right) \\ &\qquad - G(r_1,\psi)G(r_2,\chi_0) - G(r_1,\chi_0)G(r_2,\overline{\psi}) \\ &\quad= \phi(q)\sideset{}{'}\sum_{a=1}^{q}\psi(a)\sideset{}{'}\sum_{\substack{b=1 \\ ab\equiv 1(\bmod q)}}^{q} e\left(\frac{ar_1+br_2}{q}\right) \\ &\quad= \phi(q)S(r_1,r_2,\psi;q) - G(r_1,\psi)G(r_2,\chi_0) - G(r_1,\chi_0)G(r_2,\overline{\psi}) \\ &\quad\ll \phi(q)q^{1/2}(r_1,r_2,q)^{1/2}d(q) + q^{1/2}(r_1,q)(r_2,q) \\ &\quad\ll \phi(q)q^{1/2}(r_1,q)^{1/2}(r_2,q)^{1/2}d(q). \qquad \square\end{aligned}$$

**Lemma 4** *Let* $0 < \rho \leq \frac{1}{2}$, $x_0, x_1, \ldots, x_k$ *be a sequence of real numbers such that*

$$\langle x_k - x_{k'} \rangle \geq \rho, \qquad x_k \neq x_{k'},$$

*and* $\langle x_0 \rangle = \min\{\langle x_1 \rangle, \ldots, \langle x_k \rangle\}$. *Then we have*

$$\sum_{k=1}^{K} \frac{1}{\langle x_k \rangle} \ll \rho^{-1} \log(K+1).$$

*Proof* See Lemma 2 of Chapter 5.1 in [11]. □

**Lemma 5** *Let* $q \geq 3$ *be an integer,* $\psi$ *be a character* mod $q$, $n \geq 2$ *be a fixed integer with* $(n,q) = 1$, $l$ *be an integer with* $1 \leq l \leq n$. *Then we have*

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \psi(a) e\left(\frac{(a+b)l}{n}\right) \ll q^{1/2} \phi(q) d^2(q) \log q.$$

*Proof* The relations

$$1 \leq l \leq n, \qquad 1 \leq r \leq q-1, \qquad (n,q) = 1$$

imply that

$$\frac{l}{n} - \frac{r}{q} \neq 0.$$

And also

$$\psi(a) = \frac{1}{q} \sum_{r=1}^{q} G(r,\psi) e\left(-\frac{ar}{q}\right) = \frac{1}{q} \sum_{r=1}^{q-1} G(r,\psi) e\left(-\frac{ar}{q}\right).$$

Thus

$$\begin{aligned}
&\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' \psi(a) e\left(\frac{(a+b)l}{n}\right) \\
&\quad = \sum_{a=1}^{q} \psi(a) e\left(\frac{al}{n}\right) \sum_{b=1}^{q}{}' e\left(\frac{bl}{n}\right) \\
&\quad = \sum_{a=1}^{q} \frac{1}{q} \sum_{r=1}^{q-1} G(r,\psi) e\left(-\frac{ar}{q}\right) e\left(\frac{al}{n}\right) \sum_{b=1}^{q}{}' e\left(\frac{bl}{n}\right) \\
&\quad = \frac{1}{q} \sum_{r=1}^{q-1} G(r,\psi) \sum_{b=1}^{q}{}' e\left(\frac{bl}{n}\right) \sum_{a=1}^{q} e\left(\left(\frac{l}{n} - \frac{r}{q}\right) a\right) \\
&\quad = \frac{1}{q} \sum_{b=1}^{q}{}' e\left(\frac{bl}{n}\right) \left(\sum_{r=1}^{q-1} G(r,\psi) \frac{f(l,r,n,q)}{e(\frac{r}{q} - \frac{l}{n}) - 1}\right),
\end{aligned}$$

where $f(l,r,n,q) = 1 - e((\frac{l}{n} - \frac{r}{q})q)$.

Apply the upper bound

$$\left|G(r,\psi)\right| \le q^{1/2}(r,q),$$

we have

$$\begin{aligned}
\sum_{r=1}^{q-1} G(r,\psi)\frac{f(l,r,n,q)}{e(\frac{r}{q}-\frac{l}{n})-1} &\ll q^{1/2}\sum_{r=1}^{q-1}\frac{(r,q)}{|e(\frac{r}{q}-\frac{l}{n})-1|} \\
&\ll q^{1/2}\sum_{r=1}^{q-1}\frac{(r,q)}{|\sin\pi(\frac{r}{q}-\frac{l}{n})|} \ll q^{1/2}\sum_{r=1}^{q-1}\frac{(r,q)}{\langle\frac{r}{q}-\frac{l}{n}\rangle} \\
&= q^{1/2}\sum_{\substack{d|q\\ d<q}}\sum_{\substack{r\le q-1\\ (r,q)=d}}\frac{d}{\langle\frac{r}{q}-\frac{l}{n}\rangle} = q^{1/2}\sum_{\substack{d|q\\ d<q}} d\sum_{\substack{m\le\frac{q-1}{d}\\ (m,q)=1}}\frac{1}{\langle\frac{md}{q}-\frac{l}{n}\rangle} \\
&= q^{1/2}\sum_{\substack{d|q\\ d<q}} d\sum_{k|q}\mu(k)\sum_{m\le\frac{q-1}{kd}}\frac{1}{\langle\frac{mkd}{q}-\frac{l}{n}\rangle}.
\end{aligned}$$

Now write $\frac{k}{q/d}=\frac{h_0}{q_0}$, where $q_0\ge 1$, $(h_0,q_0)=1$, we have $\frac{q}{kd}=\frac{q_0}{h_0}\le q_0\le\frac{q}{d}$. Then Lemma 4 implies

$$\left\langle\frac{m_ikd}{q}-\frac{m_jkd}{q}\right\rangle=\left\langle\frac{(m_i-m_j)h_0}{q_0}\right\rangle\ge\frac{1}{q_0}\quad \text{if } i\ne j, 1\le i,j\le\frac{q-1}{kd}.$$

So we get

$$\begin{aligned}
\sum_{r=1}^{q-1} G(r,\psi)\frac{f(l,r,n,q)}{e(\frac{r}{q}-\frac{l}{n})-1} &\ll q^{1/2}\sum_{\substack{d|q\\ d<q}} d\sum_{k|q} q_0\log\left(\frac{q-1}{kd}+1\right) \\
&\ll q^{1/2}\sum_{\substack{d|q\\ d<q}} d\sum_{k|q}\frac{q}{d}\log q \ll q^{3/2}d^2(q)\log q. \qquad (2.1)
\end{aligned}$$

Thus

$$\sum_{a=1}^{q}{}'\sum_{b=1}^{q}{}'\chi_1(a)e\left(\frac{(a+b)l}{n}\right)\ll q^{1/2}\phi(q)d^2(q)\log q. \qquad \square$$

## 3 Proof of the theorem

In this section, we shall complete the proof of the theorem.

*Proof of the theorem* From the orthogonality relation for Dirichlet characters mod $q$ and the trigonometric sum identity, we can get

$$\begin{aligned}
\sum_{a\in\mathcal{C}(1,q)}\psi(a) &= \sum_{a=1}^{q}\psi(a)-\sum_{\substack{a=1\\ n|(a+\overline{a})}}^{q}\psi(a) \\
&= \sum_{a=1}^{q}\psi(a)-\sum_{a=1}^{q}{}'\sum_{\substack{b=1\\ n|(a+b)\\ ab\equiv 1(\bmod q)}}^{q}{}'\psi(a)
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\phi(q)} \sum_{\chi \bmod q} \sideset{}{'}\sum_{\substack{a=1\\ n|(a+b)}}^{q} \sideset{}{'}\sum_{b=1}^{q} \psi(a)\chi(ab) \\
&= -\frac{1}{n\phi(q)} \sum_{\chi \bmod q} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \psi(a)\chi(ab) \sum_{l=1}^{n} e\left(\frac{(a+b)l}{n}\right) \\
&= -\frac{1}{n\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0\\ \chi\neq\overline{\psi}}} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \psi(a)\chi(ab) \sum_{l=1}^{n} e\left(\frac{(a+b)l}{n}\right) \\
&\quad - \frac{1}{n\phi(q)} \sum_{l=1}^{n} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \psi(a) e\left(\frac{(a+b)l}{n}\right) \\
&\quad - \frac{1}{n\phi(q)} \sum_{l=1}^{n} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \overline{\psi}(b) e\left(\frac{(a+b)l}{n}\right) \\
&:= -E_1 - E_2 - E_3.
\end{aligned}
$$

First of all, we shall estimate $E_1$. Making use of Lemma 3, we get

$$
\begin{aligned}
E_1 &= \frac{1}{n\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0\\ \chi\neq\overline{\psi}}} \sideset{}{'}\sum_{a=1}^{q} \sideset{}{'}\sum_{b=1}^{q} \psi(a)\chi(ab) \sum_{l=1}^{n} e\left(\frac{(a+b)l}{n}\right) \\
&= \frac{1}{n\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0\\ \chi\neq\overline{\psi}}} \sum_{l=1}^{n} \sum_{a=1}^{q} \chi\psi(a) e\left(\frac{al}{n}\right) \sum_{b=1}^{q} \chi(b) e\left(\frac{bl}{n}\right) \\
&= \frac{1}{n\phi(q)} \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0\\ \chi\neq\overline{\psi}}} \sum_{l=1}^{n} \sum_{a=1}^{q} \frac{1}{q} \sum_{r_1=1}^{q-1} G(r_1, \chi\psi) e\left(-\frac{ar_1}{q}\right) e\left(\frac{al}{n}\right) \\
&\quad \times \sum_{b=1}^{q} \frac{1}{q} \sum_{r_2=1}^{q-1} G(r_2, \chi) e\left(-\frac{br_2}{q}\right) e\left(\frac{bl}{n}\right) \\
&= \frac{1}{n\phi(q)q^2} \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0\\ \chi\neq\overline{\psi}}} \sum_{l=1}^{n} \sum_{r_1=1}^{q-1} G(r_1, \chi\psi) \sum_{r_2=1}^{q-1} G(r_2, \chi) \\
&\quad \times \sum_{a=1}^{q} e\left(\left(\frac{l}{n} - \frac{r_1}{q}\right)a\right) \sum_{b=1}^{q} e\left(\left(\frac{l}{n} - \frac{r_2}{q}\right)b\right) \\
&= \frac{1}{n\phi(q)q^2} \sum_{l=1}^{n} \sum_{r_1=1}^{q-1} \sum_{r_2=1}^{q-1} \frac{f_1(l, r_1, n, q) f_2(l, r_2, n, q)}{(e(\frac{l}{n} - \frac{r_1}{q}) - 1)(e(\frac{l}{n} - \frac{r_2}{q}) - 1)} \\
&\quad \times \sum_{\substack{\chi \bmod q\\ \chi\neq\chi_0\\ \chi\neq\overline{\psi}}} G(r_1, \chi\psi) G(r_2, \chi)
\end{aligned}
$$

$$
\begin{aligned}
&\ll \frac{1}{\phi(q)q^2}\sum_{l=1}^{n}\sum_{r_1=1}^{q-1}\sum_{r_2=1}^{q-1}\frac{\phi(q)q^{1/2}(r_1,q)^{1/2}(r_2,q)^{1/2}d(q)}{|e(\frac{l}{n}-\frac{r_1}{q})-1||e(\frac{l}{n}-\frac{r_2}{q})-1|} \\
&= \frac{d(q)}{q^{3/2}}\sum_{l=1}^{n}\sum_{r_1=1}^{q-1}\sum_{r_2=1}^{q-1}\frac{(r_1,q)^{1/2}(r_2,q)^{1/2}}{|e(\frac{l}{n}-\frac{r_1}{q})-1||e(\frac{l}{n}-\frac{r_2}{q})-1|} \\
&\ll \frac{d(q)}{q^{3/2}}\sum_{l=1}^{n}\left(\sum_{r=1}^{q-1}\frac{(r,q)^{1/2}}{|e(\frac{l}{n}-\frac{r}{q})-1|}\right)^2.
\end{aligned}
$$

Similar to (2.1), we have

$$
\sum_{r=1}^{q-1}\frac{(r,q)^{1/2}}{|e(\frac{l}{n}-\frac{r}{q})-1|} \ll \sum_{\substack{d\mid q\\ d<q}} d^{1/2}\sum_{k\mid q}\frac{q}{d}\log q = q\log q\sum_{\substack{d\mid q\\ d<q}} d^{-1/2}\sum_{k\mid q}1 \ll qd^2(q)\log q.
$$

Then

$$
E_1 \ll \frac{d(q)}{q^{3/2}}q^2d^4(q)\log^2 q = q^{1/2}d^5(q)\log^2 q. \tag{3.1}
$$

Second, we estimate $E_2$. By Lemma 5, we have

$$
E_2 \ll \frac{1}{\phi(q)}q^{1/2}\phi(q)d^2(q)\log q = q^{1/2}d^2(q)\log q. \tag{3.2}
$$

In the same way we can get the estimate

$$
E_3 \ll q^{1/2}d^2(q)\log q. \tag{3.3}
$$

Combining (3.1), (3.2), and (3.3), we obtain the result. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
HC and ZZQ drafted the manuscript. YKM and TPZ participated in its design and coordination and helped to draft the manuscript. All authors read and approved the final manuscript.

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