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Endpoint estimates for the commutators of multilinear Calderón-Zygmund operators with Dini type kernels

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Abstract

Let $T_{\vec{b}}$ and $T_{\Pi b}$ be the commutators in the jth entry and iterated commutators of the multilinear Calderón-Zygmund operators, respectively. It was well known that the commutators of linear Calderón-Zygmund operators were not of weak type $(1,1)$ and (H^1, L^1), but they did satisfy certain endpoint $L\log L$ type estimates. In this paper, our aim is to give more natural sharp endpoint results. We show that $T_{\vec{b}}$ and $T_{\Pi b}$ are bounded from the product Hardy space $H^1 \times \cdots \times H^1$ to weak $L^{\frac{1}{m},\infty}$ space, whenever the kernel satisfies a class of Dini type condition. This was done by using a key lemma given by Christ, a very complex decomposition of the integrand domains, and carefully splitting the commutators into several terms.

Keywords: commutators; multilinear Calderón-Zygmund operator; C-Z kernel of ω type; Dini type conditions; Hardy spaces

1 Introduction

1.1 Commutators of classical C-Z operators

In 1976, Coifman, Rochberg, and Weiss [1] first introduced and studied the commutator of classical linear Calderón-Zygmund singular integrals, which was defined by

$$T_b f = [b, T]f = bT(f) - T(bf).$$

The L^p boundedness of T_b was given in [1] for $1 < p < \infty$ when $b \in BMO(\mathbb{R}^n)$. It is well known that T_b fails to be of weak type $(1,1)$ and is not bounded from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. Counterexamples were given by Pérez [2] and Paluszyński [3]. As an alternative result of the weak $(1,1)$ estimate of T_b, Pérez [2] obtained the following $L(\log L)$ type endpoint estimate:

$$\left|\left\{x \in \mathbb{R}^n : \left|T_b f(x)\right| > \lambda\right\}\right| \le C \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left(1 + \log^+\left(\frac{|f(x)|}{\lambda}\right)\right) dx, \quad \lambda > 0.$$

Moreover, alternative results of the (H^1, L^1) boundedness were also considered in the work of Alvarez [4], Pérez [2], and Liang, Ky, and Yang [5], which concerned with the boundedness of T_b on the subspace of atomic Hardy spaces, or concerned with the (H^1_w, L^1_w)

boundedness of T_b if b belongs to a subspace of BMO which is associated to the weight function w.

On the other hand, another more reasonable and alternative result of weak type $(1,1)$ and (H^1, L^1) estimate was given by Liu and Lu [6] in 2002. The authors [6] showed that T_b is bounded from $H^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$ if $b \in BMO(\mathbb{R}^n)$. We note that T_b also fails to be bounded from $H^p(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$ for $0 < p < 1$ by the generalized interpolation theorem [7], pp.63. Therefore, the $(H^1, L^{1,\infty})$ boundedness of T_b becomes a sharp endpoint estimate. Moreover, always $L(\log L)(\mathbb{S}^{n-1}) \subsetneq H^1(\mathbb{S}^{n-1})$ if f vanishes on the unit sphere. However, there is no such inclusion relationship on $\mathbb{R}^n$. Moreover, the inverse including relationship is still not true, since the following example shows that $H^1(\mathbb{R}^n) \not\subseteq L(\log L)(\mathbb{R}^n)$.

Example 1.1 Let

$$f(x) = \frac{\chi_{[-\frac{1}{2},\frac{1}{2}]}}{x \log_2^{1+\varepsilon} \frac{1}{|x|}} \quad \text{for some } \varepsilon > 0,$$

$$a_j(x) = \frac{f(x)}{f(\frac{1}{2^{j+1}})} \{\chi_{[-\frac{1}{2^j},-\frac{1}{2^{j+1}}]} + \chi_{[\frac{1}{2^{j+1}},\frac{1}{2^j}]}\} \times 2^j, \quad \lambda_j = \frac{f(\frac{1}{2^{j+1}})}{2^j}.$$

Thus, $f(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x)$, and it is easy to verify that each a_j is a $(1,\infty,0)$-atom. Notice that

$$\sum_{j=1}^{\infty} |\lambda_j| = \sum_{j=1}^{\infty} \frac{|f(\frac{1}{2^{j+1}})|}{2^j} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{1}{\frac{1}{2^{j+1}} \log_2^{1+\varepsilon} 2^{j+1}} = 2 \sum_{j=1}^{\infty} \frac{1}{(j+1)^{1+\varepsilon}} < \infty,$$

then we have $f \in H^1(\mathbb{R}^n)$. Obviously, $f \notin L(\log L)(\mathbb{R}^n)$.

Thus, the $(H^1, L^{1,\infty})$ boundedness and the $L \log L$ type estimate of T_b are independent in the sense that one cannot cover the results of the other.

1.2 Commutators of multilinear operators

In recent years, the theory of multilinear Calderón-Zygmund operators with standard kernels have been developed very quickly and a lot of work has been done. Among such achievements is the celebrated work of Coifman and Meyer [8–10], Christ and Journé [11], Kenig and Stein [12], Grafakos and Torres [13, 14], and Lerner *et al.* [15]. In order to state some well-known results, we need to introduce some definitions.

Definition 1.2 (C-Z kernel of ω type [16, 17]) Let $\omega(t)$ be a non-negative and non-decreasing function on $\mathbb{R}^+$. Let $K(x, y_1, \dots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$. Denote $(x, \vec{y}) = (x, y_1, \dots, y_m)$, we say K is an m-linear Calderón-Zygmund kernel of ω type, if there exists a positive constant C_0 such that

$$\bigl|K(x,\vec{y})\bigr| \leq \frac{C_0}{(\sum_{j=1}^m |x-y_j|)^{mn}}, \tag{1.1}$$

$$\bigl|K(x,\vec{y}) - K(x',\vec{y})\bigr| \leq \frac{C_0}{(\sum_{j=1}^m |x-y_j|)^{mn}} \omega\left(\frac{|x-x'|}{\sum_{j=1}^m |x-y_j|}\right), \tag{1.2}$$

whenever $|x - x'| \leq \frac{1}{2}\max_{1\leq j\leq m}|x - y_j|$, and

$$\begin{aligned}&\bigl|K(x,y_1,\dots,y_i,\dots,y_m) - K\bigl(x,y_1,\dots,y_i',\dots,y_m\bigr)\bigr|\\&\quad\leq \frac{C_0}{(\sum_{j=1}^m|x-y_j|)^{mn}}\omega\biggl(\frac{|y_i-y_i'|}{\sum_{j=1}^m|x-y_j|}\biggr),\end{aligned} \tag{1.3}$$

whenever $|y_i - y_i'| \leq \frac{1}{2}\max_{1\leq j\leq m}|x - y_j|$.

Definition 1.3 (Multilinear C-Z singular integral operators [16, 17]) Let $K(x,\vec{y})$ be a C-Z kernel of ω type. For any $\vec{f} = (f_1,\dots,f_m) \in \mathscr{S}(\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n)\times\cdots\times\mathscr{S}(\mathbb{R}^n)$ and all $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$, we define the multilinear Calderón-Zygmund singular integral operators as follows:

$$T(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x,y_1,\dots,y_m)f_1(y_1),\dots,f_m(y_m)\,dy_1\cdots dy_m.$$

Definition 1.4 (Commutators of multilinear C-Z operators) Let $b_j \in BMO(\mathbb{R}^n)$ and T be the operator defined in Definition 1.3. The commutators in the jth entry and the iterated commutators of T are defined by

$$\begin{aligned}T_{\vec{b}}(\vec{f})(x) &= \sum_{j=1}^m T_{\vec{b}}^j(\vec{f})(x)\\&= \sum_{j=1}^m\bigl[b_j(x)T(f_1,\dots,f_j,\dots,f_m)(x) - T(f_1,\dots,b_jf_j,\dots,f_m)(x)\bigr]\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}T_{\Pi b}(\vec{f}) &= \bigl[b_1,\bigl[b_2,\dots\bigl[b_{m-1},[b_m,T]_m,\bigr]_{m-1}\cdots\bigr]_2\bigr]_1(\vec{f})\\&= \int_{(\mathbb{R}^n)^m}\prod_{j=1}^m\bigl(b_j(x)-b_j(y_j)\bigr)K(x,y_1,\dots,y_m)f_1(y_1)\cdots f_m(y_m)\,d\vec{y}.\end{aligned} \tag{1.5}$$

Remark 1.5 Obviously, in the special case, $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$, then the operator T defined in Definition 1.3 coincides with the standard multilinear Calderón-Zygmund operator defined and studied by Grafakos and Torres [13]. Moreover, if $\omega(t) = t^\varepsilon$, the weighted strong and $L(\log L)$ type endpoint estimates for $T_{\vec{b}}(f_1,\dots,f_m)(x) = \sum_{j=1}^m T_{\vec{b}}^j(\vec{f})$ and $T_{\Pi b}$ have already been studied in [15] and [18], respectively.

Definition 1.6 (Dini(a) type conditions) Let $\omega(t)$ be a non-negative and non-decreasing function on $\mathbb{R}^+$. ω is said to satisfy the Dini(a) condition if

$$\int_0^1\frac{\omega^a(t)}{t}\,dt < \infty.$$

ω is said to satisfy the log-Dini(a) condition if the following inequality holds:

$$\int_0^1\frac{\omega^a(t)}{t}\biggl(1+\log\frac{1}{t}\biggr)\,dt < \infty. \tag{1.6}$$

Remark 1.7 It is easy to see that the log-Dini(a) condition is stronger than the Dini(a) condition and if $0 < a_1 < a_2$, then Dini(a_1) $\subset$ Dini(a_2).

In 2009, Maldonado and Naibo [17] showed that, when ω is concave and $\omega \in$ Dini(1/2), the bilinear Calderón-Zygmund operator of ω type is bounded from $L^1 \times L^1$ to $L^{\frac{1}{2},\infty}$. In 2014, Lu and Zhang [16] improved the results in [17] by removing the hypothesis that ω is concave and reducing the condition $\omega \in$ Dini(1/2) to the weaker condition $\omega \in$ Dini(1). Lu and Zhang [16] also extended the weighted strong and $L(\log L)$ type endpoint estimates to the commutators defined in (1.4) whenever ω satisfies the log-Dini(1) condition, which is stronger than Dini(1) condition but it is much weaker than the standard kernel $\omega(t) = t^{\varepsilon}$. More previous work on the commutators of multilinear operators with $\omega(t) = t^{\varepsilon}$ can be found in [18–21] and [22].

1.3 Main results

In this paper, we will consider the sharp endpoint estimates for both the commutator in the jth entry defined in (1.4) and the iterated commutators defined in (1.5) with a C-Z kernel of ω type. We show that they are bounded from a product Hardy space $H^1 \times \cdots \times H^1$ to a weak $L^{\frac{1}{m},\infty}$ space, whenever the kernel satisfies a class of Dini type condition. However, the proof is very difficult and complex. In particular, in the case of iterated commutators, we need to control six summations and three integrals at the same time even for $m = 2$. We formulate our main results as follows.

Theorem 1.1 *Let T be a multilinear Calderón-Zygmund operators with a C-Z kernel of ω type and $T_{\vec{b}}$ be the commutators of the jth entries defined in* (1.4) *with $\vec{b} \in BMO^m$. If $\omega(t)$ satisfies the* log-Dini(1) *condition, then there exists a constant $C > 0$, such that the following inequality holds*:

$$\left|\left\{x \in \mathbb{R}^n : \left|T_{\vec{b}}(\vec{f})(x)\right| > \lambda\right\}\right| \le C_{\|\vec{b}\|_{BMO^m}} \lambda^{-\frac{1}{m}} \prod_{j=1}^{m} \|f_j\|_{H^1(\mathbb{R}^n)}^{\frac{1}{m}}. \tag{1.7}$$

With a stronger condition assumed on the function $\omega(t)$ than in Theorem 1.1, but a weaker condition than the standard kernel $\omega(t) = t^{\varepsilon}$, we obtain the following theorem for the iterated commutators.

Theorem 1.2 *Let $\omega(t)$ be a doubling function, satisfying the* log-Dini($1/2m$) *condition, that is*,

$$\int_0^1 \omega(t)^{\frac{1}{2m}} t^{-1}\left(1 + \log\frac{1}{t}\right) dt < \infty.$$

Let T be a multilinear Calderón-Zygmund operators with a C-Z kernel of ω type and $T_{\Pi b}$ be the iterated commutators defined in (1.5) *with $\vec{b} \in BMO^m$. Then there exists a constant $C > 0$, such that the following inequality holds*:

$$\left|\left\{x \in \mathbb{R}^n : \left|T_{\Pi b}(\vec{f})(x)\right| > \lambda\right\}\right| \le C_{\|\vec{b}\|_{BMO^m}} \lambda^{-\frac{1}{m}} \prod_{j=1}^{m} \|f_j\|_{H^1(\mathbb{R}^n)}^{\frac{1}{m}}. \tag{1.8}$$

This article is organized as follows. In Section 2, the proof of Theorem 1.1 will be given. Section 3 will be devoted to the proof of Theorem 1.2.

2 Proofs of Theorem 1.1

To prove Theorem 1.1, we need the following key lemma given by Chirst [23], which provides a foundation for our analysis.

Lemma 2.1 ([23]) *For any $\alpha > 0$ and any finite collection of dyadic cubes Q and associated positive scalars λ_Q, there exists a collection of pairwise disjoint dyadic cubes S such that*

(1) $\sum_{Q\subset S}\lambda_Q \le 2^n\alpha|S|$, *for all S*;
(2) $\sum|S| \le \alpha^{-1}\sum\lambda_Q$;
(3) $\|\sum_{Q\not\subseteq \text{ any } S}\lambda_Q|Q|^{-1}\chi_Q\|_{L^\infty(\mathbb{R}^n)} \le \alpha$.

Proof of Theorem 1.1 For simplicity, we only consider the case for $m = 2$, because there is no essential difference for the general case.

Since $T_{\vec{b}}$ is bounded from $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ [16], and finite sums of atoms are dense in $H^1(\mathbb{R}^n)$, we will work with such sums and will obtain desired estimates which is independent of the number of terms in each sum. Thus, for any given $f_j \in H^1(\mathbb{R}^n)$ $(j = 1, 2)$, we may assume that $f_j = \sum_{k_j}\lambda_{k_j}a_{k_j}$ is a finite sum of H^1-atoms, where each a_{k_j} is a $(1,\infty,0)$ atom, with $\sum_{k_j}|\lambda_{k_j}| \le C\|f_j\|_{H^1(\mathbb{R}^n)}$. Set $C_1 = \|T_{\vec{b}}\|_{L^2\times L^2\to L^{1,\infty}}$ and $C_2 = \|T\|_{L^1\times L^1\to L^{\frac{1}{2},\infty}}$. By linearity, it is sufficient to consider the commutator of T with only one symbol, that is, for $\vec{b} = b \in BMO(\mathbb{R}^n)$, we will consider the operator

$$T_b(f_1,f_2)(x) = b(x)T(f_1,f_2)(x) - T(bf_1,f_2)(x).$$

To prove inequality (1.7), without loss of generality, we may assume that $\|f_j\|_{H^1(\mathbb{R}^n)} = 1$ for $j = 1, 2$. For fix $\lambda > 0$, we only need to show that there is a constant $C > 0$, independent on the variables and f_j $(j = 1, 2)$, such that

$$\left|\left\{x\in\mathbb{R}^n : \left|T_b(f_1,f_2)(x)\right| > \lambda\right\}\right| \le C(C_0 + C_1 + C_2)^{1/2}\lambda^{-1/2}. \tag{2.1}$$

Let γ be a positive number to be determined later. Take the finite collection of dyadic cubes Q_{j,k_j}, which is associated with the positive scalars $\lambda_{Q_{j,k_j}}$ in the given atomic decomposition of f_j. Now, we take $\alpha = (\gamma\lambda)^{1/2}$ in Lemma 2.1. Then there exists a collection of pairwise disjoint dyadic cubes S_{j,l_j}, such that

$$\text{(I)}\quad \sum_{Q_{j,k_j}\subset S_{j,l_j}}\lambda_{Q_{j,k_j}} \le 2^n(\gamma\lambda)^{1/2}|S_{j,l_j}|, \quad \text{for all } S_{j,l_j};$$

$$\text{(II)}\quad \sum_{S_{j,l_j}}|S_{j,l_j}| \le (\gamma\lambda)^{-1/2}\sum_{Q_{j,k_j}\subset S_{j,l_j}}\lambda_{Q_{j,k_j}};$$

$$\text{(III)}\quad \Bigg\|\sum_{Q_{j,k_j}\not\subseteq \text{ any } S_{j,l_j}}\lambda_{Q_{j,k_j}}|Q_{j,k_j}|^{-1}\chi_{Q_{j,k_j}}\Bigg\|_{L^\infty(\mathbb{R}^n)} \le (\gamma\lambda)^{1/2}.$$

Denote $S^*_{j,l_j} = 8\sqrt{n}S_{j,l_j}$, $S^*_j = \bigcup_{l_j} S^*_{j,l_j}$ for $j = 1,2$, and $S^* = \bigcup_{j=1}^2 S^*_j$. Set

$$h_j = \sum_{S_{j,l_j}} \sum_{Q_{j,k_j}\subset S_{j,l_j}} \lambda_{Q_{j,k_j}} a_{Q_{j,k_j}} \quad \text{and} \quad g_j(x) = f_j(x) - h_j(x).$$

By the definition of g_j and h_j, (III), and the properties of the $(1,\infty,0)$ atoms, we have

$$\|g_j\|_{L^\infty(\mathbb{R}^n)} \le (\gamma\lambda)^{1/2}; \qquad \|g_j\|_{L^1(\mathbb{R}^n)} \le \sum_{Q_{j,k_j}\not\subseteq \text{ any } S_{j,l_j}} |\lambda_{Q_{j,k_j}}| \le \sum_{k_j} |\lambda_{k_j}| \le C\|f_j\|_{H^1(\mathbb{R}^n)};$$

$$\|h_j\|_{L^1(\mathbb{R}^n)} \le \sum_{S_{j,l_j}} \sum_{Q_{j,k_j}\subset S_{j,l_j}} |\lambda_{Q_{j,k_j}}| \int_{\mathbb{R}^n} |a_{Q_{j,k_j}}|\,dx \le \sum_{k_j} |\lambda_{k_j}| \le C\|f_j\|_{H^1(\mathbb{R}^n)}.$$

Now, we introduce some more notations as follows:

$$E_1 = \{x\in\mathbb{R}^n : |T_b(g_1,g_2)(x)| > \lambda/4\}; \qquad E_2 = \{x\in\mathbb{R}^n\backslash S^* : |T_b(h_1,g_2)(x)| > \lambda/4\};$$
$$E_3 = \{x\in\mathbb{R}^n\backslash S^* : |T_b(g_1,h_2)(x)| > \lambda/4\}; \qquad E_4 = \{x\in\mathbb{R}^n\backslash S^* : |T_b(h_1,h_2)(x)| > \lambda/4\}.$$

By (II), it follows that

$$|S^*| \le \sum_{j=1}^2 |S^*_j| \le \sum_{j=1}^2 \sum_{S_{j,l_j}} |S^*_{j,l_j}| \le C(\gamma\lambda)^{-1/2} \sum_{j=1}^2 \sum_{Q_{j,l_j}\subset S_{j,l_j}} \lambda_{Q_{j,l_j}} \le C(\gamma\lambda)^{-1/2}. \tag{2.2}$$

From the $L^2\times L^2 \to L^{1,\infty}$ boundedness of $T_{\vec{b}}$, the Chebyshev inequality, and $\|g_j\|_{L^\infty(\mathbb{R}^n)} \le (\gamma\lambda)^{1/2}$, one may obtain

$$\begin{aligned}|E_1| &\le C_1\lambda^{-1}\|g_1\|_{L^2(\mathbb{R}^n)}\|g_2\|_{L^2(\mathbb{R}^n)} \le C_1\lambda^{-1}(\gamma\lambda)^{\frac12}\|g_1\|^{\frac12}_{L^1(\mathbb{R}^n)}\|g_2\|^{\frac12}_{L^1(\mathbb{R}^n)}\\ &\le CC_1\gamma^{\frac12}\lambda^{-1}\|f_1\|^{\frac12}_{H^1(\mathbb{R}^n)}\|f_2\|^{\frac12}_{H^1(\mathbb{R}^n)} = CC_1\gamma^{\frac12}\lambda^{-\frac12}.\end{aligned} \tag{2.3}$$

Therefore, we get

$$\begin{aligned}\big|\{x\in\mathbb{R}^n : |T_b(\vec{f})(x)| > \lambda\}\big| &\le \sum_{s=1}^4 |E_s| + C|S^*|\\ &\le \sum_{s=2}^4 |E_s| + C(\gamma\lambda)^{-1/2} + CC_1\gamma^{\frac12}\lambda^{-\frac12}.\end{aligned} \tag{2.4}$$

Hence, to finish the proof of Theorem 1.1, we only need to consider the contributions of each $|E_s|$ for $2\le s\le 4$, separately.

• *Estimate for* $|E_2|$. By the definition of g_j and h_j, the moment condition of H^1-atoms, and employing the linearity of T_b, it now follows that

$$\begin{aligned}&T_b(h_1,g_2)(x)\\ &\quad= \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \lambda_{Q_{1,k_1}} \iint_{(\mathbb{R}^n)^2} \big(b(x) - b_{Q_{1,k_1}}\big)\big(K(x,y_1,y_2) - K(x,c_{1,k_1},y_2)\big)\end{aligned}$$

$$\begin{aligned}
&\quad \times a_{Q_{1,k_1}}(y_1)g_2(y_2)\,d\vec{y}\\
&\quad + \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \lambda_{Q_{1,k_1}} \iint_{(\mathbb{R}^n)^2} \big(b_{Q_{1,k_1}} - b(y_1)\big) K(x,y_1,y_2) a_{Q_{1,k_1}}(y_1)g_2(y_2)\,d\vec{y}\\
&=: I_{2,1}(x) + I_{2,2}(x).
\end{aligned} \tag{2.5}$$

Therefore, we have

$$\begin{aligned}
|E_2| &\le \big|\big\{x\in\mathbb{R}^n\backslash S^* : \big|I_{2,1}(x)\big| > \lambda/8\big\}\big| + \big|\big\{x\in\mathbb{R}^n\backslash S^* : \big|I_{2,2}(x)\big| > \lambda/8\big\}\big|\\
&:= |E_{2,1}| + |E_{2,2}|.
\end{aligned}$$

Thus, to show the contributions of E_2, we only need to consider the contributions of $E_{2,1}$ and $E_{2,2}$, respectively.

To estimate $|E_{2,1}|$, we fix k_1 and denote $\mathscr{R}^i_{1,k_1} = (2^{i+2}\sqrt{n}Q_{1,k_1})\backslash(2^{i+1}\sqrt{n}Q_{1,k_1})$, $i = 1,2,\ldots$. Then it is obvious that $\mathbb{R}^n\backslash S^* \subset \mathbb{R}^n\backslash Q^*_{1,k_1} \subset \bigcup_{i=1}^{\infty}\mathscr{R}^i_{1,k_1}$. Let c_{1,k_1} be the center of cube Q_{1,k_1}, $l_{Q_{1,k_1}}$ be the side length of cube Q_{1,k_1} Then, for any $y_1\in Q_{1,k_1}$ and $x\in\mathscr{R}^i_{1,k_1}$, we have

$$|y_1 - c_{1,k_1}| \le \frac{1}{2}\sqrt{n}l_{Q_{1,k_1}} \quad \text{and} \quad |x - c_{1,k_1}| \ge 2^{i-1}\sqrt{n}l_{Q_{1,k_1}}. \tag{2.6}$$

By the Chebychev inequality and (1.3), it follows that

$$\begin{aligned}
|E_{2,1}| &\le \frac{8C_0}{\lambda}\|g_2\|_{L^\infty} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \int_{\mathbb{R}^n\backslash S^*}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|b(x) - b_{Q_{1,k_1}}\big|\\
&\quad \times \frac{|a_{1,k_1}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\,\omega\bigg(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|+|x-y_2|}\bigg)\,dy_1\,dy_2\,dx.
\end{aligned} \tag{2.7}$$

Since $\mathbb{R}^n\backslash S^* \subset \bigcup_{i=1}^{\infty}\mathscr{R}^i_{1,k_1}$ and ω is non-decreasing, together with (2.6) and noticing that $a_{1,k_1}\in L^1(\mathbb{R}^n)$, one obtains

$$\begin{aligned}
&\int_{\mathbb{R}^n\backslash S^*}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|b(x) - b_{Q_{1,k_1}}\big| \frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\,\omega\bigg(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|+|x-y_2|}\bigg)\,dy_1\,dy_2\,dx\\
&\quad \le \sum_{i=1}^{\infty}\int_{\mathscr{R}^i_{1,k_1}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|b(x) - b_{Q_{1,k_1}}\big| \frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\,\omega\bigg(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|}\bigg)\,dy_1\,dy_2\,dx\\
&\quad \le C\sum_{i=1}^{\infty}\omega\big(2^{-i}\big)\int_{\mathscr{R}^i_{1,k_1}}\int_{\mathbb{R}^n} \big|b(x) - b_{Q_{1,k_1}}\big| \frac{|a_{Q_{1,k_1}}(y_1)|}{|x-y_1|^n}\,dy_1\,dx\\
&\quad \le C\sum_{i=1}^{\infty}\omega\big(2^{-i}\big)\frac{1}{|2^{i+2}Q_{1,k_1}|}\int_{2^{i+2}Q_{1,k_1}} \big|b(x) - b_{Q_{1,k_1}}\big|\,dx\\
&\quad \le C\sum_{i=1}^{\infty} i\omega\big(2^{-i}\big)\|\vec{b}\|_* \le C.
\end{aligned}$$

Putting the above estimate into (2.7) and noticing the fact that $\|g_j\|_{L^\infty(\mathbb{R}^n)} \le (\gamma\lambda)^{1/2}$, we have

$$|E_{2,1}| \le \frac{CC_0}{\lambda}(\gamma\lambda)^{\frac{1}{2}} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}. \tag{2.8}$$

Now, we are in the position to estimate $|E_{2,2}|$. The $L^1 \times L^1 \to L^{\frac{1}{2},\infty}$ boundedness of T implies that

$$\begin{aligned} |E_{2,2}| &\le CC_2^{\frac{1}{2}}\lambda^{-\frac{1}{2}} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \big\| \big(b(x) - b_{Q_{1,k_1}}\big) a_{Q_{1,k_1}} \big\|_{L^1(\mathbb{R}^n)}^{\frac{1}{2}} \|g_2\|_{L^1(\mathbb{R}^n)}^{\frac{1}{2}} \\ &\le CC_2^{\frac{1}{2}}\lambda^{-\frac{1}{2}} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \left(\frac{1}{|Q_{1,k_1}|} \int_{Q_{1,k_1}} \big|b(y_1) - b_{Q_{1,k_1}}\big| \, dy_1 \right)^{\frac{1}{2}} \|f_2\|_{H^1(\mathbb{R}^n)}^{\frac{1}{2}} \\ &\le CC_2^{\frac{1}{2}} \|\vec{b}\|_*^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \\ &\le CC_2^{\frac{1}{2}} \lambda^{-\frac{1}{2}}. \end{aligned} \tag{2.9}$$

Therefore in all, combining (2.8) and the above estimate, we conclude that

$$|E_2| \le C\big(C_0 \gamma^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + C_2^{\frac{1}{2}} \lambda^{-\frac{1}{2}}\big).$$

- *Estimate for* $|E_3|$. The estimate of $|E_3|$ is similar to $|E_2|$. In fact,

$$\begin{aligned} &T_b(g_1,h_2)(x) \\ &\quad = \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \lambda_{Q_{2,k_2}} \iint_{(\mathbb{R}^n)^2} \big(b(x) - b_{Q_{2,k_2}}\big)\big(K(x,y_1,y_2) - K(x,y_1,c_{2,k_2})\big) \\ &\qquad \times g_1(y_1) a_{Q_{2,k_2}}(y_2) \, d\vec{y} \\ &\qquad + \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \lambda_{Q_{2,k_2}} \iint_{(\mathbb{R}^n)^2} \big(b_{Q_{2,k_2}} - b(y_2)\big) K(x,y_1,y_2) g_1(y_1) a_{Q_{2,k_2}}(y_2) \, d\vec{y} \\ &\quad =: I_{3,1}(x) + I_{3,2}(x). \end{aligned}$$

Repeating the same steps as we have done for $|E_2|$, we may obtain

$$|E_3| \le C\big(C_0 \gamma^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + C_2^{\frac{1}{2}} \lambda^{-\frac{1}{2}}\big).$$

- *Estimate for* $|E_4|$. First, we split $T_b(h_1,h_2)$ in the form as follows:

$$\begin{aligned} &T_b(h_1,h_2)(x) \\ &\quad = \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \iint_{(\mathbb{R}^n)^2} \big(b(x) - b_{Q_{1,k_1}}\big)\big(K(x,y_1,y_2) - K(x,c_{1,k_1},y_2)\big) \\ &\qquad \times \lambda_{Q_{1,k_1}} a_{Q_{1,k_1}}(y_1) \lambda_{Q_{2,k_2}} a_{Q_{2,k_2}}(y_2) \, d\vec{y} \\ &\qquad + \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \iint_{(\mathbb{R}^n)^2} \big(b_{Q_{1,k_1}} - b(y_1)\big) K(x,y_1,y_2) \lambda_{Q_{1,k_1}} a_{Q_{1,k_1}}(y_1) h_2(y_2) \, d\vec{y} \\ &\quad =: I_{4,1}(x) + I_{4,2}(x). \end{aligned}$$

Hence, we have

$$|E_4| \le \big|\big\{x \in \mathbb{R}^n \backslash S^* : \big|I_{4,1}(x)\big| > \lambda/8\big\}\big| + \big|\big\{x \in \mathbb{R}^n \backslash S^* : \big|I_{4,2}(x)\big| > \lambda/8\big\}\big|. \tag{2.10}$$

For fixed k_2, denote $\mathscr{R}^h_{2,k_2} = (2^{h+2}\sqrt{n}Q_{2,k_2})\backslash(2^{h+1}\sqrt{n}Q_{2,k_2})$, $h = 1, 2, \ldots$. Recalling the definition of $\mathscr{R}^i_{1,k_1}$, it is easy to check

$$(S^*)^c := \mathbb{R}^n\backslash S^* \subset \mathbb{R}^n\backslash\Bigl(Q^*_{1,k_1} \cup Q^*_{1,k_2}\Bigr) \subset \bigcup_{h=1}^{\infty}\bigcup_{i=1}^{\infty}\Bigl(\mathscr{R}^i_{1,k_1} \cap \mathscr{R}^h_{2,k_2}\Bigr).$$

Therefore, one may obtain

$$(S^*)^c = (S^*)^c \cap \left(\bigcup_{h=1}^{\infty}\bigcup_{i=1}^{\infty}\Bigl(\mathscr{R}^i_{1,k_1} \cap \mathscr{R}^h_{2,k_2}\Bigr)\right) = \bigcup_{h=1}^{\infty}\bigcup_{i=1}^{\infty}\Bigl((S^*)^c \cap \Bigl(\mathscr{R}^i_{1,k_1} \cap \mathscr{R}^h_{2,k_2}\Bigr)\Bigr). \tag{2.11}$$

By the Chebychev inequality, (1.3), and (2.11), it follows that

$$\begin{aligned}
&\bigl|\bigl\{x \in \mathbb{R}^n\backslash S^* : \bigl|I_{4,1}(x)\bigr| > \lambda/8\bigr\}\bigr| \\
&\quad\le \frac{8C_0}{\lambda}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\sum_{S_{2,l_2}}\sum_{Q_{2,k_2}\subset S_{2,l_2}}\int_{\mathbb{R}^n\backslash S^*}\iint_{(\mathbb{R}^n)^2}\bigl|b(x) - b_{Q_{1,k_1}}\bigr| \\
&\qquad\times \frac{|\lambda_{Q_{1,k_1}}||a_{Q_{1,k_1}}(y_1)||\lambda_{Q_{2,k_2}}||a_{Q_{2,k_2}}(y_2)|}{(|x-y_1| + |x-y_2|)^{2n}}\omega\left(\frac{|y_1 - c_{1,k_1}|}{|x-y_1| + |x-y_2|}\right)dy_1\,dy_2\,dx.
\end{aligned} \tag{2.12}$$

Moreover, by (2.11), the integrals in the above summations can be controlled by

$$\begin{aligned}
&\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}\cap\mathscr{R}^h_{2,k_2}}\iint_{(\mathbb{R}^n)^2}\bigl|b(x) - b_{Q_{1,k_1}}\bigr| \\
&\qquad\times \frac{|\lambda_{Q_{1,k_1}}||a_{Q_{1,k_1}}(y_1)||\lambda_{Q_{2,k_2}}||a_{Q_{2,k_2}}(y_2)|}{(|x-y_1| + |x-y_2|)^{2n}}\omega\left(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|}\right)dy_1\,dy_2\,dx \\
&\quad\le \sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\omega\bigl(2^{-i}\bigr)\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}\cap\mathscr{R}^h_{2,k_2}}\iint_{(\mathbb{R}^n)^2}\bigl|b(x) - b_{Q_{1,k_1}}\bigr| \\
&\qquad\times |\lambda_{Q_{1,k_1}}|\bigl|a_{Q_{1,k_1}}(y_1)\bigr||\lambda_{Q_{2,k_2}}|\bigl|a_{Q_{2,k_2}}(y_2)\bigr| \\
&\qquad\times \sup_{y_1,y_2\in S}\frac{1}{(|x-y_1| + |x-y_2|)^{2n}}\,dy_1\,dy_2\,dx.
\end{aligned} \tag{2.13}$$

For fixed $x \in (S^*)^c$, and any $y_1, y_2 \in S$, we have

$$\inf_{y_1\in S}|x-y_1| \approx |x-y_1|, \qquad \inf_{y_2\in S}|x-y_2| \approx |x-y_2|.$$

This implies that

$$\begin{aligned}
\sup_{y_1,y_2\in S}\frac{1}{(|x-y_1| + |x-y_2|)^{2n}} &= \frac{1}{(\inf_{y_1\in S}|x-y_1| + \inf_{y_2\in S}|x-y_2|)^{2n}} \\
&\approx \frac{1}{(|x-y_1| + |x-y_2|)^{2n}}.
\end{aligned} \tag{2.14}$$

Note that $\{S_{j,l_j}\}_{l_j}$ are pairwise disjoint dyadic cubes, by (I) and (2.14), it now follows that

$$
\begin{aligned}
&\sum_{S_{2,l_2}}\sum_{Q_{2,k_2}\subset S_{2,l_2}}\int_{\mathbb{R}^n}|\lambda_{Q_{2,k_2}}|\left|a_{Q_{2,k_2}}(y_2)\right|\sup_{y_1,y_2\in S}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\,dy_2\\
&\quad=\sum_{S_{2,l_2}}\sum_{Q_{2,k_2}\subset S_{2,l_2}}|\lambda_{Q_{2,k_2}}|\sup_{y_1,y_2\in S}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\int_{\mathbb{R}^n}\left|a_{Q_{2,k_2}}(y_2)\right|dy_2\\
&\quad\le C\sum_{S_{2,l_2}}\biggl(\sum_{Q_{2,k_2}\subset S_{2,l_2}}|\lambda_{Q_{2,k_2}}|\biggr)\sup_{y_1,y_2\in S}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\\
&\quad\le\sum_{S_{2,l_2}}2^n(\gamma\lambda)^{1/2}|S_{2,l_2}|\sup_{y_1,y_2\in S}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\\
&\quad\le C(\gamma\lambda)^{1/2}\sum_{S_{2,l_2}}\int_{S_{2,l_2}}\frac{1}{(|x-y_1|+|x-y_2|)^{2n}}\,dy_2\\
&\quad\le C(\gamma\lambda)^{1/2}\frac{1}{|x-y_1|^n}.
\end{aligned}
\tag{2.15}
$$

Combining (2.12), (2.13), and (2.15), we obtain

$$
\begin{aligned}
&\left|\left\{x\in\mathbb{R}^n\backslash S^*:\left|I_{4,1}(x)\right|>\lambda/8\right\}\right|\\
&\quad\le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\omega\bigl(2^{-i}\bigr)\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}\cap\mathscr{R}^h_{2,k_2}}\int_{\mathbb{R}^n}\left|b(x)-b_{Q_{1,k_1}}\right|\\
&\qquad\times\frac{|\lambda_{Q_{1,k_1}}||a_{Q_{1,k_1}}(y_1)|}{|x-y_1|^{2n}}\,dy_1\,dx\\
&\quad\le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\sum_{i=1}^{\infty}\omega\bigl(2^{-i}\bigr)\int_{\mathscr{R}^i_{1,k_1}}\int_{\mathbb{R}^n}\left|b(x)-b_{Q_{1,k_1}}\right|\\
&\qquad\times\frac{|\lambda_{Q_{1,k_1}}||a_{Q_{1,k_1}}(y_1)|}{|x-y_1|^{2n}}\,dy_1\,dx\\
&\quad\le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega\bigl(2^{-i}\bigr)\frac{1}{|2^{i+2}Q_{1,k_1}|}\int_{2^{i+2}Q_{1,k_1}}\left|b(x)-b_{Q_{1,k_1}}\right|dx\\
&\quad\le CC_0\|\vec{b}\|_*\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega\bigl(2^{-i}\bigr)i\\
&\quad\le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}.
\end{aligned}
\tag{2.16}
$$

The estimate of $|\{x\in\mathbb{R}^n\backslash S^*:|I_{4,2}(x)|>\lambda/8\}|$ is similar to (2.9). In fact, we only need to replace g_2 by h_2 in (2.9), and noting that $\|h_2\|_{L^1}\le C\|f_2\|_{H^1}$, we have

$$
\left|\left\{x\in\mathbb{R}^n\backslash S^*:\left|I_{4,2}(x)\right|>\lambda/8\right\}\right|\le CC_2^{\frac{1}{2}}\lambda^{-\frac{1}{2}}.
\tag{2.17}
$$

Putting (2.16) and (2.17) into (2.10), it yields

$$
|E_4|\le C\bigl(C_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}+C_2^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\bigr).
$$

Thus, we have proved that

$$|E_s| \leq C\big(C_0 \gamma^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + C_2^{\frac{1}{2}} \lambda^{-\frac{1}{2}}\big) \quad \text{for } s = 2, 3, 4. \tag{2.18}$$

Set $\gamma = (C_0 + C_1 + C_2)^{-1}$, by (2.4) and (2.18), we have

$$\begin{aligned} \big|\big\{x \in \mathbb{R}^n : \big|T_b(\vec{f})(x)\big| > \lambda\big\}\big| &\leq \sum_{s=2}^{4} |E_s| + C(\gamma\lambda)^{-1/2} + CC_1\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}} \\ &\leq C(C_0 + C_1 + C_2)^{1/2}\lambda^{-1/2}. \end{aligned}$$

The proof of (2.1) is finished. Since we have reduced the proof of Theorem 1.1 to (2.1), the proof of Theorem 1.1 is completed. □

3 Proof of Theorem 1.2

Proof of Theorem 1.2 Since there is no essential difference for the general case, we will also only consider Theorem 1.2 for the case $m = 2$. Thus, it is sufficient to consider the following operator:

$$\begin{aligned} T_{\pi b}(f_1, f_2)(x) &= \big[b_1, [b_2, T]_2, \big]_1(f_1, f_2) \\ &= \int_{(\mathbb{R}^n)^m} \prod_{j=1}^{2} \big(b_j(x) - b_j(y_j)\big) K(x, y_1, y_2) f_1(y_1) f_2(y_2)\, dy_1\, dy_2, \end{aligned}$$

where $f_j \in H^1(\mathbb{R}^n)$ $(j = 1, 2)$ with $\|f_j\|_{H^1(\mathbb{R}^n)} = 1$ for $j = 1, 2$. Since $T_{\pi b}(f_1, f_2)(x)$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ [18], we may set $C_1' = \|T_{\pi b}\|_{L^2 \times L^2 \to L^{1,\infty}}$. Recall $C_2 = \|T\|_{L^1 \times L^1 \to L^{\frac{1}{2},\infty}}$, following the same argument as in the proof of Theorem 1.1, it is also sufficient to show that

$$\big|\big\{x \in \mathbb{R}^n : \big|T_{\pi b}(f_1, f_2)(x)\big| > \lambda\big\}\big| \leq C\big(C_0 + C_1' + C_2\big)^{1/2} \lambda^{-1/2}. \tag{3.1}$$

The same decomposition for $f_j \in H^1(\mathbb{R}^n)$ $(j = 1, 2)$ as in Theorem 1.1 yields

$$h_j = \sum_{S_{j,l_j}} \sum_{Q_{j,k_j} \subset S_{j,l_j}} \lambda_{Q_{j,k_j}} a_{Q_{j,k_j}}, \qquad f_j(x) = g_j(x) + h_j(x), \tag{3.2}$$

where g_j and h_j enjoy the same properties as in Theorem 1.1.

With abuse of notations, we may still set

$$\begin{aligned} E_1 &= \big\{x \in \mathbb{R}^n : \big|T_{\pi b}(g_1, g_2)(x)\big| > \lambda/4\big\}; \\ E_2 &= \big\{x \in \mathbb{R}^n \backslash S^* : \big|T_{\pi b}(h_1, g_2)(x)\big| > \lambda/4\big\}; \\ E_3 &= \big\{x \in \mathbb{R}^n \backslash S^* : \big|T_{\pi b}(g_1, h_2)(x)\big| > \lambda/4\big\}; \\ E_4 &= \big\{x \in \mathbb{R}^n \backslash S^* : \big|T_{\pi b}(h_1, h_2)(x)\big| > \lambda/4\big\}. \end{aligned}$$

Then (2.2) still gives

$$\left|S^*\right| \le C(\gamma\lambda)^{-1/2}.$$

Note that $C_1' = \|T_{\pi b}\|_{L^2\times L^2\to L^{1,\infty}}$, repeating the arguments as in the estimates of (2.3), we may obtain

$$|E_1| \le CC_1'\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}.$$

Therefore,

$$\left|\left\{x\in\mathbb{R}^n : \left|T_{\pi b}(\vec{f})(x)\right| > \lambda\right\}\right| \le \sum_{s=1}^{4}|E_s| + C\left|S^*\right| \le \sum_{s=2}^{4}|E_s| + C(\gamma\lambda)^{-1/2} + CC_1\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}.$$

Thus, to show Theorem 1.2 is true, we only have to show that

$$|E_s| \le C\left(C_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}} + C_2^{-\frac{1}{2}}\lambda^{-\frac{1}{2}}\right), \quad \text{for } s = 2,3,4. \tag{3.3}$$

In fact, let $\gamma = (C_0 + C_1' + C_2)^{-\frac{1}{2}}$, it is easy to check that the inequality (3.1) is true.

• *Estimate for* $|E_2|$. Employing the linearity of $T_{\pi b}$ and the atomic decomposition of h_1, we may get

$$\begin{aligned}
&T_{\pi b}(h_1,g_2)(x)\\
&\quad= \int_{(\mathbb{R}^n)^m}\prod_{j=1}^{2}\bigl(b_j(x)-b_j(y_j)\bigr)K(x,y_1,y_2)h_1(y_1)g_2(y_2)\,dy_1\,dy_2\\
&\quad= \sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\lambda_{Q_{1,k_1}}\bigl(b_1(x)b_2(x)T(a_{Q_{1,k_1}},g_2)(x) - b_2(x)T(b_1a_{Q_{1,k_1}},g_2)(x)\\
&\qquad - b_1(x)T(a_{Q_{1,k_1}},b_2g_2)(x) + T(b_1a_{Q_{1,k_1}},b_2g_2)(x)\bigr)\\
&\quad= \sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\lambda_{Q_{1,k_1}}\bigl(b_1(x)-b_{1,Q_{1,k_1}}\bigr)\bigl(b_2(x)-b_{2,Q_{1,k_1}}\bigr)T(a_{Q_{1,k_1}},g_2)(x)\\
&\qquad - \sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\lambda_{Q_{1,k_1}}\bigl(b_2(x)-b_{2,Q_{1,k_1}}\bigr)T\bigl((b_1-b_{1,Q_{1,k_1}})a_{Q_{1,k_1}},g_2\bigr)(x)\\
&\qquad - \sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\lambda_{Q_{1,k_1}}\bigl(b_1(x)-b_{1,Q_{1,k_1}}\bigr)T\bigl(a_{Q_{1,k_1}},(b_2-b_{2,Q_{1,k_1}})g_2\bigr)(x)\\
&\qquad + \sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\lambda_{Q_{1,k_1}}T\bigl((b_1-b_{1,Q_{1,k_1}})a_{Q_{1,k_1}},(b_2-b_{2,Q_{1,k_1}})g_2\bigr)(x)\\
&\quad=: I_{2,1}(x) + I_{2,2}(x) + I_{2,3}(x) + I_{2,4}(x).
\end{aligned}$$

Thus

$$\begin{aligned}
|E_2| &= \left|\left\{x\in\mathbb{R}^n : \left|T_{\pi b}(g_1,h_2)(x)\right| > \lambda/4\right\}\right|\\
&\le \left|\left\{x\in\mathbb{R}^n : \left|I_{2,1}(x)\right| > \lambda/16\right\}\right| + \left|\left\{x\in\mathbb{R}^n : \left|I_{2,2}(x)\right| > \lambda/16\right\}\right|
\end{aligned}$$

$$
\begin{aligned}
&+ \left|\left\{x \in \mathbb{R}^n : \left|I_{2,3}(x)\right| > \lambda/16\right\}\right| + \left|\left\{x \in \mathbb{R}^n : \left|I_{2,4}(x)\right| > \lambda/16\right\}\right| \\
&=: |E_{2,1}| + |E_{2,2}| + |E_{2,3}| + |E_{2,4}|.
\end{aligned}
$$

By the definition of $I_{2,1}$ and the moment condition of H^1-atoms, we have

$$
\begin{aligned}
I_{2,1}(x) = {} & \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} \lambda_{Q_{1,k_1}} \big(b_1(x) - b_{1,Q_{1,k_1}}\big)\big(b_2(x) - b_{2,Q_{1,k_1}}\big) \\
& \times \iint_{(\mathbb{R}^n)^2} \big(K(x, y_1, y_2) - K(x, c_{1,k_1}, y_2)\big) a_{Q_{1,k_1}}(y_1) g_2(y_2)\, dy_1\, dy_2.
\end{aligned}
$$

Putting the above identity into the definition of $|E_{2,1}|$ and noting that $\|g_2\|_{L^\infty(\mathbb{R}^n)} \leq (\gamma\lambda)^{1/2}$, $\mathbb{R}^n \backslash S^* \subset \bigcup_{i=1}^\infty \mathscr{R}^i_{1,k_1}$, together with the Chebyshev inequality and condition (1.3), we have

$$
\begin{aligned}
|E_{2,1}| \leq {} & \frac{16}{\lambda} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \int_{(S^*)^c} \iint_{(\mathbb{R}^n)^2} \big|b_1(x) - b_{1,Q_{1,k_1}}\big| \big|b_2(x) - b_{2,Q_{1,k_1}}\big| \\
& \times \big|K(x, y_1, y_2) - K(x, c_{1,k_1}, y_2)\big| \big|a_{Q_{1,k_1}}(y_1)\big| \big|g_2(y_2)\big|\, dy_1\, dy_2\, dx \\
\leq {} & CC_0 \lambda^{1/2} \gamma^{-1/2} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \sum_{i=1}^\infty \int_{\mathscr{R}^i_{1,k_1}} \iint_{(\mathbb{R}^n)^2} \big|b_1(x) - b_{1,Q_{1,k_1}}\big| \\
& \times \big|b_2(x) - b_{2,Q_{1,k_1}}\big| \frac{|a_{1,k_1}(y_1)|}{(|x - y_1| + |x - y_2|)^{2n}} \omega \\
& \times \left(\frac{|y_1 - c_{1,k_1}|}{|x - y_1| + |x - y_2|}\right) dy_1\, dy_2\, dx.
\end{aligned} \tag{3.4}
$$

By (2.6) and the non-decreasing property of ω, we have

$$
\begin{aligned}
|E_{2,1}| \leq {} & CC_0 \gamma^{1/2} \lambda^{-1/2} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \sum_{i=1}^\infty \int_{\mathscr{R}^i_{1,k_1}} \iint_{(\mathbb{R}^n)^2} \big|b_1(x) - b_{1,Q_{1,k_1}}\big| \\
& \times \big|b_2(x) - b_{2,Q_{1,k_1}}\big| \frac{|a_{1,k_1}(y_1)|}{(|x - y_1| + |x - y_2|)^{2n}} \omega\big(2^{-i}\big)\, dy_1\, dy_2\, dx \\
\leq {} & CC_0 \gamma^{1/2} \lambda^{-1/2} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \sum_{i=1}^\infty \int_{(S^*)^c \cap \mathscr{R}^i_{1,k_1}} \iint_{(\mathbb{R}^n)^2} \big|b_1(x) - b_{1,Q_{1,k_1}}\big| \\
& \times \big|b_2(x) - b_{2,Q_{1,k_1}}\big| \frac{|a_{1,k_1}(y_1)|}{|x - y_1|^n} \omega\big(2^{-i}\big)\, dy_1\, dx \\
\leq {} & CC_0 \gamma^{1/2} \lambda^{-1/2} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \sum_{i=1}^\infty \int_{(S^*)^c \cap \mathscr{R}^i_{1,k_1}} \int_{\mathbb{R}^n} \big|b_1(x) - b_{1,Q_{1,k_1}}\big| \\
& \times \big|b_2(x) - b_{2,Q_{1,k_1}}\big| \frac{|a_{1,k_1}(y_1)|}{|2^{i+2}\sqrt{n}Q_{1,k_1}|} \omega\big(2^{-i}\big)\, dy_1\, dx \\
\leq {} & CC_0 \gamma^{1/2} \lambda^{-1/2} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \sum_{i=1}^\infty \omega\big(2^{-i}\big) \frac{1}{|2^{i+2}\sqrt{n}Q_{1,k_1}|} \\
& \times \int_{\mathscr{R}^i_{1,k_1}} \big|b_1(x) - b_{1,Q_{1,k_1}}\big| \big|b_2(x) - b_{2,Q_{1,k_1}}\big|\, dx.
\end{aligned} \tag{3.5}
$$

By the Hölder inequality, one obtains

$$\begin{aligned}
&\frac{1}{|2^{i+2}\sqrt{n}Q_{1,k_1}|}\int_{\mathscr{R}_{1,k_1}^i}|b_1(x)-b_{1,Q_{1,k_1}}||b_2(x)-b_{2,Q_{1,k_1}}|\,dx \\
&\quad\le\left(\frac{1}{|2^{i+2}\sqrt{n}Q_{1,k_1}|}\int_{2^{i+2}\sqrt{n}Q_{1,k_1}}|b_1(x)-b_{1,Q_{1,k_1}}|^2\,dx\right)^{1/2} \\
&\qquad\times\left(\frac{1}{|2^{i+2}\sqrt{n}Q_{1,k_1}|}\int_{2^{i+2}\sqrt{n}Q_{1,k_1}}|b_2(x)-b_{2,Q_{1,k_1}}|^2\,dx\right)^{1/2} \\
&\quad\le Ci\|b\|_*. \qquad (3.6)
\end{aligned}$$

Combining (3.5) and (3.6), we get

$$|E_{2,1}|\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega(2^{-i})i\le CC_0\gamma^{1/2}\lambda^{-1/2}.$$

Now we begin to estimate $|E_{2,2}|$.

Similarly to our dealing with $|E_{2,1}|$, and together with the size condition of H^1-atoms, it follows that

$$\begin{aligned}
|E_{2,2}|&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}_{1,k_1}^i}\iint_{(\mathbb{R}^n)^2}|b_1(y_1)-b_{1,Q_{1,k_1}}| \\
&\qquad\times|b_2(x)-b_{2,Q_{1,k_1}}|\frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)dy_1\,dy_2\,dx \\
&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}_{1,k_1}^i}\int_{\mathbb{R}^n}|b_1(y_1)-b_{1,Q_{1,k_1}}| \\
&\qquad\times|b_2(x)-b_{2,Q_{1,k_1}}|\frac{1}{(|x-y_1|)^n|Q_{1,k_1}|}\omega(2^{-i})\,dy_1\,dx \\
&\le CC_0\gamma^{1/2}\lambda^{-1/2}\|b_1\|_*\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega(2^{-i})\frac{1}{(|2^{i+2}Q_{1,k_1}|)^n} \\
&\qquad\times\int_{(S^*)^c\cap\mathscr{R}_{1,k_1}^i}|b_2(x)-b_{2,Q_{1,k_1}}|\,dx \\
&\le CC_0\gamma^{1/2}\lambda^{-1/2}\|b_1\|_*\|b_2\|_*\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega(2^{-i})i \\
&\le CC_0\gamma^{1/2}\lambda^{-1/2}.
\end{aligned}$$

The estimate for $|E_{2,3}|$ is more complicated, and we need to split the domain of the variable y_2. First, similar to our dealing with $|E_{2,1}|$ in (3.4) and (3.5), we may get

$$\begin{aligned}
|E_{2,3}|&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}_{1,k_1}^i}\iint_{(\mathbb{R}^n)^2}|b_1(x)-b_{1,Q_{1,k_1}}| \\
&\qquad\times|b_2(y_2)-b_{2,Q_{1,k_1}}|\frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)dy_1\,dy_2\,dx.
\end{aligned}$$

Denote $\mathscr{R}^h_{1,k_1} = (2^{h+2}\sqrt{n}Q_{1,k_1})\backslash(2^{h+1}\sqrt{n}Q_{1,k_1})$ and recall that $Q^*_{1,k_1} = 4\sqrt{n}Q_{1,k_1}$, then

$$y_2 \in \mathbb{R}^n \subset \left(\bigcup_{h=1}^{\infty}\mathscr{R}^h_{1,k_1}\right) \cup Q^*_{1,k_1}.$$

Thus $|E_{2,3}|$ can be controlled by

$$\begin{aligned}
&CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\int_{\bigcup_{i=1}^{\infty}\mathscr{R}^h_{1,k_1}}\int_{\mathbb{R}^n}\left|b_1(x)-b_{1,Q_{1,k_1}}\right| \\
&\quad\times\left|b_2(y_2)-b_{2,Q_{1,k_1}}\right|\frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)dy_1\,dy_2\,dx \\
&\quad+CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\int_{Q^*_{1,k_1}}\int_{\mathbb{R}^n}\left|b_1(x)-b_{1,Q_{1,k_1}}\right| \\
&\quad\times\left|b_2(y_2)-b_{2,Q_{1,k_1}}\right|\frac{|a_{Q_{1,k_1}}(y_1)||}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)dy_1\,dy_2\,dx \\
&\quad=:\left|E^1_{2,3}\right|+\left|E^2_{2,3}\right|.
\end{aligned}$$

For any $h\in\mathbb{N}$, if $y_2\in\mathscr{R}^h_{1,k_1}$, note that $y_1\in Q_{1,k_1}$, then

$$|x-y_1|+|x-y_2|\geq|y_1-y_2|\sim|y_2-c_{1,k_1}|\sim l_{2^{h+2}Q_{1,k_1}}.$$

On the other hand, for any $i\in\mathbb{N}$, if $x\in\mathscr{R}^i_{1,k_1}$ and $y_1\in Q_{1,k_1}$, then

$$|x-y_1|+|x-y_2|\geq|x-y_1|\sim l_{2^{i+2}Q_{1,k_1}}. \tag{3.7}$$

By the geometric properties of y_1, y_2, x above, we may obtain

$$\begin{aligned}
&\left|E^1_{2,3}\right| \\
&\quad\leq CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\int_{\mathscr{R}^h_{1,k_1}}\int_{\mathbb{R}^n}\left|b_1(x)-b_{1,Q_{1,k_1}}\right| \\
&\qquad\times\left|b_2(y_2)-b_{2,Q_{1,k_1}}\right|\frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)dy_1\,dy_2\,dx \\
&\quad\leq CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\int_{\mathscr{R}^h_{1,k_1}}\int_{\mathbb{R}^n}\left|b_1(x)-b_{1,Q_{1,k_1}}\right| \\
&\qquad\times\left|b_2(y_2)-b_{2,Q_{1,k_1}}\right|\frac{|a_{Q_{1,k_1}}(y_1)|}{|2^{i+2}Q_{1,k_1}||2^{h+2}Q_{1,k_1}|}\omega\left(2^{-i}\right)^{1/2}\omega\left(2^{-h}\right)^{1/2}dy_1\,dy_2\,dx.
\end{aligned}\tag{3.8}$$

It is easy to see that

$$\sum_{h=1}^{\infty}\omega\left(2^{-h}\right)^{1/2}\int_{\mathscr{R}^h_{1,k_1}}\frac{|b_2(y_2)-b_{2,Q_{1,k_1}}|}{|2^{h+2}Q_{1,k_1}|}dy_2\leq C\sum_{h=1}^{\infty}\omega\left(2^{-h}\right)^{1/2}h\|b_2\|_*\leq C. \tag{3.9}$$

Since $a(y_1) \in L^1(\mathbb{R}^n)$, putting the above estimate into (3.8), we have

$$\begin{aligned}\left|E_{2,3}^1\right| &\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega\left(2^{-i}\right)^{1/2}\int_{2^{i+2}Q_{1,k_1}}\frac{|b_1(x)-b_{1,Q_{1,k_1}}|}{|2^{i+2}Q_{1,k_1}|}\,dx\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega\left(2^{-i}\right)^{1/2}i\|b_1\|_*\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}.\end{aligned}$$

If $y_2 \in Q_{1,k_1}^*$, note that $x \in (8\sqrt{n}Q_{1,k_1})^c$, then

$$|x-y_1|+|x-y_2| \ge |x-y_2| \ge Cl_{Q_{1,k_1}}.$$

By the definition of $|E_{2,3}^2|$ and (3.7), we have

$$\begin{aligned}\left|E_{2,3}^2\right| &\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}_{1,k_1}^i}\int_{Q_{1,k_1}^*}\int_{\mathbb{R}^n}\left|b_1(x)-b_{1,Q_{1,k_1}}\right|\\&\quad\times\left|b_2(y_2)-b_{2,Q_{1,k_1}}\right|\frac{|a_{Q_{1,k_1}}(y_1)|}{|2^{i+2}Q_{1,k_1}||Q_{1,k_1}^*|}\omega\left(2^{-i}\right)dy_1\,dy_2\,dx\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{2^{i+2}Q_{1,k_1}}\int_{\mathbb{R}^n}\left|b_1(x)-b_{1,Q_{1,k_1}}\right|\\&\quad\times\frac{|a_{Q_{1,k_1}}(y_1)|}{|2^{i+2}Q_{1,k_1}|}\omega\left(2^{-i}\right)dy_1\,dx\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega\left(2^{-i}\right)\frac{1}{|2^{i+2}Q_{1,k_1}|}\\&\quad\times\int_{2^{i+2}Q_{1,k_1}}\left|b_1(x)-b_{1,Q_{1,k_1}}\right|dx\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\omega\left(2^{-i}\right)i\|b_1\|_*\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}.\end{aligned}$$

Hence, we obtain

$$|E_{2,3}| \le \left|E_{2,3}^1\right| + \left|E_{2,3}^2\right| \le CC_0\gamma^{1/2}\lambda^{-1/2}.$$

Now we begin to consider $|E_{2,4}|$. Similarly,

$$\begin{aligned}|E_{2,4}| &\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}_{1,k_1}^i}\iint_{(\mathbb{R}^n)^2}\left|b_1(y_1)-b_{1,Q_{1,k_1}}\right|\\&\quad\times\left|b_2(y_2)-b_{2,Q_{1,k_1}}\right|\frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)dy_1\,dy_2\,dx.\end{aligned}$$

Repeating the same steps as in the estimate of $|E_{2,3}|$, we have

$$\begin{aligned}|E_{2,4}| &\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\int_{\bigcup_{i=1}^{\infty}\mathscr{R}^h_{1,k_1}}\int_{\mathbb{R}^n}\bigl|b_1(y_1)-b_{1,Q_{1,k_1}}\bigr|\\&\quad\times\bigl|b_2(y_2)-b_{2,Q_{1,k_1}}\bigr|\frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\omega\biggl(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\biggr)\,dy_1\,dy_2\,dx\\&\quad+CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\int_{Q^*_{1,k_1}}\int_{\mathbb{R}^n}\bigl|b_1(y_1)-b_{1,Q_{1,k_1}}\bigr|\\&\quad\times\bigl|b_2(y_2)-b_{2,Q_{1,k_1}}\bigr|\frac{|a_{Q_{1,k_1}}(y_1)|}{(|x-y_1|+|x-y_2|)^{2n}}\omega\biggl(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\biggr)\,dy_1\,dy_2\,dx\\&=:\bigl|E^1_{2,4}\bigr|+\bigl|E^2_{2,4}\bigr|.\end{aligned}$$

By the definition of $|E^1_{2,4}|$, one may obtain

$$\begin{aligned}\bigl|E^1_{2,4}\bigr| &\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\int_{\mathscr{R}^h_{1,k_1}}\int_{\mathbb{R}^n}\bigl|b_1(x)-b_{1,Q_{1,k_1}}\bigr|\\&\quad\times\bigl|b_2(y_2)-b_{2,Q_{1,k_1}}\bigr|\frac{|a_{Q_{1,k_1}}(y_1)|}{|x-y_1|^n|2^{h+2}Q_{1,k_1}|}\omega\biggl(\frac{y_1-c_{1,k_1}}{|x-y_1|}\biggr)^{1/2}\omega\bigl(2^{-h}\bigr)^{1/2}\,dy_1\,dy_2\,dx.\end{aligned}$$

By (3.9), and taking the integral for x first, we have

$$\begin{aligned}\bigl|E^1_{2,4}\bigr| &\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\sum_{i=1}^{\infty}\int_{\mathscr{R}^i_{1,k_1}}\int_{Q_{1,k_1}}\frac{|b_1(y_1)-b_{1,Q_{1,k_1}}|}{|Q_{1,k_1}||x-y_1|^n}\\&\quad\times\omega\biggl(\frac{y_1-c_{1,k_1}}{|x-y_1|}\biggr)^{1/2}\,dy_1\,dx\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\int_{Q_{1,k_1}}\frac{|b_1(y_1)-b_{1,Q_{1,k_1}}|}{|Q_{1,k_1}|}\,dy_1\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}|\lambda_{Q_{1,k_1}}|\|b_1\|_*\\&\le CC_0\gamma^{1/2}\lambda^{-1/2}.\end{aligned}$$

The estimate for $|E^2_{2,4}|$ is quite similar to $|E^2_{2,3}|$, we may get $|E^2_{2,4}|\le CC_0\gamma^{1/2}\lambda^{-1/2}$.

- *Estimate for* $|E_3|$. Since $|E_3|$ is a symmetrical case of $|E_2|$, we can obtain

$$|E_3|\le CC_0\gamma^{1/2}\lambda^{-1/2}.$$

- *Estimate for* $|E_4|$.

$$\begin{aligned}T_{\Pi b}(h_1,h_2)&=\bigl[b_1,[b_2,T]_2,\bigr]_1(h_1,h_2)\\&=\int_{(\mathbb{R}^n)^m}\prod_{j=1}^{2}\bigl(b_j(x)-b_j(y_j)\bigr)K(x,y_1,y_2)h_1(y_1)h_2(y_2)\,dy_1\,dy_2\end{aligned}$$

$$\begin{aligned}
&= \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \lambda_{Q_{1,k_1}} \lambda_{Q_{2,k_2}} \bigl(b_1(x) - b_{1,Q_{1,k_1}}\bigr)\bigl(b_2(x) - b_{2,Q_{1,k_1}}\bigr) \\
&\quad \times T(a_{Q_{1,k_1}}, a_{Q_{2,k_2}})(x) \\
&\quad - \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \lambda_{Q_{1,k_1}} \lambda_{Q_{2,k_2}} \bigl(b_2(x) - b_{2,Q_{1,k_1}}\bigr) \\
&\quad \times T\bigl((b_1 - b_{1,Q_{1,k_1}})a_{Q_{1,k_1}}, a_{Q_{2,k_2}}\bigr)(x) \\
&\quad - \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \lambda_{Q_{1,k_1}} \lambda_{Q_{2,k_2}} \bigl(b_1(x) - b_{1,Q_{1,k_1}}\bigr) \\
&\quad \times T\bigl(a_{Q_{1,k_1}}, (b_2 - b_{2,Q_{1,k_1}})a_{Q_{2,k_2}}\bigr)(x) \\
&\quad + \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \lambda_{Q_{1,k_1}} \lambda_{Q_{2,k_2}} \\
&\quad \times T\bigl((b_1 - b_{1,Q_{1,k_1}})a_{Q_{1,k_1}}, (b_2 - b_{2,Q_{1,k_1}})a_{Q_{2,k_2}}\bigr)(x) \\
&=: I_{4,1}(x) + I_{4,2}(x) + I_{4,3}(x) + I_{4,4}(x).
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
|E_4| &= \bigl|\bigl\{x \in \mathbb{R}^n/S^* : \bigl|T_{\pi b}(h_1, h_2)(x)\bigr| > \lambda/4\bigr\}\bigr| \\
&\le \bigl|\bigl\{x \in \mathbb{R}^n/S^* : \bigl|I_{4,1}(x)\bigr| > \lambda/16\bigr\}\bigr| + \bigl|\bigl\{x \in \mathbb{R}^n/S^* : \bigl|I_{4,2}(x)\bigr| > \lambda/16\bigr\}\bigr| \\
&\quad + \bigl|\bigl\{x \in \mathbb{R}^n/S^* : \bigl|I_{4,3}(x)\bigr| > \lambda/16\bigr\}\bigr| + \bigl|\bigl\{x \in \mathbb{R}^n/S^* : \bigl|I_{4,4}(x)\bigr| > \lambda/16\bigr\}\bigr| \\
&=: |E_{4,1}| + |E_{4,2}| + |E_{4,3}| + |E_{4,4}|.
\end{aligned}$$

Now we begin considering $|E_{4,1}|$. By the definition of $I_{4,1}(x)$, we can write

$$\begin{aligned}
\bigl|I_{4,1}(x)\bigr| &\le \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} |\lambda_{Q_{1,k_1}}||\lambda_{Q_{2,k_2}}| \biggl| \iint_{(\mathbb{R}^n)^2} \bigl(b_1(x) - b_{1,Q_{1,k_1}}\bigr) \\
&\quad \times \bigl(b_2(x) - b_{2,Q_{1,k_1}}\bigr) K(x, y_1, y_2) a_{Q_{1,k_1}}(y_1) a_{Q_{2,k_2}}(y_2)\, dy_1\, dy_2 \biggr|.
\end{aligned}$$

Fix for a moment k_1, k_2 and assume, without loss of generality, that $l(Q_{1,k_1}) \le l(Q_{2,k_2})$. By the moment condition of H^1-atoms and the regularity condition (1.3) of the kernel K, we have

$$\begin{aligned}
&\biggl| \int_{\mathbb{R}^n} K(x, y_1, y_2) a_{1,k_1}(y_1)\, dy_1 \biggr| \\
&\quad = \biggl| \int_{\mathbb{R}^n} \bigl(K(x, y_1, y_2) - K(x, c_{1,k_1}, y_2)\bigr) a_{1,k_1}(y_1)\, dy_1 \biggr| \\
&\quad \le \biggl| \int_{\mathbb{R}^n} \frac{C_0}{(|x - y_1| + |x - y_2|)^{2n}} \omega\biggl(\frac{|y_1 - c_{1,k_1}|}{|x - y_1| + |x - y_2|} \biggr) a_{Q_{1,k_1}}(y_1)\, dy_1 \biggr|.
\end{aligned}$$

Recalling the definition of $\mathscr{R}^i_{1,k_1}$, $\mathscr{R}^h_{2,k_2}$, and note that $y_1 \in Q_{1,k_1}$, $y_2 \in Q_{2,k_2}$, it is obvious that, for any fixed i, h, k_1, k_2, if $x \in (S^*)^c \cap \mathscr{R}^i_{1,k_1} \cap \mathscr{R}^h_{2,k_2}$, then we have

$$|x - y_1| \sim 2^i l_{Q_{1,k_1}}, \qquad |x - y_2| \sim 2^h l_{Q_{2,k_2}}.$$

This and the non-decreasing property of ω give

$$\frac{\omega(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|})^{\frac{1}{2}}}{(|x-y_1|+|x-y_2|)^n} \leq \frac{\omega(\frac{l_{Q_{1,k_1}}}{|x-y_1|+|x-y_2|})^{\frac{1}{2}}}{(|x-y_1|+|x-y_2|)^n} \lesssim \prod_{i=1}^{2} \frac{\omega(\frac{l_{Q_{i,k_i}}}{|x-y_i|})^{\frac{1}{4}}}{|x-y_i|^{\frac{n}{2}}} \lesssim \frac{\omega(2^{-i})^{\frac{1}{4}}\omega(2^{-h})^{\frac{1}{4}}}{(2^i l_{Q_{1,k_1}} 2^h l_{Q_{2,k_2}})^{\frac{n}{2}}}.$$

By (2.11), the Chebychev inequality and the estimate above, we control $|E_{4,1}|$ by

$$\begin{aligned}
&\frac{CC_0}{\lambda} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \sum_{i=1}^{\infty}\sum_{h=1}^{\infty} |\lambda_{Q_{1,k_1}}||\lambda_{Q_{2,k_2}}| \int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}\cap\mathscr{R}^h_{2,k_2}} \\
&\quad\times \iint_{(\mathbb{R}^n)^2} |b_1(x)-b_{1,Q_{1,k_1}}||b_2(x)-b_{2,Q_{1,k_1}}| \frac{|a_{Q_{1,k_1}}(y_1)||a_{Q_{2,k_2}}(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}} \\
&\quad\times \omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right) dy_1\, dy_2\, dx \\
&\leq \frac{CC_0}{\lambda} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1}\subset S_{1,l_1}} \sum_{S_{2,l_2}} \sum_{Q_{2,k_2}\subset S_{2,l_2}} \sum_{i=1}^{\infty}\sum_{h=1}^{\infty} \omega(2^{-i})^{\frac{1}{4}}\omega(2^{-h})^{\frac{1}{4}} |\lambda_{Q_{1,k_1}}||\lambda_{Q_{2,k_2}}| \\
&\quad\times \int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}\cap\mathscr{R}^h_{2,k_2}} \frac{|b_1(x)-b_{1,Q_{1,k_1}}|}{(2^i l_{Q_{1,k_1}} 2^h l_{Q_{2,k_2}})^{\frac{n}{2}}} \left(\iint_{(\mathbb{R}^n)^2} |b_2(x)-b_{2,Q_{1,k_1}}|\right. \\
&\quad\left.\times \frac{|a_{Q_{1,k_1}}(y_1)||a_{Q_{2,k_2}}(y_2)|}{(|x-y_1|+|x-y_2|)^n} \omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}} dy_1\, dy_2\right) dx.
\end{aligned} \tag{3.10}$$

Let us first consider the inside integrals, by the Hölder inequality, we may have

$$\begin{aligned}
&\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}\cap\mathscr{R}^h_{2,k_2}} \frac{|b_1(x)-b_{1,Q_{1,k_1}}|}{(2^i l_{Q_{1,k_1}} 2^h l_{Q_{2,k_2}})^{\frac{n}{2}}} \left(\iint_{(\mathbb{R}^n)^2} |b_2(x)-b_{2,Q_{1,k_1}}|\right. \\
&\quad\left.\times \frac{|a_{Q_{1,k_1}}(y_1)||a_{Q_{2,k_2}}(y_2)|}{(|x-y_1|+|x-y_2|)^n} \omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}} dy_1\, dy_2\right) dx \\
&\leq \left(\frac{1}{(2^h l_{Q_{2,k_2}})^n} \int_{\mathscr{R}^h_{2,k_2}} |b_1(x)-b_{1,Q_{1,k_1}}|^2\, dx\right)^{\frac{1}{2}} \\
&\quad\times \left(\frac{1}{(2^i l_{Q_{1,k_1}})^n} \int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}} \left|\iint_{(\mathbb{R}^n)^2} |b_2(x)-b_{2,Q_{1,k_1}}|\right.\right. \\
&\quad\left.\left.\times \frac{|a_{Q_{1,k_1}}(y_1)||a_{Q_{2,k_2}}(y_2)|}{(|x-y_1|+|x-y_2|)^n} \omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}} dy_1\, dy_2\right|^2 dx\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.11}$$

Note that $a_{2,k_2}(y_2) \in L^1(\mathbb{R}^n)$, a similar argument to (2.15) yields

$$\begin{aligned}
(3.11) &\leq h^{\frac{1}{2}} \|b_2\|_*^{\frac{1}{2}} \left[\frac{1}{(2^i l_{Q_{1,k_1}})^n} \int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}} \left|\int_{\mathbb{R}^n} |b_2(x)-b_{2,Q_{1,k_1}}|\right.\right. \\
&\quad\left.\left.\times \sup_{y_1,y_2\in S}\left(\frac{1}{(|x-y_1|+|x-y_2|)^n} \omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}}\right) |a_{Q_{1,k_1}}(y_1)|\, dy_1\right|^2 dx\right]^{\frac{1}{2}}.
\end{aligned}$$

Note that the integrals in the above inequality are independent of S_{2,l_2} and Q_{2,k_2} and ω is doubling, similar to what we have done with (2.14), for fixed $x \in (S^*)^c$ and any $y_1, y_2 \in S$, we have

$$\begin{aligned}
&\sup_{y_1,y_2\in S}\left(\frac{1}{(|x-y_1|+|x-y_2|)^n}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}}\right)\\
&\quad\approx\frac{1}{(|x-y_1|+|x-y_2|)^n}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}}. \qquad (3.12)
\end{aligned}$$

Recalling (I) in Theorem 1.1 and putting the inequality above into (3.10), we may get

$$\begin{aligned}
|E_{4,1}| &\le \frac{CC_0}{\lambda}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\omega(2^{-i})^{\frac{1}{4}}\omega(2^{-h})^{\frac{1}{4}}h^{\frac{1}{2}}|\lambda_{Q_{1,k_1}}|\left(\frac{1}{(2^il_{Q_{1,k_1}})^n}\right.\\
&\quad\times\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\left|\int_{\mathbb{R}^n}|b_2(x)-b_{2,Q_{1,k_1}}|\left(\sum_{S_{2,l_2}}\sum_{Q_{2,k_2}\subset S_{2,l_2}}|\lambda_{Q_{2,k_2}}|\right)\right.\\
&\quad\left.\left.\times\sup_{y_1,y_2\in S}\left(\frac{1}{(|x-y_1|+|x-y_2|)^n}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}}\right)|a_{Q_{1,k_1}}(y_1)|\,dy_1\right|^2dx\right)^{\frac{1}{2}}\\
&\le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\omega(2^{-i})^{\frac{1}{4}}\omega(2^{-h})^{\frac{1}{4}}h^{\frac{1}{2}}|\lambda_{Q_{1,k_1}}|\left(\frac{1}{(2^il_{Q_{1,k_1}})^n}\right.\\
&\quad\times\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\left|\int_{\mathbb{R}^n}|b_2(x)-b_{2,Q_{1,k_1}}|\left(\sum_{S_{2,l_2}}\int_{S_{2,l_2}}\frac{1}{(|x-y_1|+|x-y_2|)^n}\right.\right.\\
&\quad\left.\left.\left.\times\,\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}}dy_2\right)|a_{Q_{1,k_1}}(y_1)|\,dy_1\right|^2dx\right)^{\frac{1}{2}}\\
&\le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\omega(2^{-i})^{\frac{1}{4}}\omega(2^{-h})^{\frac{1}{4}}h^{\frac{1}{2}}|\lambda_{Q_{1,k_1}}|\\
&\quad\times\left(\frac{1}{(2^il_{Q_{1,k_1}})^n}\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}|b_2(x)-b_{2,Q_{1,k_1}}|^2\left(\int_{\mathbb{R}^n}|a_{Q_{1,k_1}}(y_1)|\,dy_1\right)^2dx\right)^{\frac{1}{2}}\\
&\le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}\sum_{i=1}^{\infty}\sum_{h=1}^{\infty}\omega(2^{-i})^{\frac{1}{4}}\omega(2^{-h})^{\frac{1}{4}}h^{\frac{1}{2}}i^{\frac{1}{2}}\\
&\le CC_0\gamma^{\frac{1}{2}}\lambda^{-\frac{1}{2}}.
\end{aligned}$$

Now we begin with the estimate for $|E_{4,2}|$.

Recalling the definition of $I_{4,2}(x)$, the moment condition of H^1-atoms and smoothness condition (1.3). Similar to the estimates in (3.10), we may obtain

$$\begin{aligned}
|E_{4,2}| &\le \frac{CC_0}{\lambda}\sum_{S_{1,l_1}}\sum_{Q_{1,k_1}\subset S_{1,l_1}}\sum_{S_{2,l_2}}\sum_{Q_{2,k_2}\subset S_{2,l_2}}\sum_{i=1}^{\infty}|\lambda_{Q_{1,k_1}}||\lambda_{Q_{2,k_2}}|\\
&\quad\times\int_{(S^*)^c\cap\mathscr{R}^i_{1,k_1}}\iint_{(\mathbb{R}^n)^2}|b_1(x)-b_{1,Q_{1,k_1}}||b_2(y_2)-b_{2,Q_{1,k_1}}|\frac{|a_{Q_{1,k_1}}(y_1)||a_{Q_{2,k_2}}(y_2)|}{|x-y_1|^n}\\
&\quad\times\frac{1}{(|x-y_1|+|x-y_2|)^n}\omega\left(\frac{|y_1-c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)dy_1\,dy_2\,dx. \qquad (3.13)
\end{aligned}$$

First, we consider the following summation.

$$\sum_{S_{2,l_2}} \sum_{Q_{2,k_2} \subset S_{2,l_2}} \int_{\mathbb{R}^n} \left|b_2(y_2) - b_{2,Q_{1,k_1}}\right| \frac{|\lambda_{Q_{2,k_2}}||a_{Q_{2,k_2}}(y_2)|}{(|x-y_1|+|x-y_2|)^n} \times \omega\left(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right) dy_2. \tag{3.14}$$

Property (I) in Theorem 1.1, inequality (3.12), and the size condition of H^1-atoms, that is, $\|a_{Q_{2,k_2}}\|_{L^\infty} \leq |Q_{2,k_2}|^{-1}$, together with the Hölder inequality, enable us to obtain

$$\begin{aligned}
(3.14) &\leq \sum_{S_{2,l_2}} \sum_{Q_{2,k_2} \subset S_{2,l_2}} |\lambda_{Q_{2,k_2}}| \left(\int_{\mathbb{R}^n} \left|b_2(y_2) - b_{2,Q_{1,k_1}}\right|^2 \left|a_{Q_{2,k_2}}(y_2)\right| dy_2\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^n} \frac{1}{(|x-y_1|+|x-y_2|)^{2n}} \omega\left(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^2 \left|a_{Q_{2,k_2}}(y_2)\right| dy_2\right)^{\frac{1}{2}} \\
&\leq \omega\left(2^{-i}\right) \sum_{S_{2,l_2}} \sum_{Q_{2,k_2} \subset S_{2,l_2}} |\lambda_{Q_{2,k_2}}| \|b_2\|_*^{\frac{1}{2}} \sup_{y_1, y_2 \in S} \left(\frac{1}{(|x-y_1|+|x-y_2|)^n} \right. \\
&\quad \left. \times \omega\left(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}}\right) \\
&\leq C(\gamma\lambda)^{\frac{1}{2}} \omega\left(2^{-i}\right)^{\frac{1}{2}} \sum_{S_{2,l_2}} \int_{S_{2,l_2}} \frac{1}{(|x-y_1|+|x-y_2|)^n} \omega\left(\frac{|y_1 - c_{1,k_1}|}{|x-y_1|+|x-y_2|}\right)^{\frac{1}{2}} dy_2 \\
&\leq C(\gamma\lambda)^{\frac{1}{2}} \omega\left(2^{-i}\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore, by (3.13) and noting that $a_{Q_{1,k_1}}(y_2) \in L^1(\mathbb{R}^n)$, we have

$$\begin{aligned}
|E_{4,2}| &\leq CC_0 \gamma^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} \sum_{i=1}^{\infty} \omega\left(2^{-i}\right)^{\frac{1}{2}} |\lambda_{Q_{1,k_1}}| \int_{(S^*)^c \cap \mathscr{R}_{1,k_1}^i} \int_{\mathbb{R}^n} \frac{1}{|x-y_1|^2} \\
&\quad \times \left|b_1(x) - b_{1,Q_{1,k_1}}\right| \left|a_{Q_{1,k_1}}(y_1)\right| dy_1\, dx \\
&\leq CC_0 \|b_1\|_* \gamma^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \sum_{S_{1,l_1}} \sum_{Q_{1,k_1} \subset S_{1,l_1}} |\lambda_{Q_{1,k_1}}| \sum_{i=1}^{\infty} \omega\left(2^{-i}\right)^{\frac{1}{2}} i^{\frac{1}{2}} \leq CC_0 \gamma^{\frac{1}{2}} \lambda^{-\frac{1}{2}}.
\end{aligned}$$

Since $|E_{4,3}|$ is a symmetrical case of $|E_{4,2}|$ we may also obtain

$$|E_{4,3}| \leq CC_0 \gamma^{\frac{1}{2}} \lambda^{-\frac{1}{2}}.$$

A similar argument still works as in (2.9), we may have

$$|E_{4,4}| \leq CC_2^{\frac{1}{2}} \lambda^{-\frac{1}{2}}.$$

This completes the estimate for $|E_4|$. Thus, we have proved inequality (3.3) and the proof of Theorem 1.2 is finished. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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