Lu and Tao *Journal of Inequalities and Applications* (2016) 2016:259
DOI 10.1186/s13660-016-1203-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fractional type Marcinkiewicz integrals over non-homogeneous metric measure spaces

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Abstract

The main goal of the paper is to establish the boundedness of the fractional type Marcinkiewicz integral $\mathcal{M}_{\beta,\rho,q}$ on non-homogeneous metric measure space which includes the upper doubling and the geometrically doubling conditions. Under the assumption that the kernel satisfies a certain Hörmander-type condition, the authors prove that $\mathcal{M}_{\beta,\rho,q}$ is bounded from Lebesgue space $L^1(\mu)$ into the weak Lebesgue space $L^{1,\infty}(\mu)$, from the Lebesgue space $L^\infty(\mu)$ into the space RBLO(μ), and from the atomic Hardy space $H^1(\mu)$ into the Lebesgue space $L^1(\mu)$. Moreover, the authors also get a corollary, that is, $\mathcal{M}_{\beta,\rho,q}$ is bounded on $L^p(\mu)$ with $1 < p < \infty$.

MSC: non-homogeneous metric measure space; fractional type Marcinkiewicz integral; Lebesgue space; Hardy space; RBLO(μ)

Keywords: 42B35; 47B47; 30L99

1 Introduction

In 2010, Hytönen in [1] first introduced a new class of metric measure spaces which satisfy the so-called upper doubling and the geometrically doubling conditions (see also Definitions 1.1 and 1.2 below, respectively), for convenience, the new spaces are called *non-homogeneous metric measure spaces*. As special cases, the new spaces not only contain the homogeneous type spaces (see [2]), but also they include metric spaces endowed with measures satisfying the polynomial growth condition (see, for example, [3–9]). Further, it is meaningful to pay much attention to a study of the properties of some classical operators, commutators, and function spaces on non-homogeneous metric measure spaces; see [10–16]. In addition, we know that the harmonic analysis has important applications in many fields including geometrical analysis, functional analysis, partial differential equations, and fuzzy fractional differential equations, we refer the reader to [17–20] and the references therein.

In the present paper, let $(\mathcal{X}, d, \mu)$ be a non-homogeneous metric measure space in the sense of Hytönen [1]. In 2007, Hu *et al.* [5] obtained the boundedness of the Marcinkiewicz with non-doubling measure. Besides, Lin and Yang [13] established some equivalent boundedness of Marcinkiewicz integral on $(\mathcal{X}, d, \mu)$. Inspired by this, we will mainly consider the boundedness of the fractional type Marcinkiewicz integrals introduced in [21] on $(\mathcal{X}, d, \mu)$.

To state the main consequences of this article, we first of all recall some necessary notions and notation. Hytönen [1] originally introduced the following notions of the upper doubling condition and the geometrically doubling condition.

Definition 1.1 ([1]) A metric measure space $(\mathcal{X}, d, \mu)$ is said to be upper doubling if μ is a Borel measure on $\mathcal{X}$ and there exist a dominating function $\lambda : \mathcal{X} \times (0,\infty) \to (0,\infty)$ and a positive constant C_λ such that, for each $x \in \mathcal{X}$, $r \to \lambda(x,r)$ is non-decreasing and, for all $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$\mu\big(B(x,r)\big) \le \lambda(x,r) \le C_\lambda \lambda\left(x, \frac{r}{2}\right). \tag{1.1}$$

Hytönen *et al.* [16] have proved that there is another dominating function $\tilde{\lambda}$ such that $\tilde{\lambda} \le \lambda$, $C_{\tilde{\lambda}} \le C_\lambda$, and

$$\tilde{\lambda}(x,r) \le C_{\tilde{\lambda}} \tilde{\lambda}(y,r), \tag{1.2}$$

where $x, y \in \mathcal{X}$ and $d(x,y) \le r$. Based on this, we also assume the dominating function λ that in (1.1) satisfies (1.2) in this paper.

Definition 1.2 ([1]) A metric space $(\mathcal{X}, d)$ is said to be geometrically doubling, if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x,r) \subset \mathcal{X}$, there exists a finite ball covering $\{B(x_i, \frac{r}{2})\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most N_0.

Remark 1.3 Let $(\mathcal{X}, d)$ be a metric space. Hytönen in [1] proved the following statements are mutually equivalent:

(1) $(\mathcal{X}, d)$ is geometrically doubling.
(2) For any $\epsilon \in (0,1)$ and any ball $B(x,r) \subset \mathcal{X}$, there is a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x,r)$ such that the cardinality of this covering is at most $N_0 \epsilon^{-n}$, where $n := \log_2 N_0$.
(3) For any $\epsilon \in (0,1)$, any ball $B(x,r) \subset \mathcal{X}$ contains at most $N_0 \epsilon^{-n}$ centers of disjoint balls $\{B(x_i, \epsilon r)\}_i$.
(4) There is $M \in \mathbb{N}$ such that any ball $B(x,r) \subset \mathcal{X}$ contains at most M centers $\{x_i\}_i$ of disjoint balls $\{B(x_i, \frac{r}{4})\}_{i=1}^M$.

Now we recall the definition of coefficient $K_{B,S}$ introduced by Hytönen in [1], which is analogous to the quantity $K_{Q,R}$ introduced in [4], that is, for any two balls $B \subset S$ in $\mathcal{X}$, define

$$K_{B,S} := 1 + \int_{2S \setminus B} \frac{1}{\lambda(c_B, d(x, c_B))} \,\mathrm{d}\mu(x), \tag{1.3}$$

where c_B is the center of the ball B.

Though the measure doubling condition is not assumed uniformly for all balls on $(\mathcal{X}, d, \mu)$, it was proved in [1] that there still exist many balls satisfying the property of the (α, η)-doubling, namely, we say that a ball $B \subset \mathcal{X}$ is (α, η)-doubling if $\mu(\alpha B) \le \eta \mu(B)$, for $\alpha, \eta > 1$. In the rest of this paper, unless α and η_α are specified, otherwise, by an (α, η_α)-doubling ball we mean a $(6, \beta_6)$-doubling ball with a fixed number $\eta_6 > \max\{C_\lambda^{3\log_2 6}, 6^n\}$, where $n := \log_2 N_0$ is viewed as a geometric dimension of the space. Moreover, the smallest

$(6, \eta_6)$-doubling ball of the from $6^j B$ with $j \in \mathbb{N}$ is denoted by $\tilde{B}^6$, and $\tilde{B}^6$ is simply denoted by $\tilde{B}$.

Next, we recall the following definition of RBMO(μ) from [1].

Definition 1.4 ([1]) Let $\kappa > 1$ be a fixed constant. A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space RBMO(μ) if there exist a positive constant C and, for any ball B, a number f_B such that

$$\frac{1}{\mu(\kappa B)} \int_B \left|f(y) - f_B\right| \mathrm{d}\mu(y) \leq C$$

and

$$|f_B - f_R| \leq C K_{B,R}$$

for any two balls B and R such that $B \subset R$. Moreover, the RBMO(μ) norm of f is defined to be the minimal constant C as above and denoted by $\|f\|_{\mathrm{RBMO}(\mu)}$.

From [1], Hytönen showed that the space RBMO(μ) is not dependent on the choice of κ. Lin and Yang [14] introduced the following definition of the space RBLO(μ) and proved that RBLO$(\mu) \subset$ RBMO(μ).

Definition 1.5 ([14]) A function $f \in L^1_{\mathrm{loc}}(\mu)$ is said to belong to the space RBLO(μ) if there exists a positive constant C such that. for any $(6, \beta_6)$-doubling ball B,

$$\frac{1}{\mu(\sigma B)} \int_B \left[f(y) - \operatorname*{ess\,inf}_{\tilde{B}} f\right] \mathrm{d}\mu(y) \leq C$$

and

$$\operatorname*{ess\,inf}_B f - \operatorname*{ess\,inf}_S f \leq C K_{B,S}$$

for any two $(6, \beta_6)$-doubling balls $B \subset S$. The minimal constant C above is defined to be the norm of f in RBLO(μ) and denoted by $\|f\|_{\mathrm{RBLO}(\mu)}$.

Now we give the notion of the fractional type Marcinkiewicz integral slightly changed from [21].

Definition 1.6 Let $\Delta = \{(x,x) : x \in \mathcal{X}\}$. A *stand kernel* is a mapping $K : \mathcal{X} \times \mathcal{X} \setminus \Delta \to \mathbb{C}$ for which there exist positive constants $\delta \in (0,1]$, $\beta \geq 0$, and C such that, for $x, y \in \mathcal{X}$ with $x \neq y$,

$$\left|K(x,y)\right| \leq C \frac{[d(x,y)]^{1+\beta}}{\lambda(x, d(x,y))}, \tag{1.4}$$

and for all $x, \tilde{x}, y \in \mathcal{X}$ with $d(x,y) \geq 2d(x,\tilde{x})$,

$$\left|K(x,y) - K(\tilde{x},y)\right| + \left|K(y,x) - K(y,\tilde{x})\right| \leq C \frac{[d(x,\tilde{x})]^{\delta+1+\beta}}{[d(x,y)]^{\delta} \lambda(x, d(x,y))}. \tag{1.5}$$

The fractional type Marcinkiewicz integral $\mathcal{M}_{\beta,\rho,q}(f)$ related to the above kernel $K(x,y)$ is formally defined by

$$\mathcal{M}_{\beta,\rho,q}(f)(x) := \left(\int_0^\infty \left| \frac{1}{t^{\beta+\rho}} \int_{d(x,y)<t} \frac{K(x,y)}{[d(x,y)]^{1-\rho}} f(y)\,\mathrm{d}\mu(y) \right|^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}, \tag{1.6}$$

where $x \in \mathcal{X}$, $\rho > 0$, $\beta \geq 0$, and $q > 1$.

Recently, many authors have studied the properties of the fractional type Marcinkiewicz integrals; see [22–24]. To the fractional type Marcinkiewicz integral operator $\mathcal{M}_{\beta,\rho,q}$ as in (1.6), one can return to the Marcinkiewicz integrals on different function spaces when the indices are replaced by some fixed numbers; see the following remark.

Remark 1.7

(1) When $\rho = 1$, $\beta = 0$, and $q = 2$, the operator $\mathcal{M}_{\beta,\rho,q}(f)$ as in (1.6) is just the Marcinkiewicz integral on $(\mathcal{X}, d, \mu)$ in [13].

(2) If we take $(\mathcal{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mu)$, $\rho = 1$, $\beta = 0$, and $q = 2$, the operator $\mathcal{M}_{\beta,\rho,q}(f)$ as in (1.6) is just the Marcinkiewicz integral with non-doubling measures (see [5]).

(3) If we take $(\mathcal{X}, d, \mu) = (\mathbb{R}^n, |\cdot|, \mathrm{d}x)$, $K(x,y) = \frac{\Omega(x-y)}{|x-y|^{n-1}}$, $\rho = 1$, $\beta = 0$, and $q = 2$, then the operator $\mathcal{M}_{\beta,\rho,q}(f)$ as in (1.6) is just the classical Marcinkiewicz integral introduced in [25] and its form is as follows:

$$\mathcal{M}_{\Omega}(f)(x) := \left(\int_0^\infty \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)\,\mathrm{d}y \right|^2 \frac{\mathrm{d}t}{t^3} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^n;$$

for more about behaviors of the $\mathcal{M}_{\Omega}$, see [26–30].

Further, we recall the notion of the atomic Hardy spaces given in [16].

Definition 1.8 ([16]) Let $\zeta \in (1,\infty)$ and $p > 1$. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a $(p,1)_\tau$-*atomic block* if

(1) there exists a ball S such that $\operatorname{supp} b \subset S$;

(2) $\int_{\mathcal{X}} b(x)\,\mathrm{d}\mu(x) = 0$;

(3) for any $i \in \{1,2\}$, there exists a function a_i supported on a ball $B_i \subset S$ and $\tau_i \in \mathbb{C}$ such that $b = \tau_1 a_1 + \tau_2 a_2$ and

$$\|a_i\|_{L^p(\mu)} \leq \left[\mu(\zeta B_i)\right]^{\frac{1}{p}-1} K^{-1}_{B_i,S}. \tag{1.7}$$

Moreover, let

$$|b|_{H^{1,p}_{\mathrm{atb}}(\mu)} := |\tau_1| + |\tau_2|.$$

We say that a function $f \in L^1(\mu)$ belongs to the atomic Hardy space $H^{1,p}_{\mathrm{atb}}(\mu)$, if there exist $(p,1)_\tau$-atomic blocks $\{b_i\}_{i=1}^\infty$ such that $f = \sum_{i=1}^\infty b_i$ in $L^1(\mu)$ and $\sum_{i=1}^\infty |b_i|_{H^{1,p}_{\mathrm{atb}}(\mu)} < \infty$. The norm of f in $H^{1,p}_{\mathrm{atb}}(\mu)$ is defined by $\|f\|_{H^{1,p}_{\mathrm{atb}}(\mu)} := \inf\{\sum_i |b_i|_{H^{1,p}_{\mathrm{atb}}(\mu)}\}$, where the infimum is taken over all the possible decompositions of f as above.

Also, in [16], Hytönen *et al.* proved that, for each $p \in (1,\infty]$, the atomic Hardy space $H^{1,p}_{\mathrm{atb}}(\mu)$ is independent of the choice of ζ and that the spaces $H^{1,p}_{\mathrm{atb}}(\mu)$ and $H^{1,\infty}_{\mathrm{atb}}(\mu)$ have the same norms for all $p \in (1,\infty]$. Thus, we always denote $H^{1,p}_{\mathrm{atb}}(\mu)$ simply by $H^1(\mu)$.

Finally, we state the main results of this article.

Theorem 1.9 *Let $K(x,y)$ satisfy* (1.4) *and* (1.5), *and $\mathcal{M}_{\beta,\rho,q}$ be as in* (1.6), *where $\rho > 0$, $\beta \geq 0$, and $q > 1$. If $\mathcal{M}_{\beta,\rho,q}$ is bounded on $L^2(\mu)$, then it is also bounded from $L^1(\mu)$ into $L^{1,\infty}(\mu)$, that is, there exists a positive constant C such that, for all $t > 0$ and $f \in L^1(\mu)$,*

$$\mu\bigl(\bigl\{x \in \mathcal{X} : \mathcal{M}_{\beta,\rho,q}(f)(x) > t\bigr\}\bigr) \leq C\frac{\|f\|_{L^1(\mu)}}{t}. \tag{1.8}$$

Theorem 1.10 *Let $K(x,y)$ satisfy* (1.4) *and* (1.5), $\rho > 0$, $\beta \geq 0$, *and* $q > 1$. *Suppose that $\mathcal{M}_{\beta,\rho,q}$ is as in* (1.6) *and bounded on $L^2(\mu)$. Then for $f \in L^\infty(\mu)$, $\mathcal{M}_{\beta,\rho,q}$ is either infinite everywhere or finite μ-finite almost everywhere; more precisely, if $\mathcal{M}_{\beta,\rho,q}$ is finite at some point $x_0 \in \mathcal{X}$, then $\mathcal{M}_{\beta,\rho,q}$ is μ-almost everywhere and*

$$\bigl\|\mathcal{M}_{\beta,\rho,q}(f)\bigr\|_{\mathrm{RBLO}(\mu)} \leq C\|f\|_{L^\infty(\mu)},$$

where the positive constant C is not dependent on f.

By Theorem 1.9, Theorem 1.10, and Theorem 1.1 in [15], it is easy to obtain the following corollary.

Corollary 1.11 *Under the assumption of Theorem* 1.9, *then $\mathcal{M}_{\beta,\rho,q}$ is bounded on $L^p(\mu)$ for any $p \in (1,\infty)$.*

Theorem 1.12 *Let $K(x,y)$ satisfy* (1.4) *and* (1.5), *and $\mathcal{M}_{\beta,\rho,q}$ be as in* (1.6). *If $\mathcal{M}_{\beta,\rho,q}$ is bounded on $L^2(\mu)$, then it is also bounded from $H^1(\mu)$ into $L^1(\mu)$.*

Throughout the paper, C represents for a positive constant which is independent of the main parameters involved, but it may be different from line to line. For a μ-measurable set E, χ_E denotes its characteristic function. For any $p \in [1,\infty]$, we denote by p' its conjugate index, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

2 Preliminaries

In this section, in order to prove our main theorems, we need some lemmas. First, we recall some useful properties of $K_{B,S}$ as in (1.3) (see [1]).

Lemma 2.1 ([1])

1. *For all balls $B \subset R \subset S$, it holds true that $K_{B,R} \leq K_{B,S}$.*
2. *For any $\xi \in [1,\infty)$, there exists a positive constant C_ξ, depending on ξ, such that, for all balls $B \subset S$ with $r_S \leq \xi r_B$, $K_{B,S} \leq C_\xi$.*
3. *For any $\varrho \in (1,\infty)$, there exists a positive constant C_ϱ, depending on ϱ, such that, for all balls B, $K_{B,\tilde{B}^\varrho} \leq C_\varrho$.*
4. *There is a positive constant c such that, for all balls $B \subset R \subset S$, $K_{B,S} \leq K_{B,R} + cK_{R,S}$. In particular, if B and R are concentric, then $c = 1$.*

(5) *There exists a positive constant* $\tilde{c}$ *such that, for all balls* $B \subset R \subset S$, $K_{B,R} \leq \tilde{c}K_{B,S}$; *moreover, if* B *and* R *are concentric, then* $K_{R,S} \leq K_{B,S}$.

Next, we recall the Calderón-Zygmund decomposition theorem from [31] as follows. Let γ_0 be a fixed non-negative constant and satisfy $\gamma_0 > \max\{C_\lambda^{3\log_2 6}, 6^{3n}\}$, where C_λ is as in (1.1) and n as in Remark 1.3.

Lemma 2.2 ([31]) *Let* $p \in [1,\infty)$, $f \in L^p(\mu)$, *and* $t \in (0,\infty)$ ($t > \frac{\gamma_0\|f\|_{L^p(\mu)}}{\mu(\mathcal{X})}$ *when* $\mu(\mathcal{X}) < \infty$). *Then*:

(1) *There exists a family of finite overlapping balls* $\{6B_i\}_i$ *such that* $\{B_i\}_i$ *is pairwise disjoint,*

$$\frac{1}{\mu(6^2B_i)}\int_{B_i}|f(x)|^p\,\mathrm{d}\mu(x) > \frac{t^p}{\gamma_0} \quad \textit{for all } i, \tag{2.1}$$

$$\frac{1}{\mu(6^2\upsilon B_i)}\int_{\upsilon B_i}|f(x)|^p\,\mathrm{d}\mu(x) \leq \frac{t^p}{\gamma_0} \quad \textit{for all } i \textit{ and all } \upsilon \in (2,\infty),$$

and

$$|f(x)| \leq t \quad \textit{for } \mu\textit{-almost every } x \in \mathcal{X} \setminus \Bigl(\bigcup_i 6B_i\Bigr). \tag{2.2}$$

(2) *For each* i, *let* S_i *be the smallest* $(3\times 6^2, C_\lambda^{\log_2(3\times 6^2)+1})$*-doubling ball of the family* $\{(3\times 6^2)^k B_i\}_{k\in\mathbb{N}}$, *and* $\omega_i = \chi_{6B_i}/(\sum_k \chi_{6B_k})$. *Then there exist a family* $\{\varphi_i\}_i$ *of functions that, for each* i, $\operatorname{supp}(\varphi_i) \subset S_i$, φ_i *has a constant sign on* S_i *and*

$$\int_{\mathcal{X}}\varphi_i(x)\,\mathrm{d}\mu(x) = \int_{6B_i} f(x)\omega_i(x)\,\mathrm{d}\mu(x), \tag{2.3}$$

$$\sum_i|\varphi_i(x)| \leq \gamma t \quad \textit{for } \mu\textit{-almost every } x \in \mathcal{X}, \tag{2.4}$$

where γ *is some positive constant depending only on* $(\mathcal{X},\mu)$, *and there exists a positive constant* C, *independent of* f, t, *and* i, *such that, if* $p = 1$, *then*

$$\|\varphi_i\|_{L^\infty(\mu)}\mu(S_i) \leq C\int_{\mathcal{X}}|f(x)\omega_i(x)|\,\mathrm{d}\mu(x), \tag{2.5}$$

and if $p \in (1,\infty)$,

$$\biggl(\int_{S_i}|\varphi_i(x)|^p\,\mathrm{d}\mu(x)\biggr)^{\frac{1}{p}}\bigl[\mu(S_i)\bigr]^{\frac{1}{p'}} \leq \frac{C}{t^{p-1}}\int_{\mathcal{X}}|f(x)\omega_i(x)|^p\,\mathrm{d}\mu(x). \tag{2.6}$$

Finally, we recall the following characterizations of $\mathrm{RBLO}(\mu)$ given in [14].

Lemma 2.3 ([14]) *If* $f \in L^1_{\mathrm{loc}}(\mu)$ *is said to be in the space* $\mathrm{RBLO}(\mu)$, *then there exists a non-negative constant* C *satisfying that, for all* $(6,\beta_6)$*-doubling balls* B,

$$\frac{1}{\mu(B)}\int_B\Bigl[f(y) - \operatorname*{ess\,inf}_B f\Bigr]\,\mathrm{d}\mu(y) \leq C$$

and, for all $(6, \beta_6)$*-doubling balls* $B \subset S$,

$$m_B(f) - m_S(f) \le CK_{B,S}, \tag{2.7}$$

in this paper, $m_B(f)$ *represents the mean of* f *over* B, *that is,*

$$m_B(f) := \frac{1}{\mu(B)} \int_B f(y)\,\mathrm{d}\mu(y).$$

Moreover, the minimal constant C *above equals* $\|f\|_{\mathrm{RBLO}(\mu)}$.

3 Proofs of the main theorems

Proof of Theorem 1.9 Without loss of generality, by homogeneity, we can assume that $\|f\|_{L^1(\mu)} = 1$. It is easy to see that the conclusion of Theorem 1.9 automatically holds true if $t \le \eta_6\|f\|_{L^1(\mu)}/\mu(\mathcal{X})$ when $\mu(\mathcal{X}) < \infty$. Therefore, we only need consider the case $t > \eta_6\|f\|_{L^1(\mu)}/\mu(\mathcal{X})$. For any given $f \in L^1(\mu)$ and $t > \eta_6\|f\|_{L^1(\mu)}/\mu(\mathcal{X})$, applying Lemma 2.2 to f and t, and letting S_i be as in Lemma 2.2, we may write $f = g + h$, where $g := f\chi_{\mathcal{X}\setminus\bigcup_i 6B_i} + \sum_i \varphi_i$ and $h := \sum_i(\omega_i f - \varphi_i) =: \sum_i h_i$. By applying (2.2), (2.4), and the assumption $\|f\|_{L^1(\mu)} = 1$, we easily obtain $\|g\|_{L^\infty(\mu)} \le Ct$ and $\|g\|_{L^1(\mu)} \le C$. Thus, by the $L^2(\mu)$-boundedness of $\mathcal{M}_{\beta,\rho,q}$, we conclude that

$$\mu\bigl(\bigl\{x \in \mathcal{X} : \mathcal{M}_{\beta,\rho,q}(g)(x) > t\bigr\}\bigr) \le Ct^{-2}\bigl\|\mathcal{M}_{\beta,\rho,q}(g)\bigr\|_{L^2(\mu)} \le Ct^{-2}\|g\|_{L^2(\mu)} \le Ct^{-1}.$$

On the other hand, by (2.1) with $p = 1$, and the fact that $\{B_i\}_i$ is a sequence of pairwise disjoint balls, we have

$$\mu\biggl(\bigcup_i 6^2B_i\biggr) \le \sum_i \mu\bigl(6^2B_i\bigr) \le Ct^{-1}\sum_i \int_{B_i} \bigl|f(x)\bigr|\,\mathrm{d}\mu(x) \le Ct^{-1},$$

and therefore, the proof of Theorem 1.9 can be reduced to proving

$$\mu\biggl(\biggl\{x \in \mathcal{X} \setminus \bigcup_i \bigl(6^2B_i\bigr) : \mathcal{M}_{\beta,\rho,q}(h)(x) > t\biggr\}\biggr) \le Ct^{-1}. \tag{3.1}$$

Notice that

$$\begin{aligned}
&\mu\biggl(\biggl\{x \in \mathcal{X} \setminus \bigcup_i \bigl(6^2B_i\bigr) : \mathcal{M}_{\beta,\rho,q}(h)(x) > t\biggr\}\biggr)\\
&\quad\le t^{-1}\sum_i \int_{\mathcal{X}\setminus 6S_i} \mathcal{M}_{\beta,\rho,q}(h_i)(x)\,\mathrm{d}\mu(x) + t^{-1}\sum_i \int_{6S_i\setminus 6^2B_i} \mathcal{M}_{\beta,\rho,q}(h_i)(x)\,\mathrm{d}\mu(x)\\
&\quad=: \mathrm{E}_1 + \mathrm{E}_2.
\end{aligned}$$

For E_1. Let S_i be as in Lemma 2.2. Denote its center and radius by c_{S_i} and r_{S_i}, respectively. Write

$$\begin{aligned}
&\int_{\mathcal{X}\setminus 6S_i} \mathcal{M}_{\beta,\rho,q}(h_i)(x)\,\mathrm{d}\mu(x)\\
&\quad\le \int_{\mathcal{X}\setminus 6S_i} \biggl(\int_0^{d(x,c_{S_i})+r_{S_i}} \biggl|\frac{1}{t^{\beta+\rho}} \int_{d(x,y)<t} \frac{K(x,y)}{[d(x,y)]^{1-\rho}} h_i(y)\,\mathrm{d}\mu(y)\biggr|^q \frac{\mathrm{d}t}{t}\biggr)^{\frac{1}{q}}\,\mathrm{d}\mu(x)
\end{aligned}$$

$$+\int_{\mathcal{X}\setminus 6S_i}\left(\int_{d(x,c_{S_i})+r_{S_i}}^{\infty}\left|\frac{1}{t^{\beta+\rho}}\int_{d(x,y)<t}\frac{K(x,y)}{[d(x,y)]^{1-\rho}}h_i(y)\,\mathrm{d}\mu(y)\right|^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\mathrm{d}\mu(x)$$

$$=: \mathrm{F}_1+\mathrm{F}_2.$$

Applying the Minkowski inequality and (1.4), we have

$$\begin{aligned}
\mathrm{F}_1 &\le \int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\frac{|h_i(y)|[d(x,y)]^{1+\beta}}{\lambda(x,d(x,y))[d(x,y)]^{1-\rho}}\left(\int_{d(x,y)}^{d(x,c_{S_i})+r_{S_i}}\frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}}\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le C\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\frac{|h_i(y)|[d(x,y)]^{\rho+\beta}}{\lambda(x,d(x,y))}\\
&\quad\times\left(\frac{1}{[d(x,y)]^{q(\beta+\rho)}}-\frac{1}{[d(x,c_{S_i})+r_{S_i}]^{q(\beta+\rho)}}\right)^{\frac{1}{q}}\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le C\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\frac{|h_i(y)|[d(x,y)]^{\rho+\beta}}{\lambda(x,d(x,y))}\frac{(r_{S_i})^{\frac{1}{q}(\beta+\rho)}}{[d(x,y)]^{(\beta+\rho)}[d(x,c_{S_i})+r_{S_i}]^{\frac{1}{q}(\beta+\rho)}}\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le C(r_{S_i})^{\frac{1}{q}(\beta+\rho)}\int_{\mathcal{X}}\left|h_i(y)\right|\mathrm{d}\mu(y)\int_{\mathcal{X}\setminus 6S_i}\frac{\mathrm{d}\mu(x)}{\lambda(x,d(x,c_{S_i}))[d(x,c_{S_i})+r_{S_i}]^{\frac{1}{q}(\beta+\rho)}}\\
&\le C\int_{\mathcal{X}}\left|h_i(y)\right|\mathrm{d}\mu(y)\sum_{k=1}^{\infty}6^{-k\frac{1}{q}(\beta+\rho)}\int_{6^{k+1}S_i\setminus 6^kS_i}\frac{\mathrm{d}\mu(x)}{\lambda(x,d(x,c_{S_i}))}\\
&\le C\|h_i\|_{L^1(\mu)}.
\end{aligned}$$

For $x\in\mathcal{X}\setminus 6S_i$ and $y\in S_i$, it holds true that $d(x,y)<d(x,c_{S_i})+r_{S_i}$. Thus, by the vanishing moment of h_i and (1.5), we can conclude

$$\begin{aligned}
\mathrm{F}_2 &= \int_{\mathcal{X}\setminus 6S_i}\left(\int_{d(x,c_{S_i})+r_{S_i}}^{\infty}\left|\frac{1}{t^{\beta+\rho}}\int_{\mathcal{X}}\left[\frac{K(x,y)}{[d(x,y)]^{1-\rho}}-\frac{K(x,c_{S_i})}{[d(x,c_{S_i})]^{1-\rho}}\right]\right.\right.\\
&\quad\left.\left.\times h_i(y)\,\mathrm{d}\mu(y)\right|^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\mathrm{d}\mu(x)\\
&\le \int_{\mathcal{X}\setminus 6S_i}\left(\int_{d(x,c_{S_i})+r_{S_i}}^{\infty}\left|\frac{1}{t^{\beta+\rho}}\int_{\mathcal{X}}\left[\frac{K(x,y)}{[d(x,y)]^{1-\rho}}-\frac{K(x,y)}{[d(x,c_{S_i})]^{1-\rho}}\right]\right.\right.\\
&\quad\left.\left.\times h_i(y)\,\mathrm{d}\mu(y)\right|^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\mathrm{d}\mu(x)\\
&\quad+\int_{\mathcal{X}\setminus 6S_i}\left(\int_{d(x,c_{S_i})+r_{S_i}}^{\infty}\left|\frac{1}{t^{\beta+\rho}}\int_{\mathcal{X}}\left[\frac{K(x,y)}{[d(x,y)]^{1-\rho}}-\frac{K(x,c_{S_i})}{[d(x,y)]^{1-\rho}}\right]\right.\right.\\
&\quad\left.\left.\times h_i(y)\,\mathrm{d}\mu(y)\right|^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}\mathrm{d}\mu(x)\\
&=: \mathrm{F}_{21}+\mathrm{F}_{22}.
\end{aligned}$$

For F_{21}. By applying the Minkowski inequality, (1.2), and (1.4), we have

$$\begin{aligned}
\mathrm{F}_{21} &\le \int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\left|h_i(y)\right|\left|\frac{K(x,y)}{[d(x,y)]^{1-\rho}}-\frac{K(x,y)}{[d(x,c_{S_i})]^{1-\rho}}\right|\\
&\quad\times\left(\int_{d(x,c_{S_i})+r_{S_i}}^{\infty}\frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}}\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)
\end{aligned}$$

$$\begin{aligned}
&\le C\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\left|h_i(y)\right|\left|\frac{K(x,y)}{[d(x,y)]^{1-\rho}}-\frac{K(x,y)}{[d(x,c_{S_i})]^{1-\rho}}\right|\frac{1}{[d(x,c_{S_i})+r_{S_i}]^{\beta+\rho}}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le Cr_{S_i}\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\frac{|h_i(y)|[d(x,y)]^{\rho+\beta-1}}{\lambda(x,d(x,y))}\frac{1}{[d(x,c_{S_i})+r_{S_i}]^{\beta+\rho}}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le Cr_{S_i}\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\frac{|h_i(y)|}{\lambda(x,d(x,c_{S_i}))}\frac{\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)}{d(x,c_{S_i})}\\
&\le Cr_{S_i}\int_{\mathcal{X}}\left|h_i(y)\right|\mathrm{d}\mu(y)\sum_{k=1}^{\infty}\int_{6^{k+1}S_i\setminus 6^kS_i}\frac{\mathrm{d}\mu(x)}{\lambda(x,d(x,c_{S_i}))d(x,c_{S_i})}\\
&\le C\int_{\mathcal{X}}\left|h_i(y)\right|\mathrm{d}\mu(y)\sum_{k=1}^{\infty}6^{-k}\int_{6^{k+1}S_i\setminus 6^kS_i}\frac{\mathrm{d}\mu(x)}{\lambda(c_{S_i},d(x,c_{S_i}))}\\
&\le C\|h_i\|_{L^1(\mu)}.
\end{aligned}$$

Next we estimate F_{22}. By the Minkowski inequality and (1.5), we deduce that

$$\begin{aligned}
\mathrm{F}_{22}&\le C\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\left|h_i(y)\right|\left|\frac{K(x,y)}{[d(x,y)]^{1-\rho}}-\frac{K(x,c_{S_i})}{[d(x,y)]^{1-\rho}}\right|\\
&\quad\times\left(\int_{d(x,c_{S_i})+r_{S_i}}^{\infty}\frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}}\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le C\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\left|h_i(y)\right|\left|K(x,y)-K(x,c_{S_i})\right|\frac{1}{[d(x,y)]^{\beta+1}}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le C\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\left|h_i(y)\right|\frac{[d(c_{S_i},y)]^{\delta+\beta+1}}{[d(x,y)]^{\delta}\lambda(x,d(x,y))}\frac{1}{[d(x,y)]^{\beta+1}}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le C\int_{\mathcal{X}\setminus 6S_i}\int_{\mathcal{X}}\left|h_i(y)\right|\frac{r_{S_i}^{\delta+\beta+1}}{[d(x,y)]^{\delta+\beta+1}\lambda(x,d(x,c_{S_i}))}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\le C\int_{\mathcal{X}}\left|h_i(y)\right|\mathrm{d}\mu(y)\left(\sum_{k=1}^{\infty}6^{-k(\delta+\beta+1)}\int_{6^{k+1}S_i\setminus 6^kS_i}\frac{\mathrm{d}\mu(x)}{\lambda(x,d(x,c_{S_i}))}\right)\\
&\le C\|h_i\|_{L^1(\mu)}.
\end{aligned}$$

Combining the estimates for F_{21}, F_{22}, F_1, and the fact that

$$\|h_i\|_{L^1(\mu)}\le\int_{\mathcal{X}}\left|f(y)\omega_i(y)\right|\mathrm{d}\mu(y),$$

we have $\mathrm{E}_1\le Ct^{-1}$.

Now we turn to an estimate of E_2. Let N_1 be the positive integer satisfying $S_i=(3\times 6^2)^{N_1}B_i$. By $h_i:=\omega_i f-\varphi_i$, (1.4), the Minkowski inequality, the Hölder inequality, and (2.5) together with the $L^2(\mu)$-boundedness of $\mathcal{M}_{\beta,\rho,q}$, we get

$$\begin{aligned}
\mathrm{E}_2&\le t^{-1}\sum_i\int_{6S_i\setminus 6^2B_i}\left|\mathcal{M}_{\beta,\rho,q}(\omega_i f)(x)\right|\mathrm{d}\mu(x)+t^{-1}\sum_i\int_{6S_i\setminus 6^2B_i}\left|\mathcal{M}_{\beta,\rho,q}(\varphi_i)(x)\right|\mathrm{d}\mu(x)\\
&\le Ct^{-1}\sum_i\int_{6S_i\setminus 6^2B_i}\int_{\mathcal{X}}\frac{[d(x,y)]^{\rho+\beta}}{\lambda(x,d(x,y))}\left|f(y)\omega_i(y)\right|\left(\int_{d(x,y)}^{\infty}\frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}}\mathrm{d}\mu(y)\,\mathrm{d}\mu(x)\\
&\quad+Ct^{-1}\sum_i\left(\int_{6S_i}\left|\mathcal{M}_{\beta,\rho,q}(\varphi_i)(x)\right|^2\mathrm{d}\mu(x)\right)^{\frac{1}{2}}\mu(6S_i)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq Ct^{-1}\sum_i \int_{6S_i\setminus 6^2B_i}\int_{\mathcal{X}} \frac{|f(y)\omega_i(y)|}{\lambda(x,d(x,y))}\,\mathrm{d}\mu(y)\,\mathrm{d}\mu(x) + Ct^{-1}\sum_i \|\varphi_i\|_{L^2(\mu)}\mu(6S_i)^{\frac{1}{2}} \\
&\leq Ct^{-1}\sum_i \int_{\mathcal{X}} \bigl|f(y)\omega_i(y)\bigr|\,\mathrm{d}\mu(y)\sum_{k=1}^{N_1+1}\frac{\mu((3\times 6^2)^k B_i)}{\lambda(c_{B_i},(3\times 6^2)^k r_{B_i})} \\
&\quad + Ct^{-1}\sum_i \|\varphi_i\|_{L^\infty(\mu)}\mu(S_i) \\
&\leq Ct^{-1}\int_{\mathcal{X}}\bigl|f(y)\bigr|\,\mathrm{d}\mu(y) \leq Ct^{-1}.
\end{aligned}
$$

Combining the estimates for E_1 and E_2, we obtain (3.1) and hence the proof of Theorem 1.9 is finished. □

Proof of Theorem 1.10 We first claim that there exists a positive constant C such that, for any $f \in L^\infty(\mu)$ and $(6,\beta_6)$-doubling ball B,

$$
\frac{1}{\mu(B)}\int_B \mathcal{M}_{\beta,\rho,q}(f)(y)\,\mathrm{d}\mu(y) \leq C\|f\|_{L^\infty(\mu)} + \inf_{y\in B}\mathcal{M}_{\beta,\rho,q}(f)(y). \tag{3.2}
$$

In order to prove (3.2), for each fixed ball B, we assume that Q is the smallest ball which includes B and has the same center as B, so $2Q \subset 6B$. Decompose f as

$$
f(x) = f\chi_{2Q}(x) + f\chi_{\mathcal{X}\setminus 2Q}(x) =: f_1(x) + f_2(x).
$$

By applying Hölder inequality and the $L^2(\mu)$-boundedness of $\mathcal{M}_{\beta,\rho,q}$, we have

$$
\begin{aligned}
&\frac{1}{\mu(B)}\int_B \mathcal{M}_{\beta,\rho,q}(f_1)(y)\,\mathrm{d}\mu(y) \\
&\quad \leq \frac{1}{\mu(B)}\left(\int_B \bigl[\mathcal{M}_{\beta,\rho,q}(f_1)(y)\bigr]^2\,\mathrm{d}\mu(y)\right)^{\frac{1}{2}}\mu(B)^{\frac{1}{2}} \\
&\quad \leq \frac{1}{[\mu(B)]^{\frac{1}{2}}}\|f_1\|_{L^2(\mu)} \leq C\|f\|_{L^\infty(\mu)}\frac{[\mu(6B)]^{\frac{1}{2}}}{[\mu(B)]^{\frac{1}{2}}} \\
&\quad \leq C\|f\|_{L^\infty(\mu)}.
\end{aligned} \tag{3.3}
$$

Let r_Q be the radius of the ball Q. Noticing that $d(y,z) \geq r_Q$ for any $y \in B$, and $z \in \mathcal{X}\setminus 2Q$, by the Minkowski inequality, (1.2), and (1.4), we can deduce

$$
\begin{aligned}
\mathcal{M}_{\beta,\rho,q}(f_2)(y) &\leq \left(\int_{r_Q}^\infty \left|\frac{1}{t^{\beta+\rho}}\int_{d(y,z)<t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}f(z)\,\mathrm{d}\mu(z)\right|^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\quad + \left(\int_{r_Q}^\infty \left|\frac{1}{t^{\beta+\rho}}\int_{d(y,z)<t}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}f_1(z)\,\mathrm{d}\mu(z)\right|^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\leq \mathcal{M}_{\beta,\rho,q}(f)(y) + \left(\int_{r_Q}^\infty \left|\frac{1}{t^{\beta+\rho}}\int_{d(y,z)<4r_Q}\frac{K(y,z)}{[d(y,z)]^{1-\rho}}f_1(z)\,\mathrm{d}\mu(z)\right|^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\leq \mathcal{M}_{\beta,\rho,q}(f)(y) + C\int_{4Q}\frac{|K(y,z)|}{[d(y,z)]^{1-\rho}}\bigl|f_1(z)\bigr|\left(\int_{r_Q}^\infty \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}}\,\mathrm{d}\mu(z)
\end{aligned}
$$

$$
\begin{aligned}
&\le \mathcal{M}_{\beta,\rho,q}(f)(y) + C\int_{4Q} \frac{[d(y,z)]^{1+\beta}}{\lambda(y,d(y,z))} \frac{1}{[d(y,z)]^{1-\rho}} \frac{1}{r_Q^{\beta+\rho}} \left|f(z)\right| \mathrm{d}\mu(z) \\
&\le \mathcal{M}_{\beta,\rho,q}(f)(y) + C\|f\|_{L^\infty(\mu)} \int_{4Q} \frac{1}{\lambda(c_B,d(c_B,z))} \mathrm{d}\mu(z) \\
&\le \mathcal{M}_{\beta,\rho,q}(f)(y) + C\|f\|_{L^\infty(\mu)}.
\end{aligned} \tag{3.4}
$$

Therefore, the estimate for (3.2) can be reduced to proving

$$
\left|\mathcal{M}_{\beta,\rho,q}(f_2)(x) - \mathcal{M}_{\beta,\rho,q}(f_2)(y)\right| \le C\|f\|_{L^\infty(\mu)}. \tag{3.5}
$$

Write

$$
\begin{aligned}
&\left|\mathcal{M}_{\beta,\rho,q}(f_2)(x) - \mathcal{M}_{\beta,\rho,q}(f_2)(y)\right| \\
&\quad\le \Bigg(\int_0^\infty \Bigg|\int_{d(x,z)<t} \frac{K(x,z)}{[d(x,z)]^{1-\rho}} f_2(z)\,\mathrm{d}\mu(z) \\
&\qquad - \int_{d(y,z)<t} \frac{K(y,z)}{[d(y,z)]^{1-\rho}} f_2(z)\,\mathrm{d}\mu(z)\Bigg|^q \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\Bigg)^{\frac{1}{q}} \\
&\quad\le \left(\int_0^\infty \left[\int_{d(x,z)<t\le d(y,z)} \frac{|K(x,z)|}{[d(x,z)]^{1-\rho}} \left|f_2(z)\right| \mathrm{d}\mu(z)\right]^q \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}} \\
&\qquad + \left(\int_0^\infty \left[\int_{d(y,z)<t\le d(x,z)} \frac{|K(y,z)|}{[d(y,z)]^{1-\rho}} \left|f_2(z)\right| \mathrm{d}\mu(z)\right]^q \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}} \\
&\qquad + \left(\int_0^\infty \left[\int_{d(y,z)<td(x,z)<t} \left|\frac{K(x,z)}{[d(x,z)]^{1-\rho}} - \frac{K(y,z)}{[d(y,z)]^{1-\rho}}\right| \left|f_2(z)\right| \mathrm{d}\mu(z)\right]^q \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}} \\
&\quad=: \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3.
\end{aligned}
$$

For any $x, y \in B$, applying the Minkowski inequality, (1.2), and (1.4), we have

$$
\begin{aligned}
\mathrm{I}_1 &\le C\int_{\mathcal{X}} \frac{[d(y,z)]^{\beta+\rho}}{\lambda(y,d(y,z))} \left|f_2(z)\right| \left(\int_{d(x,z)}^{d(y,z)} \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}} \mathrm{d}\mu(z) \\
&\le C\int_{\mathcal{X}} \frac{[d(y,z)]^{\beta+\rho}}{\lambda(y,d(y,z))} \left|f_2(z)\right| \frac{[d(x,y)]^{\beta+\rho}}{[d(x,z)]^{\beta+\rho}[d(y,z)]^{\beta+\rho}} \mathrm{d}\mu(z) \\
&\le Cr_B^{\beta+\rho} \int_{\mathcal{X}\setminus 2Q} \frac{|f(z)|}{\lambda(c_B,d(c_B,z))} \frac{1}{[d(c_B,z)]^{\beta+\rho}} \mathrm{d}\mu(z) \\
&\le Cr_B^{\beta+\rho} \sum_{i=1}^\infty \int_{2^{i+1}Q\setminus 2^iQ} \frac{|f(z)|}{\lambda(c_B,d(c_B,z))} \frac{1}{[d(c_B,z)]^{\beta+\rho}} \mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)} \sum_{i=1}^\infty 2^{-i(\beta+\rho)} \frac{\mu(2^{i+1}Q)}{\lambda(c_B,2^ir_Q)} \\
&\le C\|f\|_{L^\infty(\mu)}.
\end{aligned}
$$

With a similar argument to that used in the proof of I_1, it is not difficult to obtain

$$
\mathrm{I}_2 \le C\|f\|_{L^\infty(\mu)}.
$$

Now we turn to an estimate of I_3. Write

$$\begin{aligned}
\mathrm{I}_3 &\le \int_{\mathcal{X}} \left| \frac{K(x,z)}{[d(x,z)]^{1-\rho}} - \frac{K(y,z)}{[d(y,z)]^{1-\rho}} \right| \left|f_2(z)\right| \left(\int_{d(y,z)}^{\infty} \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} \mathrm{d}\mu(z) \\
&\le \int_{\mathcal{X}} \left| \frac{K(x,z)}{[d(x,z)]^{1-\rho}} - \frac{K(y,z)}{[d(x,z)]^{1-\rho}} \right| \left|f_2(z)\right| \left(\int_{d(y,z)}^{\infty} \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} \mathrm{d}\mu(z) \\
&\quad + \int_{\mathcal{X}} \left| \frac{K(y,z)}{[d(x,z)]^{1-\rho}} - \frac{K(y,z)}{[d(y,z)]^{1-\rho}} \right| \left|f_2(z)\right| \left(\int_{d(y,z)}^{\infty} \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}} \right)^{\frac{1}{q}} \mathrm{d}\mu(z) \\
&\le C \int_{\mathcal{X}} \left|K(x,z) - K(y,z)\right| \left|f_2(z)\right| \frac{1}{[d(y,z)]^{\beta+1}} \,\mathrm{d}\mu(z) \\
&\quad + C \int_{\mathcal{X}} \left| \frac{1}{[d(x,z)]^{1-\rho}} - \frac{1}{[d(y,z)]^{1-\rho}} \right| \frac{|f_2(z)|}{[d(y,z)]^{\rho-1}\lambda(y,d(y,z))} \,\mathrm{d}\mu(z) \\
&=: \mathrm{I}_{31} + \mathrm{I}_{32}.
\end{aligned}$$

By applying (1.5), we have

$$\begin{aligned}
\mathrm{I}_{31} &\le C\|f\|_{L^\infty(\mu)} \int_{\mathcal{X}\setminus 2Q} \frac{r_B^{\delta+1+\beta}}{\lambda(c_B,d(c_B,z))} \frac{1}{[d(c_B,z)]^{\delta+\beta+1}} \,\mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)} \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} \frac{r_B^{\delta+1+\beta}}{\lambda(c_B,d(c_B,z))} \frac{1}{[d(c_B,z)]^{\delta+\beta+1}} \,\mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)} \sum_{k=1}^{\infty} 2^{-k(1+\beta+\delta)} \frac{\mu(2^{k+1}Q)}{\lambda(c_B,2^kQ)} \\
&\le C\|f\|_{L^\infty(\mu)}.
\end{aligned}$$

Now we turn to an estimate of I_{32} by two steps: $0 < \rho < 1$ and $\rho \ge 1$.
As $0 < \rho < 1$, we get

$$\begin{aligned}
\mathrm{I}_{32} &\le C\|f\|_{L^\infty(\mu)} \int_{\mathcal{X}\setminus 2Q} \frac{r_B[d(x,z)]^{-\rho}}{[d(x,z)]^{1-\rho}} \frac{1}{\lambda(c_B,d(c_B,z))} \,\mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)} \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} \frac{r_B}{d(x,z)} \frac{1}{\lambda(c_B,d(c_B,z))} \,\mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)} \sum_{k=1}^{\infty} 2^{-k} \frac{\mu(2^{k+1}Q)}{\lambda(c_B,2^kr_Q)} \\
&\le C\|f\|_{L^\infty(\mu)}.
\end{aligned}$$

As $\rho \ge 1$, we deduce that

$$\begin{aligned}
\mathrm{I}_{32} &\le C\|f\|_{L^\infty(\mu)} \int_{\mathcal{X}\setminus 6B} \frac{r_B[d(y,z)]^{\rho-2}}{[d(y,z)]^{\rho-1}} \frac{1}{\lambda(c_B,d(c_B,z))} \,\mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)} \sum_{k=1}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} \frac{r_B}{d(c_B,z)} \frac{1}{\lambda(c_B,d(c_B,z))} \,\mathrm{d}\mu(z) \\
&\le C\|f\|_{L^\infty(\mu)} \sum_{k=1}^{\infty} 2^{-k} \frac{\mu(2^{k+1}Q)}{\lambda(c_B,2^kr_Q)} \le C\|f\|_{L^\infty(\mu)}.
\end{aligned}$$

Thus, we have

$$\mathrm{I}_3 \le C\|f\|_{L^\infty(\mu)},$$

which, together with I_1 and I_2, implies (3.5). Therefore, the estimate of (3.2) is completed.

By (3.2), it follows that, for $f \in L^\infty(\mu)$, if $\mathcal{M}_{\beta,\rho,q}(f)(x_0) < \infty$ with some point $x_0 \in \mathcal{X}$, we can get $\mathcal{M}_{\beta,\rho,q}(f)$ is μ-finite a.e., and in this case, for any $(6,\beta_6)$-doubling ball B, we have

$$\frac{1}{\mu(B)} \int_B \bigl[\mathcal{M}_{\beta,\rho,q}(f)(x) - \operatorname*{ess\,inf}_B \mathcal{M}_{\beta,\rho,q}(f)\bigr]\,\mathrm{d}\mu(x) \le C\|f\|_{L^\infty(\mu)}.$$

To prove that $\mathcal{M}_{\beta,\rho,q}(f) \in \mathrm{RBLO}(\mu)$, applying Lemma 2.3, it only suffices to prove that $\mathcal{M}_{\beta,\rho,q}(f)$ satisfies (2.7), namely, for any two $(6,\beta_6)$-doubling balls B and S with $B \subset S$,

$$m_B\bigl(\mathcal{M}_{\beta,\rho,q}(f)\bigr) - m_S\bigl(\mathcal{M}_{\beta,\rho,q}(f)\bigr) \le CK_{B,S}. \tag{3.6}$$

For any $x \in B$ and $y \in S$, write

$$\begin{aligned}\mathcal{M}_{\beta,\rho,q}(f)(x) &\le \mathcal{M}_{\beta,\rho,q}(f\chi_{2B})(x) + \mathcal{M}_{\beta,\rho,q}(f\chi_{2S\setminus 2B})(x) \\ &\quad + \bigl[\mathcal{M}_{\beta,\rho,q}(f\chi_{\mathcal{X}\setminus 2S})(x) - \mathcal{M}_{\beta,\rho,q}(f\chi_{\mathcal{X}\setminus 2S})(y)\bigr] + \mathcal{M}_{\beta,\rho,q}(f\chi_{\mathcal{X}\setminus 2S})(y).\end{aligned}$$

With an argument similar to that used in the proof of $\mathcal{M}_{\beta,\rho,q}(f_2)(y)$ in (3.4), we have

$$\mathcal{M}_{\beta,\rho,q}(f\chi_{\mathcal{X}\setminus 2S})(y) \le \mathcal{M}_{\beta,\rho,q}(f)(y) + C\|f\|_{L^\infty(\mu)}.$$

By (3.5), for any $x, y \in S$, it is not difficult to get

$$\bigl|\mathcal{M}_{\beta,\rho,q}(f\chi_{\mathcal{X}\setminus 2S})(x) - \mathcal{M}_{\beta,\rho,q}(f\chi_{\mathcal{X}\setminus 2S})(y)\bigr| \le C\|f\|_{L^\infty(\mu)}.$$

For any $x \in B$, by the Minkowski inequality and (1.4), we obtain

$$\begin{aligned}&\mathcal{M}_{\beta,\rho,q}(f\chi_{2S\setminus 2B})(x) \\ &\quad \le C\int_{\mathcal{X}} \frac{|f\chi_{2S\setminus 2B}(y)|[d(x,y)]^{\rho+\beta}}{\lambda(x,d(x,y))} \left(\int_{d(x,y)}^\infty \frac{\mathrm{d}t}{t^{q(\beta+\rho)+1}}\right)^{\frac{1}{q}} \mathrm{d}\mu(y) \\ &\quad \le C\int_{2S\setminus 2B} \frac{|f(y)|}{\lambda(x,d(x,y))}\,\mathrm{d}\mu(y) \\ &\quad \le C\|f\|_{L^\infty(\mu)} \int_{2S\setminus B} \frac{1}{\lambda(c_B,d(c_B,y))}\,\mathrm{d}\mu(y) \\ &\quad \le CK_{B,S}\|f\|_{L^\infty(\mu)}.\end{aligned}$$

Thus, for any $x \in B$ and $y \in S$, we have

$$\mathcal{M}_{\beta,\rho,q}(f)(x) \le \mathcal{M}_{\beta,\rho,q}(f\chi_{2B})(x) + \mathcal{M}_{\beta,\rho,q}(f)(y) + CK_{B,S}\|f\|_{L^\infty(\mu)}. \tag{3.7}$$

For (3.7), taking the mean value over B for x and over S for y, it follows that

$$\begin{aligned} m_B(\mathcal{M}_{\beta,\rho,q}(f)) - m_S(\mathcal{M}_{\beta,\rho,q}(f)) &\leq C\big[m_B(\mathcal{M}_{\beta,\rho,q}(f\chi_{2B})) + K_{B,S}\|f\|_{L^\infty(\mu)}\big] \\ &\leq CK_{B,S}\|f\|_{L^\infty(\mu)}, \end{aligned}$$

which, together with (3.3), we finish the proof of Theorem 1.10. □

Proof of Theorem 1.12 Because the definition of $H^1(\mu)$ is independent of the choice of ς, thus, for convenience, we assume $\zeta = 2$ as in (1.7). By a standard argument, it suffices to prove that

$$\big\|\mathcal{M}_{\beta,\rho,q}(f)\big\|_{L^1(\mu)} \leq C|b|_{H^1(\mu)} \tag{3.8}$$

for any atomic block b with $\operatorname{supp} b \subset S$. Write

$$\begin{aligned} \int_{\mathcal{X}}\big|\mathcal{M}_{\beta,\rho,q}(b)(x)\big|\,\mathrm{d}\mu(x) &= \int_{\mathcal{X}\setminus 2S}\big|\mathcal{M}_{\beta,\rho,q}(b)(x)\big|\,\mathrm{d}\mu(x) + \int_{2S}\big|\mathcal{M}_{\beta,\rho,q}(b)(x)\big|\,\mathrm{d}\mu(x) \\ &=: \mathrm{J}_1 + \mathrm{J}_2. \end{aligned}$$

In a way similar to that used in the proof of E_1 in Theorem 1.9, we can obtain

$$\mathrm{J}_1 \leq C\|b\|_{L^1(\mu)} \leq C|b|_{H^1(\mu)}.$$

Let $b = \sum_i \tau_i a_i$ be as in Definition 1.8 and we have

$$\begin{aligned} \mathrm{J}_2 &\leq \sum_i|\tau_i|\int_{2B_i}\big|\mathcal{M}_{\beta,\rho,q}(b_i)(x)\big|\,\mathrm{d}\mu(x) + \sum_i|\tau_i|\int_{2S\setminus 2B_i}\big|\mathcal{M}_{\beta,\rho,q}(b_i)(x)\big|\,\mathrm{d}\mu(x) \\ &=: \mathrm{J}_{21} + \mathrm{J}_{22}. \end{aligned}$$

By applying the Hölder inequality, the $L^2(\mu)$-boundedness of $\mathcal{M}_{\beta,\rho,q}$, and (1.7), we have

$$\begin{aligned} \mathrm{J}_{21} &\leq \sum_i|\tau_i|\left(\int_{2B_i}\big|\mathcal{M}_{\beta,\rho,q}(a_i)(x)\big|^2\,\mathrm{d}\mu(x)\right)^{\frac{1}{2}}\mu(2B_i)^{\frac{1}{2}} \\ &\leq C\sum_i|\tau_i|\|a_i\|_{L^2(\mu)}\mu(2B_i)^{\frac{1}{2}} \\ &\leq C\sum_i|\tau_i|\|a_i\|_{L^\infty(\mu)}\mu(2B_i) \leq C\sum_i|\tau_i|. \end{aligned}$$

Now we turn to an estimate of J_{22}. Also, in a way similar to E_2 in the proof of Theorem 1.9, we have

$$\begin{aligned} \mathrm{J}_{22} &\leq C\sum_i|\tau_i|\int_{2S\setminus 2B_i}\frac{1}{\lambda(c_{B_i}, d(x, c_{B_i}))}\|a_i\|_{L^1(\mu)} \\ &\leq C\sum_i|\tau_i|K_{B_i,S}\|a_i\|_{L^\infty(\mu)}\mu(2B_i) \leq C\sum_i|\tau_i|. \end{aligned}$$

Combining with the above estimates, this implies (3.8) and hence the proof of Theorem 1.12. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
This paper is supported by National Natural Foundation of China (Grant No. 11561062).

Received: 8 March 2016 Accepted: 12 October 2016 Published online: 21 October 2016

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