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# Certain inequalities associated with Hadamard $k$-fractional integral operators

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**Abstract**
We aim to present some new Pólya-Szegö type inequalities associated with Hadamard $k$-fractional integral operators, which are also used to derive some Chebyshev type integral inequalities. Further we apply some of the results presented here to a function which is bounded by the Heaviside functions.

**MSC:** 26A33; 26D10; 26D15

## 1 Introduction and preliminaries

We begin by recalling the following Chebyshev functional which has been investigated by many authors (see, *e.g.*, [1–4]):

$$T(f,g;a,b) = \frac{1}{b-a}\int_a^b f(x)g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\int_a^b f(x)\,\mathrm{d}x\int_a^b g(x)\,\mathrm{d}x, \tag{1.1}$$

where $f,g:[a,b]\to\mathbb{R}$ are integrable functions on $[a,b]$. Here and in the following, let $\mathbb{R}$ and $\mathbb{R}^+$ be the set of real and positive real numbers, respectively, and $\mathbb{R}_0^+ := \mathbb{R}^+\cup\{0\}$. Under more conditions $n\le f(x)\le N$ and $m\le g(x)\le M$ for all $x\in[a,b]$, where $n$, $m$, $N$, $M$ are real constants, the Chebyshev functional (1.1) satisfies the following inequality, which is known as Grüss integral inequality (see [5]; see also [6], p.236):

$$\bigl|T(f,g;a,b)\bigr| \le \frac{1}{4}(M-m)(N-n), \tag{1.2}$$

where the constant $\frac{1}{4}$ is sharp. In fact, the equality in (1.2) holds, for example, by taking

$$f(x) = g(x) = \operatorname{sgn} x - \frac{a+b}{2} \quad \bigl(x\in[a,b]\bigr).$$

The Grüss inequality (1.2) has been investigated a lot and a number of its generalizations have been presented (see, *e.g.*, [7–10]).

Let $f$ and $g$ be two positive integrable functions on $[a,b]$ such that

$$0 < m \le f(x) \le M < \infty \quad \text{and} \quad 0 < n \le f(x) \le N < \infty.$$

Pólya and Szegö [11] established the following inequality:

$$\frac{\int_a^b f^2(x)\,\mathrm{d}x \int_a^b g^2(x)\,\mathrm{d}x}{(\int_a^b f(x)g(x)\,\mathrm{d}x)^2} \leq \frac{1}{4}\left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}\right)^2, \tag{1.3}$$

which was used by Dragomir and Diamond [12] who proved the following inequality:

$$\left|T(f,g;a,b)\right| \leq \frac{(M-m)(N-n)}{4(b-a)^2\sqrt{mMnN}} \int_a^b f(x)\,\mathrm{d}x \int_a^b g(x)\,\mathrm{d}x. \tag{1.4}$$

Fractional calculus is a very helpful tool to perform differentiation and integration of real or complex number orders. This subject has earned much attention from researchers and mathematicians during the last few decades (see, *e.g.*, [13–21]). Among a large number of the fractional integral operators developed, due to applications in many fields of sciences, the Riemann-Liouville fractional integral operator and Hadamard fractional integral operator have been extensively investigated.

Let $f \in L[a,b]$. Then the left-sided and the right-sided Hadamard fractional integrals of order $\alpha \geq 0$ and $a > 0$ are defined, respectively, by

$$H_{a^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{\tau}\right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau} \quad (0 < a < t \leq b) \tag{1.5}$$

and

$$H_{b^-}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\ln\frac{\tau}{t}\right)^{\alpha-1} f(\tau)\frac{\mathrm{d}\tau}{\tau} \quad (0 < a \leq t < b). \tag{1.6}$$

The theory of $k$-functions has been investigated since, about a decade ago, Diaz and Pariguan [22] introduced the following generalizations of the classical gamma and beta functions, with a new parameter $k \in \mathbb{R}^+$, which are called $k$-gamma and $k$-beta functions, respectively:

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}}\,\mathrm{d}t \quad \bigl(\Re(\alpha) > 0\bigr) \tag{1.7}$$

and

$$B_k(\alpha,\beta) = \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}\,\mathrm{d}t \quad \bigl(\min\bigl\{\Re(\alpha),\Re(\beta)\bigr\} > 0\bigr). \tag{1.8}$$

The functions $\Gamma_k$ defined on $\mathbb{R}^+$ and $B_k(x,y)$ on $(0,1)$ satisfy the following properties:

(1) $\Gamma_k(x+k) = x\Gamma_k(x)$;

(2) $\Gamma_k(k) = 1$;

(3) $\Gamma_k(x)$ is logarithmically convex;

(4) $B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$.

During the past several years, certain interesting properties, identities, and inequalities involving $k$-functions have been presented (see, *e.g.*, [23–29]). Mubeen and Habibullah [30] used the $k$-gamma function $\Gamma_k$ (1.7) to introduce the following Riemann-Liouville type $k$-fractional integral:

$$I^{\alpha}_{a,k}f(t)=\frac{1}{k\Gamma_k(\alpha)}\int_a^t(t-x)^{\frac{\alpha}{k}-1}f(x)\,\mathrm{d}x \quad \bigl(t\in[a,b]\bigr). \tag{1.9}$$

Later, Romero *et al.* [31] also used the $k$-gamma function $\Gamma_k$ (1.7) to introduce the $k$-Riemann-Liouville fractional derivative whose properties including a relationship with the $k$-fractional integral (1.9) were presented.

Using the $k$-gamma function with the parameter $k$, Mubeen *et al.* [32] have introduced left-sided and right-sided Hadamard $k$-fractional integrals of order $\alpha\in\mathbb{R}^+$, respectively, as follows: For $f\in L[a,b]$ and $k,a\in\mathbb{R}^+$,

$$\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)=\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}f(\tau)\frac{\mathrm{d}\tau}{\tau}\quad (0<a<t\le b) \tag{1.10}$$

and

$$\mathcal{H}^{\alpha}_{b^-,k}\{f\}(t)=\frac{1}{k\Gamma_k(\alpha)}\int_t^b\left(\ln\frac{\tau}{t}\right)^{\frac{\alpha}{k}-1}f(\tau)\frac{\mathrm{d}\tau}{\tau}\quad (0<a\le t<b). \tag{1.11}$$

Using the Hadamard $k$-fractional integral and Proposition 6 in [22], we have

$$\mathcal{H}^{\alpha}_{a^+,k}\{1\}(t)=\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\quad \bigl(0<a<t\le b;k,\alpha\in\mathbb{R}^+\bigr) \tag{1.12}$$

and

$$\mathcal{H}^{\alpha}_{1^+,k}\{1\}(t)=\frac{(\ln(t))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\quad \bigl(1<t\le b;k,\alpha\in\mathbb{R}^+\bigr). \tag{1.13}$$

## 2 Some Pólya-Szegö and Chebyshev type inequalities involving the Hadamard *k*-fractional integrals

In this section, we derive some new Pólya-Szegö type inequalities associated with the Hadamard $k$-fractional integral operators which are also used to establish some Chebyshev type integral inequalities.

**Lemma 2.1** *Let $f$ and $g$ be two positive real integrable functions defined on $[a,\infty)$. Also let $\varphi_1$, $\varphi_2$, $\psi_1$, and $\psi_2$ be integrable functions on $[a,\infty)$ such that*

$$0<\varphi_1(\tau)\le f(\tau)\le\varphi_2(\tau)\quad \textit{and}\quad 0<\psi_1(\tau)\le g(\tau)\le\psi_2(\tau) \tag{2.1}$$

*for all $\tau\in[a,t]$ $(t>a)$. Then, for $k,\alpha\in\mathbb{R}^+$, and $a\in\mathbb{R}^+_0$, the following inequality holds true*:

$$\frac{\mathcal{H}^{\alpha}_{a^+,k}\{\psi_1\psi_2f^2\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_1\varphi_2g^2\}(t)}{(\mathcal{H}^{\alpha}_{a^+,k}\{(\varphi_1\psi_1+\varphi_2\psi_2)fg\}(t))^2}\le\frac{1}{4}. \tag{2.2}$$

*Proof* Under the given conditions, we find

$$\frac{f(\tau)}{g(\tau)}\le\frac{\varphi_2(\tau)}{\psi_1(\tau)}\quad\text{and}\quad\frac{\varphi_1(\tau)}{\psi_2(\tau)}\le\frac{f(\tau)}{g(\tau)}\quad \bigl(\tau\in[a,t]\ (t>a)\bigr), \tag{2.3}$$

from which we have

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} - \frac{f(\tau)}{g(\tau)}\right)\left(\frac{f(\tau)}{g(\tau)} - \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right) \geq 0,$$

and so

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\tau)} + \frac{\varphi_1(\tau)}{\psi_2(\tau)}\right)\frac{f(\tau)}{g(\tau)} \geq \frac{f^2(\tau)}{g^2(\tau)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\tau)\psi_2(\tau)}. \tag{2.4}$$

The inequality (2.4) can also be written as follows:

$$\bigl(\varphi_1(\tau)\psi_1(\tau) + \varphi_2(\tau)\psi_2(\tau)\bigr)f(\tau)g(\tau) \geq \psi_1(\tau)\psi_2(\tau)f^2(\tau) + \varphi_1(\tau)\varphi_2(\tau)g^2(\tau). \tag{2.5}$$

Here, multiplying each side of the inequality (2.5) by the following non-negative factor:

$$\frac{1}{k\Gamma_k(\alpha)}\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\frac{1}{\tau} \quad \bigl(\tau \in [a,t]\ (t > a)\bigr)$$

and integrating the resulting inequality with respective to $\tau$ on $[a,t]$, we obtain

$$\mathcal{H}^{\alpha}_{a^+,k}\bigl\{(\varphi_1\psi_1 + \varphi_2\psi_2)fg\bigr\}(t) \geq \mathcal{H}^{\alpha}_{a^+,k}\bigl\{\psi_1\psi_2 f^2\bigr\}(t) + \mathcal{H}^{\alpha}_{a^+,k}\bigl\{\varphi_1\varphi_2 g^2\bigr\}(t). \tag{2.6}$$

Applying the AM-GM (the arithmetic-geometric mean) inequality,

$$a + b \geq 2\sqrt{ab} \quad \bigl(a, b \in \mathbb{R}^+_0\bigr) \tag{2.7}$$

to the right-hand side of (2.6), we have

$$\mathcal{H}^{\alpha}_{a^+,k}\bigl\{(\varphi_1\psi_1 + \varphi_2\psi_2)fg\bigr\}(t) \geq 2\sqrt{\mathcal{H}^{\alpha}_{a^+,k}\bigl\{\psi_1\psi_2 f^2\bigr\}(t)\mathcal{H}^{\alpha}_{a^+,k}\bigl\{\varphi_1\varphi_2 g^2\bigr\}(t)}, \tag{2.8}$$

which leads to

$$\mathcal{H}^{\alpha}_{a^+,k}\bigl\{\psi_1\psi_2 f^2\bigr\}(t)\mathcal{H}^{\alpha}_{a^+,k}\bigl\{\varphi_1\varphi_2 g^2\bigr\}(t) \leq \frac{1}{4}\bigl(\mathcal{H}^{\alpha}_{a^+,k}\bigl\{(\varphi_1\psi_1 + \varphi_2\psi_2)fg\bigr\}(t)\bigr)^2.$$

This completes the proof. □

The following corollary is easily seen to be a special case of Lemma 2.1.

**Corollary 1** *Let $f$ and $g$ be two real positive integrable functions defined on $[a,\infty)$ such that*

$$0 < m \leq f(\tau) \leq M < \infty \quad \textit{and} \quad 0 < n \leq g(\tau) \leq N < \infty \quad \bigl(\tau \in [a,t]\ (t > a)\bigr), \tag{2.9}$$

*where $n$, $N$, $m$, $M$ are real constants. Then, for all $t, k \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}^+$, we have*

$$\frac{\mathcal{H}^{\alpha}_{a^+,k}\{f^2\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{g^2\}(t)}{(\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(t))^2} \leq \frac{1}{4}\left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{nm}}\right)^2. \tag{2.10}$$

**Lemma 2.2** *Let $f$ and $g$ be two positive real integrable functions defined on $[a,\infty)$. Also let $\varphi_1$, $\varphi_2$, $\psi_1$, and $\psi_2$ be integrable functions on $[a,\infty)$ satisfying the condition* (2.1). *Then, for $t > a$ ($a \in \mathbb{R}_0^+$) and $k,\alpha,\beta \in \mathbb{R}^+$, the following inequality holds true*:

$$\frac{\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_1\varphi_2\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_1\psi_2\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{f^2\}(t)\mathcal{H}^{\beta}_{a^+,k}\{g^2\}(t)}{(\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_1 f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_1 g\}(t)+\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_2 f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_2 g\}(t))^2} \le \frac{1}{4}. \tag{2.11}$$

*Proof* We find from (2.1) that

$$\left(\frac{\varphi_2(\tau)}{\psi_1(\rho)} - \frac{f(\tau)}{g(\rho)}\right) \ge 0 \quad \text{and} \quad \left(\frac{f(\tau)}{g(\rho)} - \frac{\varphi_1(\tau)}{\psi_2(\rho)}\right) \ge 0 \quad \bigl(\tau,\rho \in [a,t]\ (t > a)\bigr),$$

which yields

$$\left(\frac{\varphi_1(\tau)}{\psi_2(\rho)} + \frac{\varphi_2(\tau)}{\psi_1(\rho)}\right)\frac{f(\tau)}{g(\rho)} \ge \frac{f^2(\tau)}{g^2(\rho)} + \frac{\varphi_1(\tau)\varphi_2(\tau)}{\psi_1(\rho)\psi_2(\rho)}. \tag{2.12}$$

Multiplying each side of the inequality (2.12) by $\psi_1(\rho)\psi_2(\rho)g^2(\rho)$, we get

$$\varphi_1(\tau)f(\tau)\psi_1(\rho)g(\rho)+\varphi_2(\tau)f(\tau)\psi_2(\rho)g(\rho) \ge \psi_1(\rho)\psi_2(\rho)f^2(\tau)+\varphi_1(\tau)\varphi_2(\tau)g^2(\rho). \tag{2.13}$$

Again multiplying each side of the inequality (2.13) by the following non-negative factor:

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}\frac{1}{\tau\rho} \quad \bigl(\tau,\rho \in [a,t](t > a)\bigr)$$

and integrating the resulting inequality with respect to $\tau$ and $\rho$ on $[a,t]$, we have

$$\begin{aligned}&\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_1 f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_1 g\}(t)+\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_2 f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_2 g\}(t)\\ &\quad\ge \mathcal{H}^{\alpha}_{a^+,k}\bigl\{f^2\bigr\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_1\psi_2\}(t)+\mathcal{H}^{\beta}_{a^+,k}\{\varphi_1\varphi_2\}(t)\mathcal{H}^{\alpha}_{a^+,k}\bigl\{g^2\bigr\}(t).\end{aligned} \tag{2.14}$$

Applying the AM-GM inequality (2.7) to (2.14), we obtain

$$\begin{aligned}&\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_1 f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_1 g\}(t)+\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_2 f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_2 g\}(t)\\ &\quad\ge 2\sqrt{\mathcal{H}^{\alpha}_{a^+,k}\bigl\{f^2\bigr\}(t)\mathcal{H}^{\beta}_{a^+,k}\{\psi_1\psi_2\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_1\varphi_2\}(t)\mathcal{H}^{\beta}_{a^+,k}\bigl\{g^2\bigr\}(t)},\end{aligned} \tag{2.15}$$

which is easily seen to yield the desired inequality (2.11). Hence the proof is complete. □

**Corollary 2** *Let $f$ and $g$ be two positive integrable functions on interval $[a,\infty)$ satisfying the conditions in* (2.9). *Then, for $t > 1$ and $k,\alpha,\beta \in \mathbb{R}^+$, we have*

$$\frac{(\ln t)^{\frac{\alpha+\beta}{k}}}{\Gamma_k(\alpha+k)\Gamma_k(\beta+k)}\frac{\mathcal{H}^{\alpha}_{1^+,k}\{f^2(t)\}\mathcal{H}^{\beta}_{1^+,k}\{g^2(t)\}}{(\mathcal{H}^{\alpha}_{1^+,k}\{f(t)\}+\mathcal{H}^{\beta}_{1^+,k}\{g(t)\})^2} \le \frac{1}{4}\left(\sqrt{\frac{mn}{MN}}+\sqrt{\frac{MN}{mn}}\right)^2. \tag{2.16}$$

**Lemma 2.3** *Suppose that all assumptions of* Lemma 2.2 *are satisfied. Then, for $t > a$ and $\alpha,\beta \in \mathbb{R}^+$, the following inequality holds true*:

$$\mathcal{H}^{\alpha}_{a^+,k}\bigl\{f^2\bigr\}(t)\mathcal{H}^{\beta}_{a^+,k}\bigl\{g^2\bigr\}(t) \le \mathcal{H}^{\alpha}_{a^+,k}\bigl\{(\varphi_2 fg)/\psi_1\bigr\}(t)\mathcal{H}^{\beta}_{a^+,k}\bigl\{(\psi_2 fg)/\varphi_1\bigr\}(t). \tag{2.17}$$

*Proof* Using the conditions (2.1), we get

$$\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}f^2(\tau)\frac{d\tau}{\tau}\leq\frac{1}{k\Gamma_k(\alpha)}\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\frac{\varphi_2(\tau)}{\psi_1(\tau)}f(\tau)g(\tau)\frac{d\tau}{\tau},$$

which implies

$$\mathcal{H}^{\alpha}_{a^+,k}\{f^2\}(t)\leq\mathcal{H}^{\alpha}_{a^+,k}\{(\varphi_2 fg)/\psi_1\}(t). \tag{2.18}$$

Similarly we have

$$\frac{1}{k\Gamma_k(\beta)}\int_a^t\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}g^2(\rho)\frac{d\rho}{\rho}\leq\frac{1}{k\Gamma_k(\beta)}\int_a^t\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}\frac{\psi_2(\rho)}{\varphi_1(\rho)}f(\rho)g(\rho)\frac{d\rho}{\rho}$$

and so

$$\mathcal{H}^{\beta}_{a^+,k}\{g^2\}(t)\leq\mathcal{H}^{\beta}_{a^+,k}\{(\psi_2 fg)/\varphi_1\}(t). \tag{2.19}$$

Multiplying the inequalities (2.18) and (2.19) side by side and considering all the involved terms are non-negative real numbers, we obtain the desired inequality (2.17). □

It is easy to see from Lemma 2.3 that the assertion in Corollary 3 holds true.

**Corollary 3** *Let $f$ and $g$ be two positive integrable functions on interval $[a,\infty)$ satisfying the conditions in* (2.9). *Then, for $t > a$ and $\alpha,\beta\in\mathbb{R}^+$, we have*

$$\frac{\mathcal{H}^{\alpha}_{a^+,k}\{f^2\}(t)\mathcal{H}^{\beta}_{a^+,k}\{g^2\}(t)}{\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(t)\mathcal{H}^{\beta}_{a^+,k}\{fg\}(t)}\leq\frac{MN}{mn}. \tag{2.20}$$

**Theorem 1** *Let $f$ and $g$ be two positive integrable functions on interval $[a,\infty)$. Suppose that there exist four positive functions $\varphi_1$, $\varphi_2$, $\psi_1$, and $\psi_2$ satisfying the conditions* (2.1). *Then, for $t > a$ and $k,\alpha,\beta\in\mathbb{R}^+$, the following inequality holds true*:

$$\begin{aligned}&\left|\frac{(\ln(t/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(t)+\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\beta}_{a^+,k}\{fg\}(t)\right.\\&\qquad\left.-\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{g\}(t)-\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t)\mathcal{H}^{\beta}_{a^+,k}\{f\}(t)\right|\\&\quad\leq\left|\mathrm{M}_1(f,\varphi_1,\varphi_2)(t)+\mathrm{M}_2(f,\varphi_1,\varphi_2)(t)\right|^{\frac{1}{2}}\\&\qquad\times\left|\mathrm{M}_1(g,\psi_1,\psi_2)(t)+\mathrm{M}_2(g,\psi_1,\psi_2)(t)\right|^{\frac{1}{2}},\end{aligned} \tag{2.21}$$

*where*

$$\mathrm{M}_1(u,v,w)(t):=\frac{(\ln(t/a))^{\frac{\beta}{k}}}{4\Gamma_k(\beta+k)}\frac{(\mathcal{H}^{\alpha}_{a^+,k}\{(v+w)u\}(t))^2}{\mathcal{H}^{\alpha}_{a^+,k}\{vw\}(t)}-\mathcal{H}^{\alpha}_{a^+,k}\{u\}(t)\mathcal{H}^{\beta}_{a^+,k}\{u\}(t)$$

*and*

$$\mathrm{M}_2(u,v,w)(t) := \frac{(\ln(t/a))^{\frac{\alpha}{k}}}{4\Gamma_k(\alpha+k)} \frac{(\mathcal{H}^{\beta}_{a^+,k}\{(v+w)u\}(t))^2}{\mathcal{H}^{\beta}_{a^+,k}\{vw\}(t)} - \mathcal{H}^{\alpha}_{a^+,k}\{u\}(t)\mathcal{H}^{\beta}_{a^+,k}\{u\}(t).$$

*Proof* Let

$$H(\tau,\rho) := \big(f(\tau)-f(\rho)\big)\big(g(\tau)-g(\rho)\big),$$

or, equivalently,

$$H(\tau,\rho) = f(\tau)g(\tau) + f(\rho)g(\rho) - f(\tau)g(\rho) - f(\rho)g(\tau). \tag{2.22}$$

Upon multiplying each side of (2.22) by

$$\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\left(\ln\frac{x}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{x}{\rho}\right)^{\frac{\beta}{k}-1}\frac{1}{\tau\rho}$$

and integrating the resulting identity with respect to $\tau$ and $\rho$ on $[a,t]$, we get

$$\begin{aligned}
&\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}H(\tau,\rho)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}\rho}{\rho}\\
&\quad=\frac{(\ln(t/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(t)+\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\beta}_{a^+,k}\{fg\}(t)\\
&\qquad-\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{g\}(t)-\mathcal{H}^{\beta}_{a^+,k}\{f\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t).
\end{aligned}\tag{2.23}$$

Making use of the weighted Cauchy-Schwarz inequality for double integrals in (2.23), we have

$$\begin{aligned}
&\left|\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}H(\tau,\rho)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}\rho}{\rho}\right|\\
&\quad\le\left[\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}f^2(\tau)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}\rho}{\rho}\right.\\
&\qquad+\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}f^2(\rho)\frac{d\tau}{\tau}\frac{\mathrm{d}\rho}{\rho}\\
&\qquad\left.-\frac{2}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}f(\tau)f(\rho)\frac{d\tau}{\tau}\frac{\mathrm{d}\rho}{\rho}\right]^{\frac{1}{2}}\\
&\qquad\times\left[\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}g^2(\tau)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}\rho}{\rho}\right.\\
&\qquad+\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}g^2(\rho)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}\rho}{\rho}\\
&\qquad\left.-\frac{2}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}g(\tau)g(\rho)\frac{\mathrm{d}\tau}{\tau}\frac{\mathrm{d}\rho}{\rho}\right]^{\frac{1}{2}}.
\end{aligned}\tag{2.24}$$

Then, upon using the Hadamard $k$-fractional integrals, we get

$$\begin{aligned}
&\left|\frac{1}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)}\int_a^t\int_a^t\left(\ln\frac{t}{\tau}\right)^{\frac{\alpha}{k}-1}\left(\ln\frac{t}{\rho}\right)^{\frac{\beta}{k}-1}H(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}\right| \\
&\quad\le\left[\frac{(\ln(t/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{H}^{\alpha}_{a^+,k}\{f^2\}(t)+\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\beta}_{a^+,k}\{f^2\}(t)-2\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{f\}(t)\right]^{\frac{1}{2}} \\
&\qquad\times\left[\frac{(\ln(t/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{H}^{\alpha}_{a^+,k}\{g^2\}(t)+\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\beta}_{a^+,k}\{g^2\}(t)\right. \\
&\qquad\left.-2\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t)\mathcal{H}^{\beta}_{a^+,k}\{g\}(t)\right]^{\frac{1}{2}}.
\end{aligned}\tag{2.25}$$

Setting $\psi_1(t)=\psi_2(t)=g(t)=1$ in Lemma 2.1, we obtain

$$\frac{(\ln(t/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{H}^{\alpha}_{a^+,k}\{f^2\}(t)\le\frac{(\ln(t/a))^{\frac{\beta}{k}}}{4\Gamma_k(\beta+k)}\frac{(\mathcal{H}^{\alpha}_{a^+,k}\{(\varphi_1+\varphi_2)f\}(t))^2}{\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_1\varphi_2\}(t)},$$

which leads to

$$\begin{aligned}
&\frac{(\ln(t/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{H}^{\alpha}_{a^+,k}\{f^2\}(t)-\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{f\}(t) \\
&\quad\le\frac{(\ln(t/a))^{\frac{\beta}{k}}}{4\Gamma_k(\beta+k)}\frac{(\mathcal{H}^{\alpha}_{a^+,k}\{(\varphi_1+\varphi_2)f\}(t))^2}{\mathcal{H}^{\alpha}_{a^+,k}\{\varphi_1\varphi_2\}(t)}-\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{f\}(t) \\
&\quad=\mathrm{M}_1(f,\varphi_1,\varphi_2)(t)
\end{aligned}\tag{2.26}$$

and

$$\begin{aligned}
&\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\beta}_{a^+,k}\{f^2\}(t)-\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{f\}(t) \\
&\quad\le\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{4\Gamma_k(\alpha+k)}\frac{(\mathcal{H}^{\beta}_{a^+,k}\{(\varphi_1+\varphi_2)f\}(t))^2}{\mathcal{H}^{\beta}_{a^+,k}\{\varphi_1\varphi_2\}(t)}-\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{f\}(t) \\
&\quad=\mathrm{M}_2(f,\varphi_1,\varphi_2)(t).
\end{aligned}\tag{2.27}$$

Similarly, taking $\varphi_1(t)=\varphi_2(t)=f(t)=1$ in Lemma 2.1, we get

$$\frac{(\ln(t/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{H}^{\alpha}_{a^+,k}\{g^2\}(t)-\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t)\mathcal{H}^{\beta}_{a^+,k}\{g\}(t)\le\mathrm{M}_1(g,\psi_1,\psi_2)(t)\tag{2.28}$$

and

$$\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\beta}_{a^+,k}\{g^2\}(t)-\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t)\mathcal{H}^{\beta}_{a^+,k}\{g\}(t)\le\mathrm{M}_2(g,\psi_1,\psi_2)(t).\tag{2.29}$$

Finally, by combining the inequalities (2.25)-(2.29), we can get the desired inequality (2.21). This completes the proof. □

The following assertion is a special case of Theorem 1 when $\alpha=\beta$.

**Theorem 2** *Suppose that the assumptions of* Theorem 1 *are satisfied. Then, for* $t > 1$ *and* $\alpha \in \mathbb{R}^+$, *the following inequality holds true*:

$$\begin{aligned}&\left|\frac{(\ln(t))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(t)-\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t)\right|\\&\quad\le\left|\mathcal{M}(f,\varphi_1,\varphi_2)(t)\mathcal{M}(g,\varphi_1,\varphi_2)(t)\right|^{\frac{1}{2}},\end{aligned}\tag{2.30}$$

*where*

$$\mathcal{M}(u,v,w)(t):=\frac{(\ln(t))^{\frac{\alpha}{k}}}{4\Gamma_k(\alpha+k)}\frac{(\mathcal{H}^{\alpha}_{a^+,k}\{(v+w)u\}(t))^2}{\mathcal{H}^{\alpha}_{a^+,k}\{vw\}(t)}-\left(\mathcal{H}^{\alpha}_{a^+,k}\{u\}(t)\right)^2.$$

**Remark 2.1** *Setting* $\varphi_1 = m$, $\varphi_2 = M$, $\psi_1 = n$, *and* $\psi_2 = N$, *we have*

$$\mathcal{M}(f,m,M)(t)=\frac{(M-m)^2}{4mM}\left(\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\right)^2$$

*and*

$$\mathcal{M}(g,n,N)(t)=\frac{(N-n)^2}{4nN}\left(\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t)\right)^2.$$

**Corollary 4** *Let* $f$ *and* $g$ *be two positive integrable functions on* $[a,\infty)$ *satisfying the condition* (2.9). *Then, for* $t > a$ *and* $\alpha \in \mathbb{R}^+$, *we have*

$$\begin{aligned}&\left|\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(t)-\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t)\right|\\&\quad\le\frac{(M-m)(N-n)}{4\sqrt{mMnN}}\mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{g\}(t).\end{aligned}\tag{2.31}$$

## 3 Applications

In this section we apply Hadamard $k$-fractional integrals to a function which is bounded by the Heaviside functions.

The simplest piecewise continuous function is the unit step function, which is known as the Heaviside function, defined by

$$u_c(t)=\begin{cases}1 & \text{if } u\ge c,\\ 0 & \text{if } u<c.\end{cases}$$

The unit step function is basically an on-off switch which is very useful in differential equations and piecewise functions when there is a large number of pieces, for example, Riemann sums as in Figure 1. Using Heaviside function, a piecewise continuous function $\varphi_1(t)$ defined on an interval $[a,T]$ can be written as follows:

$$\begin{aligned}\varphi_1(t)&=m_1\bigl(u_{t_0}(t)-u_{t_1}(t)\bigr)+m_2\bigl(u_{t_1}(t)-u_{t_2}(t)\bigr)+m_3\bigl(u_{t_3}(t)-u_{t_2}(t)\bigr)+\cdots+m_{p+1}u_{t_p}(t)\\&=m_1u_{t_0}+(m_2-m_1)u_{t_1}(t)+(m_3-m_2)u_{t_2}(t)+\cdots+(m_{p+1}-m_p)u_{t_p}(t)\\&=\sum_{k=0}^{p}(m_{k+1}-m_k)u_{t_k}(t),\end{aligned}\tag{3.1}$$

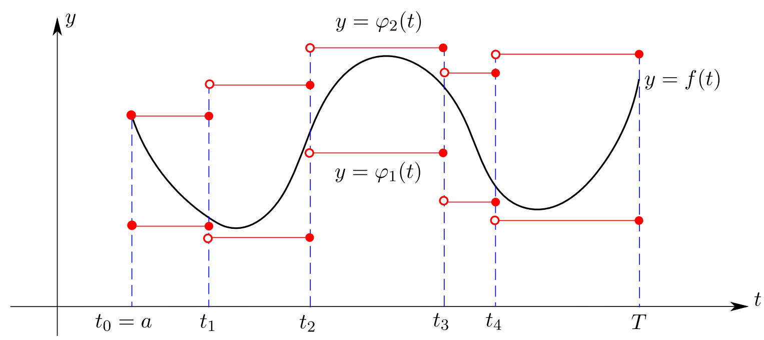


**Figure 1** Functions *f*, $\varphi_1$, and $\varphi_2$.

where $m_0 = 0$, $m_i \in \mathbb{R}$ $(i = 0, 1, \ldots, p+1)$ and $a = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = T$. Similarly we define the functions $\varphi_2$, $\psi_1$, and $\psi_2$ as follows:

$$\varphi_2(t) = \sum_{k=0}^{p} (M_{k+1} - M_k) u_{t_k}(t), \tag{3.2}$$

$$\psi_1(t) = \sum_{k=0}^{p} (n_{k+1} - n_k) u_{t_k}(t), \tag{3.3}$$

$$\psi_2(t) = \sum_{k=0}^{p} (N_{k+1} - N_k) u_{t_k}(t), \tag{3.4}$$

where $n_0 = N_0 = M_0 = 0$ and $n_i, N_i, M_i \in \mathbb{R}$ $(i = 0, 1, \ldots, p+1)$.

Let $f$ be an integrable function on $[a, T]$ which satisfies the condition (2.1) with the functions $\varphi_1$, $\varphi_2$, $\psi_1$, and $\psi_2$ in (3.1), (3.2), (3.3) and (3.4), respectively. Then we get $m_{j+1} \le f(t) \le M_{j+1}$ for each $t \in (t_j, t_{j+1})$ $(j = 0, 1, \ldots, p)$. For example, Figure 1 represents the case $p = 4$.

Then the Hadamard $k$-fractional integral of $f$ on $[a, T]$ can be defined as follows:

$$\mathcal{H}^{\alpha}_{a^+,k}\{f\}(T) = \sum_{j=0}^{p} \mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{f\}(t), \tag{3.5}$$

where

$$\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{f\}(t) := \frac{1}{k\Gamma_k(\alpha)} \int_{t_j}^{t_{j+1}} \left(\ln\frac{t}{s}\right)^{\frac{\alpha}{k}-1} f(s) \frac{\mathrm{d}s}{s} \quad (j = 0, 1, 2, \ldots, p). \tag{3.6}$$

**Proposition 1** *Let $f$ and $g$ be two positive integrable functions on $[a, T]$ which satisfy the condition* (2.1) *with the functions $\varphi_1$, $\varphi_2$, $\psi_1$, and $\psi_2$ in* (3.1), (3.2), (3.3), *and* (3.4), *respectively. Then, for $\alpha \in \mathbb{R}^+$, the following inequality holds true*:

$$\left(\sum_{j=0}^{p} n_{j+1} N_{j+1} \mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{f^2\}(T)\right)\left(\sum_{j=0}^{p} m_{j+1} M_{j+1} \mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{g^2\}(T)\right) \le \frac{1}{4} \sum_{j=0}^{p} (n_{j+1} N_{j+1} + m_{j+1} M_{j+1}) \left(\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(T)\right)^2. \tag{3.7}$$

*Proof* Using the Hadamard $k$-fractional integral in (3.5), we get

$$\mathcal{H}_{a^+,k}\{\psi_1\psi_2 f^2\}(T) = \sum_{j=0}^{p} n_{j+1}N_{j+1}\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{f^2\}(T), \tag{3.8}$$

$$\mathcal{H}_{a^+,k}\{\varphi_1\varphi_2 g^2\}(T) = \sum_{j=0}^{p} m_{j+1}M_{j+1}\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{g^2\}(T), \tag{3.9}$$

and

$$\mathcal{H}_{a^+,k}\{(\varphi_1\psi_1 + \varphi_2\psi_2 fg)\}(T) = \sum_{j=0}^{p}(m_{j+1}n_{j+1} + M_{j+1}N_{j+1})\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{fg\}(T). \tag{3.10}$$

Then substituting equalities (3.8), (3.9), and (3.10) for the result in Lemma 2.1 yields the desired result (3.7). □

**Proposition 2** *Suppose that assumptions of* Proposition 1 *are satisfied. Then, for* $k,\alpha,\beta \in \mathbb{R}^+$, *we have*

$$\begin{aligned}
&\Bigg|\frac{(\ln(t/a))^{\frac{\beta}{k}}}{\Gamma_k(\beta+k)}\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(T) + \frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\beta}_{a^+,k}\{fg\}(T) \\
&\qquad - \mathcal{H}^{\alpha}_{a^+,k}\{f\}(t)\mathcal{H}^{\beta}_{a^+,k}\{g\}(T) - \mathcal{H}^{\beta}_{a^+,k}\{f\}(t)\mathcal{H}^{\alpha}_{a^+,k}\{f\}(T)\Bigg| \\
&\quad \le \big|\mathrm{M}_1^*(f,m_{j+1},M_{j+1})(t) + \mathrm{M}_2^*(f,m_{j+1},M_{j+1})(T)\big|^{\frac{1}{2}} \\
&\qquad \times \big|\mathrm{M}_1^*(g,n_{j+1},N_{j+1})(t) + \mathrm{M}_2^*(g,n_{j+1},N_{j+1})(T)\big|^{\frac{1}{2}},
\end{aligned} \tag{3.11}$$

*where*

$$\begin{aligned}
\mathrm{M}_1^*(u,v,w)(t) &:= \frac{(\ln(t/a))^{\frac{\beta}{k}}\Gamma_k(\alpha+k)}{4\Gamma_k(\beta+k)}\frac{\sum_{j=0}^{p}(v+w)(\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{u\}(t))^2}{\sum_{j=0}^{p} vw[(\ln(t/t_j)]^{\frac{\alpha}{k}} - [\ln(t/t_{j+1}))^{\frac{\alpha}{k}}]} \\
&\quad - \big(\mathcal{H}^{\alpha}_{a^+,k}\{u\}(T)\big)\big(\mathcal{H}^{\beta}_{a^+,k}\{u\}(T)\big), \\
\mathrm{M}_2^*(u,v,w)(t) &:= \frac{(\ln(t/a))^{\frac{\alpha}{k}}\Gamma_k(\beta+k)}{4\Gamma_k(\alpha+k)}\frac{\sum_{j=0}^{p}(v+w)(\mathcal{H}^{\beta}_{t_j,t_{j+1},k}\{u\}(t))^2}{\sum_{j=0}^{p} vw[(\ln(t/t_j)]^{\frac{\beta}{k}} - [\ln(t/t_{j+1}))^{\frac{\beta}{k}}]} \\
&\quad - \big(\mathcal{H}^{\beta}_{a^+,k}\{u\}(t)\big)\big(\mathcal{H}^{\alpha}_{a^+,k}\{u\}(t)\big).
\end{aligned}$$

*Proof* Since

$$\begin{aligned}
\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{f\}(T) &= \frac{1}{k\Gamma_k(\alpha)}\int_{t_j}^{t_{j+1}}\left(\ln\frac{t}{s}\right)^{\frac{\alpha}{k}-1} f(s)\frac{\mathrm{d}s}{s} \\
&= \frac{(\ln(t/t_j))^{\frac{\alpha}{k}} - (\ln(t/t_{j+1}))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)},
\end{aligned}$$

we get

$$\mathcal{H}_{a^+,k}\{\varphi_1\varphi_2\}\{T\} = \sum_{j=0}^{p} \frac{m_{j+1}M_{j+1}}{\Gamma_k(\alpha+k)}\left[\left(\ln(t/t_j)\right)^{\frac{\alpha}{k}} - \left(\ln(t/t_{j+1})\right)^{\frac{\alpha}{k}}\right]$$

and

$$\mathcal{H}_{a^+,k}\{\psi_1\psi_2\}\{T\} = \sum_{j=0}^{p} \frac{n_{j+1}N_{j+1}}{\Gamma_k(\alpha+k)}\left[\left(\ln(t/t_j)\right)^{\frac{\alpha}{k}} - \left(\ln(t/t_{j+1})\right)^{\frac{\alpha}{k}}\right].$$

After some computations, we have

$$\begin{aligned}
\mathrm{M}_1(f,\varphi_1,\varphi_2)(T) &= \frac{(\ln(t/a))^{\frac{\beta}{k}}\Gamma_k(\alpha+k)}{4\Gamma_k(\beta+k)} \frac{\sum_{j=0}^{p}(m_{j+1}+M_{j+1})(\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{f\}(t))^2}{\sum_{j=0}^{p} m_{j+1}M_{j+1}[(\ln(t/t_j)]^{\frac{\alpha}{k}} - [\ln(t/t_{j+1}))^{\frac{\alpha}{k}}]} \\
&\quad - \left(\mathcal{H}^{\alpha}_{a^+,k}\{f\}(T)\right)\left(\mathcal{H}^{\beta}_{a^+,k}\{f\}(T)\right), \\
\mathrm{M}_1(g,\psi_1,\psi_2)(T) &= \frac{(\ln(t/a))^{\frac{\beta}{k}}\Gamma_k(\alpha+k)}{4\Gamma_k(\beta+k)} \frac{\sum_{j=0}^{p}(n_{j+1}+N_{j+1})(\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{g\}(t))^2}{\sum_{j=0}^{p} n_{j+1}N_{j+1}[(\ln(t/t_j)]^{\frac{\alpha}{k}} - [\ln(t/t_{j+1}))^{\frac{\alpha}{k}}]} \\
&\quad - \left(\mathcal{H}^{\alpha}_{a^+,k}\{g\}(T)\right)\left(\mathcal{H}^{\beta}_{a^+,k}\{g\}(T)\right), \\
\mathrm{M}_2(f,\varphi_1,\varphi_2)(T) &= \frac{(\ln(t/a))^{\frac{\alpha}{k}}\Gamma_k(\beta+k)}{4\Gamma_k(\alpha+k)} \frac{\sum_{j=0}^{p}(m_{j+1}+M_{j+1})(\mathcal{H}^{\beta}_{t_j,t_{j+1},k}\{f\}(t))^2}{\sum_{j=0}^{p} m_{j+1}M_{j+1}[(\ln(t/t_j)]^{\frac{\beta}{k}} - [\ln(t/t_{j+1}))^{\frac{\beta}{k}}]} \\
&\quad - \left(\mathcal{H}^{\alpha}_{a^+,k}\{g\}(T)\right)\left(\mathcal{H}^{\beta}_{a^+,k}\{g\}(T)\right),
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{M}_2(g,\psi_1,\psi_2)(t) &= \frac{(\ln(t/a))^{\frac{\alpha}{k}}\Gamma_k(\beta+k)}{4\Gamma_k(\alpha+k)} \frac{\sum_{j=0}^{p}(n_{j+1}+N_{j+1})(\mathcal{H}^{\beta}_{t_j,t_{j+1},k}\{g\}(T))^2}{\sum_{j=0}^{p} n_{j+1}N_{j+1}[(\ln(t/t_j)]^{\frac{\beta}{k}} - [\ln(t/t_{j+1}))^{\frac{\beta}{k}}]} \\
&\quad - \left(\mathcal{H}^{\alpha}_{a^+,k}\{g\}(T)\right)\left(\mathcal{H}^{\beta}_{a^+,k}\{g\}(T)\right).
\end{aligned}$$

By applying the results here to Theorem 1, we obtain the desired inequality (3.11). Hence the proof is complete. □

The special case of Proposition 2 when $\alpha = \beta$ is seen immediately to reduce to the result in Corollary 5.

**Corollary 5** *Suppose that the assumptions of* Proposition 2 *are satisfied. Then, for* $k,\alpha \in \mathbb{R}^+$, *the following inequality holds true*:

$$\begin{aligned}
&\left|\frac{(\ln(t/a))^{\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)}\mathcal{H}^{\alpha}_{a^+,k}\{fg\}(T) - \mathcal{H}^{\alpha}_{a^+,k}\{f\}(T)\mathcal{H}^{\alpha}_{a^+,k}\{f\}(T)\right| \\
&\quad \le \left|\mathrm{M}^*(f,m_{j+1},M_{j+1})(T)\mathrm{M}^*(g,n_{j+1},N_{j+1})(T)\right|^{\frac{1}{2}},
\end{aligned} \tag{3.12}$$

*where*

$$\mathrm{M}^*(u,v,w)(t) = \frac{(\ln(t/a))^{\frac{\alpha}{k}}}{4}\frac{\sum_{j=0}^{p}(v+w)(\mathcal{H}^{\alpha}_{t_j,t_{j+1},k}\{u\}(t))^2}{\sum_{j=0}^{p} vw[(\ln(t/t_j))]^{\frac{\alpha}{k}} - [\ln(t/t_{j+1}))^{\frac{\alpha}{k}}]} - \big(\mathcal{H}^{\alpha}_{a^+,k}\{u\}(t)\big)^2.$$

We conclude this paper by remarking that all the results presented in this paper can be converted into those for the right-sided Hadamard $k$-fractional integral.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors have contributed equally to this manuscript. They read and approved the final manuscript.

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