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Monotonicity of the incomplete gamma function with applications

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Abstract
In the article, we discuss the monotonicity properties of the function $x \to (1-e^{-ax^p})^{1/p}/\int_0^x e^{-t^p}\,dt$ for $a, p > 0$ with $p \neq 1$ on $(0,\infty)$ and prove that the double inequality $\Gamma(1+1/p)(1-e^{-ax^p})^{1/p} < \int_0^x e^{-t^p}\,dt < \Gamma(1+1/p)(1-e^{-bx^p})^{1/p}$ holds for all $x > 0$ if and only if $a \leq \min\{1, \Gamma^{-p}(1+1/p)\}$ and $b \geq \max\{1, \Gamma^{-p}(1+1/p)\}$.

MSC: 33B20; 26D07; 26D15

Keywords: incomplete gamma function; gamma function; psi function

1 Introduction

Let $a > 0$ and $x > 0$. Then the classical gamma function $\Gamma(x)$, incomplete gamma function $\Gamma(a,x)$, and psi function $\psi(x)$ are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad \Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}\,dt,$$
$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. It is well known that the identity

$$\int_0^x e^{-t^p}\,dt = \frac{1}{p}\Gamma\left(\frac{1}{p}\right) - \frac{1}{p}\Gamma\left(\frac{1}{p}, x^p\right) \tag{1.1}$$

holds for all $x, p > 0$.

Recently, the bounds for the integral $\int_0^x e^{-t^p}\,dt$ have attracted the interest of many researchers. In particular, many remarkable inequalities for the integral $\int_0^x e^{-t^p}\,dt$ can be found in the literature [1–12]. Let

$$I_p(x) = \int_0^x e^{-t^p}\,dt. \tag{1.2}$$

Then we clearly see that $I_1(x) = 1 - e^{-t}$ and that $I_p(x)$ diverges if $p \leq 0$. The functions $I_3(x)$ and $I_4(x)$ can be used to study the heat transfer problem [13] and electrical discharge in gases [14], respectively.

Komatu [15] and Pollak [16] proved the double inequality

$$\Gamma\left(1+\frac{1}{p}\right)-\frac{e^{-x^2}}{\sqrt{x^2+\frac{4}{\pi}}+x}<I_2(x)<\Gamma\left(1+\frac{1}{p}\right)-\frac{e^{-x^2}}{\sqrt{x^2+2}+x}$$

for all $x>0$.

Gautschi [17] proved that the double inequality

$$\Gamma\left(1+\frac{1}{p}\right)-\frac{e^{-x^p}}{b}\left[\left(x^p+b\right)^{1/p}-x\right]<I_p(x)<\Gamma\left(1+\frac{1}{p}\right)-\frac{e^{-x^p}}{a}\left[\left(x^p+a\right)^{1/p}-x\right] \tag{1.3}$$

holds for all $x>0$ and $p>1$ if and only if $a\geq 2$ and $b\leq\Gamma^{p/(1-p)}(1+1/p)$.

An application of inequality (1.3) in radio propagation mode was given in [18].

Alzer [19] proved that $a=\min\{1,\Gamma^{-p}(1+1/p)\}$ and $b=\max\{1,\Gamma^{-p}(1+1/p)\}$ are the best possible parameters such that the double inequality

$$\Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p}<I_p(x)<\Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-bx^p}\right)^{1/p} \tag{1.4}$$

holds for all $x>0$ and $p>0$ with $p\neq 1$.

Motivated by the Alzer's inequality (1.4), in this paper, we discuss the monotonicity of the function

$$x\to R(a,p;x)=\frac{(1-e^{-ax^p})^{1/p}}{\int_0^x e^{-t^p}\,dt} \tag{1.5}$$

and provide an alternative proof of Alzer's inequality (1.4).

2 Lemmas

In order to prove our main results, we first introduce an auxiliary function. Let $-\infty\leq a<b\leq\infty$, f and g be differentiable on (a,b), and $g'\neq 0$ on (a,b). Then the function $H_{f,g}$ [20, 21] is defined by

$$H_{f,g}(x)=\frac{f'(x)}{g'(x)}g(x)-f(x). \tag{2.1}$$

Lemma 2.1 (See [21], Theorem 8) *Let $\infty\leq a<b\leq\infty$, f and g be differentiable on (a,b) with $f(a^+)=g(a^+)=0$ and $g'(x)>0$ on (a,b), and $H_{f,g}$ be defined by* (2.1). *Then the following statements are true*:

1. *If $H_{f,g}(b^-)>0$ and there exists $\lambda\in(a,b)$ such that $f'(x)/g'(x)$ is strictly decreasing on (a,λ) and strictly increasing on (λ,b), then there exists $\mu\in(a,b)$ such that $f(x)/g(x)$ is strictly decreasing on (a,μ) and strictly increasing on (μ,b)*;
2. *If $H_{f,g}(b^-)<0$ and there exists $\lambda^*\in(a,b)$ such that $f'(x)/g'(x)$ is strictly increasing on (a,λ^*) and strictly decreasing on (λ^*,b), then there exists $\mu^*\in(a,b)$ such that $f(x)/g(x)$ is strictly increasing on (a,μ^*) and strictly decreasing on (μ^*,b).*

Lemma 2.2 (See [22], Theorem 1.25) *Let $-\infty<a<b<\infty$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing* (*decreasing*) *on* (a, b), *then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

Lemma 2.3 *We have*

$$\Gamma^{1/x}(1+x) > \frac{1+x}{2}$$

for all $x \in (0, 1)$, *and the above inequality is reversed for all* $x \in (1, \infty)$.

Proof Let $x > 0$, $\gamma = 0.577215\cdots$ be the Euler-Mascheroni constant, and

$$f(x) = \log\Gamma(x+1) - x\log(1+x) + x\log 2. \tag{2.2}$$

Then simple computations lead to

$$f(0) = f(1) = 0, \tag{2.3}$$

$$f'(x) = \psi(1+x) - \log(1+x) - \frac{x}{1+x} + \log 2,$$

$$f'(1) = \psi(1) + \frac{1}{2} = -\gamma + \frac{1}{2} < 0, \tag{2.4}$$

$$f''(x) = \psi'(1+x) - \frac{1}{1+x} - \frac{1}{(1+x)^2}. \tag{2.5}$$

It follows from the identity

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{\theta}{30x^5} \quad (0 < \theta < 1),$$

given in [23], and (2.5) that

$$f''(x) < \frac{1}{x+1} + \frac{1}{2(x+1)^2} + \frac{1}{6(x+1)^3} - \frac{1}{x+1} - \frac{1}{(x+1)^2} = -\frac{3x+2}{6(x+1)^3} < 0 \tag{2.6}$$

for all $x > 0$.

Inequality (2.6) implies that $f(x)$ is strictly concave and $f'(x)$ is strictly decreasing on the interval $(0, \infty)$.

From the concavity of $f(x)$ and monotonicity of $f'(x)$ on the interval $(0, \infty)$, together with (2.3) and (2.4), we clearly see that

$$f(x) > (1-x)f(0) + xf(1) = 0 \tag{2.7}$$

for all $x \in (0, 1)$ and

$$f(x) < 0 \tag{2.8}$$

for all $x \in (1, \infty)$.

Therefore, Lemma 2.3 follows easily from (2.2), (2.7), and (2.8). □

Lemma 2.4 *Let $a, p > 0$, $I_p(x)$ and $H_{f,g}$ be respectively defined by* (1.2) *and* (2.1), *and*

$$f(x) = \left(1 - e^{-ax^p}\right)^{1/p}. \tag{2.9}$$

Then the following statements are true:

(1) $H_{f,I_p}(\infty) = \infty$ *if* $a < 1$;

(2) $H_{f,I_p}(\infty) = -1$ *if* $a > 1$.

Proof From (1.2), (2.1), and (2.9) we get

$$\begin{aligned} H_{f,I_p}(x) &= \frac{f'(x)}{I_p'(x)} I_p(x) - f(x) \\ &= ax^{p-1} e^{(1-a)x^p} \left(1 - e^{-ax^p}\right)^{1/p-1} \int_0^x e^{-t^p}\, dt - \left(1 - e^{-ax^p}\right)^{1/p}, \end{aligned}$$

$$H_{f,I_p}(\infty) = a\Gamma\left(1 + \frac{1}{p}\right) \lim_{x\to\infty} \left[x^{p-1} e^{(1-a)x^p}\right] - 1 = \begin{cases} \infty, & a < 1, \\ -1, & a > 1. \end{cases}$$

□

3 Main results

Theorem 3.1 *Let $a, p > 0$ with $p \neq 1$, and $R(a, p; x)$ be defined by* (1.5). *Then the following statements are true*:

(1) *if* $a \leq \min\{1, 2p/(p + 1)\}$, *then the function* $x \to R(a, p; x)$ *is strictly increasing on* $(0, \infty)$;

(2) *if* $a \geq \max\{1, 2p/(p + 1)\}$, *then the function* $x \to R(a, p; x)$ *is strictly decreasing on* $(0, \infty)$;

(3) *if* $\min\{1, 2p/(p + 1)\} < a < \max\{1, 2p/(p + 1)\}$ *and* $p < 1$ $(p > 1)$, *then there exists* $x_0 \in (0, \infty)$ *such that the function* $x \to R(a, p; x)$ *is strictly decreasing* (*increasing*) *on* $(0, x_0)$ *and strictly increasing* (*decreasing*) *on* (x_0, ∞).

Proof Let $x > 0$, $u = x^p > 0$, $I_p(x)$ and $f(x)$ be respectively defined by (1.2) and (2.9), and

$$h(u) = a(1 - a)pue^{au} + a(p - 1)e^{au} + a(a - p)u + a(1 - p). \tag{3.1}$$

Then it follows from (1.2), (1.5), (2.9), and (3.1) that

$$R(a, p; x) = \frac{f(x)}{I_p(x)}, \tag{3.2}$$

$$f(0) = I_p(0) = 0, \qquad I_p'(x) = e^{-x^p} > 0, \tag{3.3}$$

$$\frac{f'(x)}{I_p'(x)} = ax^{p-1} e^{(1-a)x^p} \left(1 - e^{-ax^p}\right)^{1/p-1} = au^{1-1/p} e^{(1-a)u} \left(1 - e^{-au}\right)^{1/p-1}, \tag{3.4}$$

$$\begin{aligned} \left[\frac{f'(x)}{I_p'(x)}\right]' &= a\frac{d}{du}\left[u^{1-1/p} e^{(1-a)u} \left(1 - e^{-au}\right)^{1/p-1}\right] \frac{du}{dx} \\ &= u^{1-2/p} e^{(1-2a)u} \left(1 - e^{-au}\right)^{1/p-2} h(u), \end{aligned} \tag{3.5}$$

$$h(0) = 0, \tag{3.6}$$

$$h'(u) = a\left[(a - p)\left(1 - e^{au}\right) + a(1 - a)pue^{au}\right], \tag{3.7}$$

$$h'(0) = 0, \tag{3.8}$$

$$h''(u) = a^2\big[a(1-a)pu + 2p - a(p+1)\big]e^{au} = a^3(1-a)pe^{au}\left[u - \frac{a(p+1)-2p}{a(1-a)p}\right]. \tag{3.9}$$

We divide the proof into four cases.

Case 1: $a \leq \min\{1, 2p/(p+1)\}$. From $p \neq 1$ and (3.9) we know that $h'(u)$ is strictly increasing on $(0,\infty)$. Then (3.5), (3.6), and (3.8) lead to the conclusion that $f'(x)/I_p'(x)$ is strictly increasing on $(0,\infty)$. Therefore, $R(a,p;x)$ is strictly increasing on $(0,\infty)$, as follows from Lemma 2.2, (3.2), and (3.3) together with the monotonicity of $f'(x)/I_p'(x)$.

Case 2: $a \geq \max\{1, 2p/(p+1)\}$. From $p \neq 1$ and (3.9) we know that $h'(u)$ is strictly decreasing on $(0,\infty)$. Then (3.5), (3.6), and (3.8) lead to the conclusion that $f'(x)/I_p'(x)$ is strictly decreasing on $(0,\infty)$. Therefore, $R(a,p;x)$ is strictly decreasing on $(0,\infty)$, as follows from Lemma 2.2, (3.2), and (3.3) together with the monotonicity of $f'(x)/I_p'(x)$.

Case 3: $\min\{1, 2p/(p+1)\} < a < \max\{1, 2p/(p+1)\}$ and $p < 1$. Then we clearly see that $2p/(p+1) < a < 1$, and (3.1) and (3.7) lead to

$$h(\infty) = \infty, \tag{3.10}$$

$$h'(\infty) = \infty. \tag{3.11}$$

Let

$$u_0 = \frac{a(p+1)-2p}{a(1-a)p}. \tag{3.12}$$

Then we clearly see that $u_0 \in (0,\infty)$, and (3.9) leads to the conclusion that $h'(u)$ is strictly decreasing on $(0,u_0)$ and strictly increasing on (u_0,∞).

It follows from (3.8) and (3.11) together with the piecewise monotonicity of $h'(u)$ that there exists $u_1 \in (0,\infty)$ such that $h(u)$ is strictly decreasing on $(0,u_1)$ and strictly increasing on (u_1,∞). From (3.5), (3.6), and (3.10) together with the piecewise monotonicity of $h(u)$ we know that there exists $\lambda \in (0,\infty)$ such that $f'(x)/I_p'(x)$ is strictly decreasing on $(0,\lambda)$ and strictly increasing on (λ,∞).

Therefore, there exists $x_0 \in (0,\infty)$ such that the function $x \to R(a,p;x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on (x_0,∞), as follows from Lemma 2.1(1), Lemma 2.4(1), (3.2), (3.3), and the piecewise monotonicity of $f'(x)/I_p'(x)$.

Case 4: $\min\{1, 2p/(p+1)\} < a < \max\{1, 2p/(p+1)\}$ and $p > 1$. Then we clearly see that $1 < a < 2p/(p+1)$, and (3.1) and (3.7) lead to

$$h(\infty) = -\infty, \tag{3.13}$$

$$h'(\infty) = -\infty. \tag{3.14}$$

Let $u_0 \in (0,\infty)$ be defined by (3.12). Then from (3.9) we clearly see that $h'(u)$ is strictly increasing on $(0,u_0)$ and strictly decreasing on (u_0,∞). It follows from (3.5), (3.6), (3.8), (3.13), (3.14), and the piecewise monotonicity of $h'(u)$ that there exists $\mu \in (0,\infty)$ such that $f'(x)/I_p'(x)$ is strictly increasing on $(0,\mu)$ and strictly decreasing on (μ,∞).

Therefore, there exists $x_0 \in (0,\infty)$ such that the function $x \to R(a,p;x)$ is strictly increasing on $(0,x_0)$ and strictly decreasing on (x_0,∞), as follows from Lemma 2.1(2), Lemma 2.4(2), (3.2), (3.3), and the piecewise monotonicity of $f'(x)/I_p'(x)$. □

Remark 3.2 Let $R(a,p;x)$ be defined by (1.5). Then from (1.2), (2.9), and (3.2)-(3.4) we clearly see that

$$R(a,p;\infty) = \frac{1}{\Gamma(1+\frac{1}{p})},$$

$$R(a,p;0^+) = \lim_{x\to 0^+} \frac{f'(x)}{I_p'(x)} = a \lim_{u\to 0^+} \left(\frac{1-e^{-au}}{u}\right)^{1/p-1} = a^{1/p}.$$

From Theorem 3.1 and Remark 3.2 we immediately get Corollary 3.3.

Corollary 3.3 *Let $a,p>0$ with $p\neq 1$, $I_p(x)$ and $R(a,p;x)$ be respectively defined by* (1.2) *and* (1.5), *and x_0 be the unique solution of the equation $d[R(a,p;x)]/dx = 0$ on the interval $(0,\infty)$ in the case of $\min\{1,2p/(p+1)\} < a < \max\{1,2p/(p+1)\}$. Then the following statements are true*:

(1) *if $a \le \min\{1,2p/(p+1)\}$, then we have the double inequality*

$$\Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p} < I_p(x) < \frac{1}{a^{1/p}}\left(1-e^{-ax^p}\right)^{1/p}$$

for all $x>0$;

(2) *if $a \ge \max\{1,2p/(p+1)\}$, then we have the double inequality*

$$\frac{1}{a^{1/p}}\left(1-e^{-ax^p}\right)^{1/p} < I_p(x) < \Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p}$$

for all $x>0$;

(3) *if $\min\{1,2p/(p+1)\} < a < \max\{1,2p/(p+1)\}$ and $p<1$, then we have the double inequality*

$$\min\left\{\Gamma\left(1+\frac{1}{p}\right), \frac{1}{a^{1/p}}\right\}\left(1-e^{-ax^p}\right)^{1/p} < I_p(x) \le \frac{1}{R(a,p;x_0)}\left(1-e^{-ax^p}\right)^{1/p}$$

for all $x>0$;

(4) *if $\min\{1,2p/(p+1)\} < a < \max\{1,2p/(p+1)\}$ and $p>1$, then we have the double inequality*

$$\frac{1}{R(a,p;x_0)}\left(1-e^{-ax^p}\right)^{1/p} \le I_p(x) < \max\left\{\Gamma\left(1+\frac{1}{p}\right), \frac{1}{a^{1/p}}\right\}\left(1-e^{-ax^p}\right)^{1/p}$$

for all $x>0$.

Next, we prove Alzer's inequality (1.4) by using Theorem 3.1, Remark 3.2, and Corollary 3.3.

Theorem 3.4 *Let $a,b,p>0$ with $p\neq 1$, $a_0 = \min\{1,\Gamma^{-p}(1+1/p)\}$, $b_0 = \max\{1,\Gamma^{-p}(1+1/p)\}$, and $I_p(x)$ be defined by* (1.2). *Then the double inequality*

$$\Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p} < I_p(x) < \Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-bx^p}\right)^{1/p}$$

holds for all $x>0$ if and only if $a\le a_0$ and $b\ge b_0$.

Proof Let $R(a,p;x)$ be defined by (1.5). Then we divide the proof into four steps.

Step 1: $p<1$. We prove that the inequality

$$I_p(x) > \Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p} \tag{3.15}$$

holds for all $x>0$ if and only if $a \leq a_0$.

From $p \in (0,1)$ and Lemma 2.3 we clearly see that

$$\frac{2p}{1+p} < \Gamma^{-p}\left(1+\frac{1}{p}\right) = a_0 < 1. \tag{3.16}$$

If inequality (3.15) holds for all $x>0$, then (1.5) and Remark 3.2, together with (3.16), lead to the conclusion that

$$\begin{aligned} &R(a,p;x) < \Gamma^{-1}\left(1+\frac{1}{p}\right), \qquad a^{1/p} = R\left(a,p;0^+\right) \leq \Gamma^{-1}\left(1+\frac{1}{p}\right), \\ &a \leq \Gamma^{-p}\left(1+\frac{1}{p}\right) = a_0. \end{aligned}$$

Next, we prove inequality (3.15) for all $x>0$ if $a \leq a_0$. We divide the proof into two cases.

Case 1.1: $a \leq 2p/(1+p)$. Then from (3.16) and Corollary 3.3(1) we clearly see that $a \leq \min\{1, 2p/(1+p)\}$ and inequality (3.15) holds for all $x>0$.

Case 1.2: $2p/(1+p) < a \leq a_0 = \Gamma^{-p}(1+1/p)$. Then (3.16) and Corollary 3.3(3) lead to the conclusion that $\min\{1, 2p/(1+p)\} < a < \max\{1, 2p/(1+p)\}$ and

$$I_p(x) > \min\left\{\Gamma\left(1+\frac{1}{p}\right), \frac{1}{a^{1/p}}\right\}\left(1-e^{-ax^p}\right)^{1/p} = \Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p}$$

for all $x>0$.

Step 2: $p>1$. We prove that inequality (3.15) holds for all $x>0$ if and only if $a \leq a_0$.

From $p \in (0,1)$ and Lemma 2.3 we clearly see that

$$1 = a_0 < \Gamma^{-p}\left(1+\frac{1}{p}\right) < \frac{2p}{1+p}. \tag{3.17}$$

If $a \leq a_0$, then inequality (3.17) and Corollary 3.3(1) lead to the conclusion that $a \leq \min\{1, 2p/(1+p)\}$ and inequality (3.15) holds for all $x>0$.

Next, we prove by contradiction that $a \leq a_0$ if inequality (3.15) holds for all $x>0$. We divide the proof into two cases.

Case 2.1: $a \geq 2p/(p+1)$. Then (3.17) and Corollary 3.3(2) lead to the conclusion that $a \geq \max\{1, 2p/(p+1)\}$ and the opposite direction inequality of (3.15) holds for all $x>0$.

Case 2.2: $1 = a_0 < a < 2p/(p+1)$. Then inequality (3.17) and Theorem 3.1(3), together with Remark 3.2, lead to the conclusion that $\min\{1, 2p/(p+1)\} < a < \max\{1, 2p/(p+1)\}$ and there exists $x_0 \in (0,\infty)$ such that

$$I_p(x) < \Gamma\left(1+\frac{1}{p}\right)\left(1-e^{-ax^p}\right)^{1/p}$$

for $x \in (x_0,\infty)$, which contradicts with (3.15).

Step 3: $p < 1$. We prove that the inequality

$$I_p(x) < \Gamma\left(1 + \frac{1}{p}\right)\left(1 - e^{-ax^p}\right)^{1/p} \tag{3.18}$$

holds for all $x > 0$ if and only if $a \geq b_0$.

From $p < 1$ and Lemma 2.3 we clearly see that

$$\frac{2p}{p+1} < \Gamma^{-p}\left(1 + \frac{1}{p}\right) < 1 = b_0. \tag{3.19}$$

If $a \geq b_0$, then (3.19) and Corollary 3.3(2) lead to the conclusion that $a \geq \max\{1, 2p/(p+1)\}$ and inequality (3.18) holds for all $x > 0$.

Next, we prove by contradiction that $a \geq b_0$ if inequality (3.18) holds for all $x > 0$. We divide the proof into two cases.

Case 3.1: $a \leq 2p/(1+p)$. Then (3.19) and Corollary 3.3(1) lead to the conclusion that $a \leq \min\{1, 2p/(p+1)\}$ and the opposite direction inequality of (3.18) holds for all $x > 0$.

Case 3.2: $2p/(p+1) < a < b_0 = 1$. Then (3.19) and Theorem 3.1(3), together with Remark 3.2, lead to the conclusion that $\min\{1, 2p/(p+1)\} < a < \max\{1, 2p/(p+1)\}$ and there exists x_0 such that the opposite direction inequality of (3.18) holds for $x \in (x_0, \infty)$.

Step 4: $p > 1$. We prove that inequality (3.18) holds for all $x > 0$ if and only if $a \geq b_0$.

From $p > 1$ and Lemma 2.3 we clearly see that

$$1 < \Gamma^{-p}\left(1 + \frac{1}{p}\right) = b_0 < \frac{2p}{p+1}. \tag{3.20}$$

If inequality (3.18) holds for all $x > 0$, then (1.2), (1.5), Remark 3.2, and (3.20) lead to

$$\Gamma\left(1 + \frac{1}{p}\right)R(a, p; x) > 1, \qquad \Gamma\left(1 + \frac{1}{p}\right)R\left(a, p; 0^+\right) = a^{1/p}\Gamma\left(1 + \frac{1}{p}\right) \geq 1,$$
$$a \geq \Gamma^{-p}\left(1 + \frac{1}{p}\right) = b_0.$$

Next, we prove that inequality (3.18) holds for all $x > 0$ if $a \geq b_0$. We divide the proof into two cases.

Case 4.1: $a \geq 2p/(p+1)$. Then (3.20) and Corollary 3.3(2) lead to the conclusion that $a \geq \max\{1, 2p/(p+1)\}$ and inequality (3.18) holds for all $x > 0$.

Case 4.2: $b_0 \leq a < 2p/(p+1)$. Then (3.20) and Corollary 3.3(4) lead to the conclusion that $\min\{1, 2p/(p+1)\} < a < \max\{1, 2p/(p+1)\}$ and

$$I_p(x) < \max\left\{\frac{1}{a^{1/p}}, \Gamma\left(1 + \frac{1}{p}\right)\right\}\left(1 - e^{-ax^p}\right)^{1/p} = \Gamma\left(1 + \frac{1}{p}\right)\left(1 - e^{-ax^p}\right)^{1/p}$$

for all $x > 0$. □

Let $q = 1/p$, and $u = x^p$. Then (1.1) and (1.2), together with Corollary 3.3, lead to Corollary 3.5.

Corollary 3.5 *Let $a > 0$, $q > 0$ with $q \neq 1$, and u_0 be the unique solution of the equation*

$$\frac{d}{du}\left[\frac{(1-e^{-au})^q}{\Gamma(q)-\Gamma(q,u)}\right] = 0$$

on the interval $(0,\infty)$ *in the case of* $\min\{1, 2/(q+1)\} < a < \max\{1, 2/(q+1)\}$. *Then the following statements are true*:

(1) *if* $a \leq \min\{1, 2/(q+1)\}$, *then we have the double inequality*

$$1 - \frac{(1-e^{-au})^q}{a^q\Gamma(1+q)} < \frac{\Gamma(q,u)}{\Gamma(q)} < 1 - \left(1-e^{-au}\right)^q$$

for all $u > 0$;

(2) *if* $a \geq \max\{1, 2/(q+1)\}$, *then we have the double inequality*

$$1 - \left(1-e^{-au}\right)^q < \frac{\Gamma(q,u)}{\Gamma(q)} < 1 - \frac{(1-e^{-au})^q}{a^q\Gamma(1+q)}$$

for all $u > 0$;

(3) *if* $\min\{1, 2/(q+1)\} < a < \max\{1, 2/(q+1)\}$ *and* $q > 1$, *then we have the double inequality*

$$1 - \frac{\Gamma(q)-\Gamma(q,u_0)}{\Gamma(q)(1-e^{-au_0})^q}\left(1-e^{-au}\right)^q \leq \frac{\Gamma(q,u)}{\Gamma(q)} < 1 - \min\left\{\frac{1}{a^q\Gamma(1+q)}, 1\right\}\left(1-e^{-au}\right)^q$$

for all $u > 0$;

(4) *if* $\min\{1, 2/(q+1)\} < a < \max\{1, 2/(q+1)\}$ *and* $q < 1$, *then we have the double inequality*

$$1 - \max\left\{\frac{1}{a^q\Gamma(1+q)}, 1\right\}\left(1-e^{-au}\right)^q < \frac{\Gamma(q,u)}{\Gamma(q)} \leq 1 - \frac{\Gamma(q)-\Gamma(q,u_0)}{\Gamma(q)(1-e^{-au_0})^q}\left(1-e^{-au}\right)^q$$

for all $u > 0$.

Note that

$$\lim_{q\to 0^+}\left[\Gamma(q)\left(1-\left(1-e^{-au}\right)^q\right)\right] = -\log\left(1-e^{-au}\right), \tag{3.21}$$

$$\lim_{q\to 0^+}\left[\Gamma(q)\left(1-\frac{(1-e^{-au})^q}{a^q\Gamma(1+q)}\right)\right] = \log a - \gamma - \log\left(1-e^{-au}\right). \tag{3.22}$$

Let $Ei(u) = \Gamma(0,u)$ be the exponential integral. Then Corollary 3.5(1) and (2), together with (3.21) and (3.22), immediately lead to Corollary 3.6.

Corollary 3.6 *We have the double inequality*

$$\log a - \gamma - \log\left(1-e^{-au}\right) < Ei(u) < -\log\left(1-e^{-au}\right) \tag{3.23}$$

for all $u > 0$ *and* $0 < a \leq 1$, *and inequality* (3.23) *is reversed for all* $u > 0$ *if* $a \geq 2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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