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Boundedness and compactness of a new product-type operator from a general space to Bloch-type spaces

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Abstract

We characterize the boundedness and compactness of a product-type operator, which, among others, includes all the products of the single composition, multiplication, and differentiation operators, from a general space to Bloch-type spaces. We also give some upper and lower bounds for the norm of the operator.

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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $dA(z)$ the normalized area measure on $\mathbb{D}$ (i.e., $A(\mathbb{D}) = 1$), $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$, and $S(\mathbb{D})$ the family of all holomorphic self-maps of $\mathbb{D}$. Let

$$\eta_a(z) = \frac{a-z}{1-\overline{a}z}, \quad a, z \in \mathbb{D},$$

that is, the involutive automorphism of $\mathbb{D}$ interchanging points a and 0. Simple calculation shows that

$$\frac{1 - |\eta_a(z)|^2}{1 - |z|^2} = \frac{1 - |a|^2}{|1 - \overline{a}z|^2} = |\eta_a'(z)|.$$

A strictly positive continuous function μ on $\mathbb{D}$ is called *weight*. A weight μ is called *radial* if $\mu(z) = \mu(|z|)$ for every $z \in \mathbb{D}$. A radial weight μ is called *typical* if it is nonincreasing with respect to $|z|$ and $\mu(z) \to 0$ as $|z| \to 1$. For a weight μ, the *Bloch-type space* $\mathcal{B}_\mu = \mathcal{B}_\mu(\mathbb{D})$ is the space of all $f \in H(\mathbb{D})$ such that

$$b_\mu(f) = \sup_{z \in \mathbb{D}} \mu(z)|f'(z)| < \infty.$$

The *little Bloch-type space* $\mathcal{B}_{\mu,0} = \mathcal{B}_{\mu,0}(\mathbb{D})$ consists of all $f \in \mathcal{B}_\mu$ such that

$$\lim_{|z| \to 1} \mu(z)|f'(z)| = 0.$$

The space $\mathcal{B}_\mu$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}_\mu} = |f(0)| + b_\mu(f)$$

and if μ is a typical weight, then $\mathcal{B}_{\mu,0}$ is a closed subspace of $\mathcal{B}_\mu$. When $\mu(z) = (1 - |z|^2)^\alpha$, $\alpha > 0$, $\mathcal{B}_\mu$ reduces to the α-Bloch space, denoted by $\mathcal{B}^\alpha$, whereas $\mathcal{B}_{\mu,0}$ reduces to the little α-Bloch space $\mathcal{B}_0^\alpha$. For some information on Bloch-type spaces, see, for example, [1, 2].

Likewise, for a weight μ, the *weighted-type space* $H_\mu^\infty = H_\mu^\infty(\mathbb{D})$ consists of all $f \in H(\mathbb{D})$ such that

$$\sup_{z \in \mathbb{D}} \mu(z) |f(z)| < \infty,$$

and the *little weighted-type space* $H_{\mu,0}^\infty = H_{\mu,0}^\infty(\mathbb{D})$ consists of all $f \in H_\mu^\infty$ such that

$$\lim_{|z| \to 1} \mu(z) |f(z)| = 0$$

(see, e.g., [3]).

For $0 < p < \infty$, $-2 < q < \infty$, and $0 \le s < \infty$, the spaces $F(p,q,s)$ and $F_0(p,q,s)$ are defined as the sets of all $f \in H(\mathbb{D})$ such that

$$M(f) = \left(\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\eta_a(z)|^2)^s \, dA(z) \right)^{1/p} < \infty$$

and such that

$$\lim_{|a| \to 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\eta_a(z)|^2)^s \, dA(z) = 0$$

for $0 < s < \infty$, respectively, whereas if $s = 0$, then $F(p,q,0) = F_0(p,q,0)$ is defined naturally as the set of all $f \in H(\mathbb{D})$ such that

$$M(f) = \left(\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q \, dA(z) \right)^{1/p} < \infty.$$

The spaces $F(p,q,s)$ and $F_0(p,q,s)$ are known as general families of function spaces. For $1 \le p < \infty$, $F(p,q,s)$ is a Banach space with respect to the norm

$$\|f\|_{F(p,q,s)} = |f(0)| + M(f),$$

and $F_0(p,q,s)$ is a closed subspace of $F(p,q,s)$. The importance of these spaces stems from the fact that for appropriate parameter values of p, q, and s, they coincide with several classical function spaces. For example, $F(2,1,0)$ is the Hardy space H^2, $F(p, p + \alpha, 0)$, $\alpha > -1$, is the weighted Bergman space A_α^p, $F(p,q,s) = \mathcal{B}^{\frac{2+q}{p}}$ and $F_0(p,q,s) = \mathcal{B}_0^{\frac{2+q}{p}}$ for $s > 1$, $F(p,q,s) \subset \mathcal{B}^{\frac{2+q}{p}}$ and $F_0(p,q,s) \subset \mathcal{B}_0^{\frac{2+q}{p}}$ for $0 < s \le 1$, $F(2,0,p) = Q_p$ and $F_0(2,0,p) = Q_{p,0}$, and $F(2,0,1) = BMOA$, the space of analytic functions with bounded mean oscillation, and $F_0(2,0,1) = VMOA$, the space of analytic functions with vanishing mean oscillation.

If $q + s \leq -1$, $F(p, q, s)$ is trivial, that is, equal to the space of constant functions, whereas for $q + s > -1$, it is nontrivial [4].

Let $\varphi \in S(\mathbb{D})$. The composition operator C_φ induced by φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

For $\psi \in H(\mathbb{D})$, the multiplication operator M_ψ is defined on $H(\mathbb{D})$ by

$$M_\psi f(z) = \psi(z) f(z), \quad f \in H(\mathbb{D}).$$

The following product of these two operators $W_{\varphi, \psi} = M_\psi \circ C_\varphi$, the so-called *weighted composition operator*, has been studied a lot recently.

The differentiation operator denoted by D is defined by

$$Df = f', \quad f \in H(\mathbb{D}).$$

At first, the experts studied operator-theoretic properties of these operators on spaces of holomorphic functions in terms of their symbols separately. A systematic study of products of concrete linear operators between spaces of holomorphic functions started approximately a decade ago; see, for example, [5–32], and the related references therein. The first product-type operators different from weighted composition operators, which have been considerably studied, are the products of composition and differentiation operators (see, e.g., [6, 7, 9, 10, 13, 15, 16, 20, 25, 28] and the references therein). Quite recently, there appeared some more complex product-type operators that include some of classical operators, such as composition, differentiation, multiplication, or integral-type operators (see, e.g., [17–19, 28, 30, 33–35] and the related references therein).

The product-type operators consisting of exactly one composition, multiplication, and differentiation operator are the following:

$$
\begin{aligned}
(M_\psi C_\varphi D f)(z) &= \psi(z) f'\big(\varphi(z)\big), \\
(M_\psi D C_\varphi f)(z) &= \psi(z) \varphi'(z) f'\big(\varphi(z)\big), \\
(C_\varphi M_\psi D f)(z) &= \psi\big(\varphi(z)\big) f'\big(\varphi(z)\big), \\
(D M_\psi C_\varphi f)(z) &= \psi'(z) f\big(\varphi(z)\big) + \psi(z) \varphi'(z) f'\big(\varphi(z)\big), \\
(C_\varphi D M_\psi f)(z) &= \psi'\big(\varphi(z)\big) f\big(\varphi(z)\big) + \psi\big(\varphi(z)\big) f'\big(\varphi(z)\big), \\
(D C_\varphi M_\psi f)(z) &= \psi'\big(\varphi(z)\big) \varphi'(z) f\big(\varphi(z)\big) + \psi\big(\varphi(z)\big) \varphi'(z) f'\big(\varphi(z)\big),
\end{aligned}
\tag{1}
$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$.

To treat the operators in (1) in a unified manner, Stević et al. [23, 24] introduced a generalized operator and studied it on the weighted Bergman spaces. Motivated by that operator, here we introduce the operator

$$T^n_{\psi_1, \psi_2, \varphi} f(z) = \psi_1(z) f^{(n)}\big(\varphi(z)\big) + \psi_2(z) f^{(n+1)}\big(\varphi(z)\big), \quad f \in H(\mathbb{D}), \tag{2}$$

where $\psi_1, \psi_2 \in H(\mathbb{D})$, $\varphi \in S(\mathbb{D})$, and $n \in \mathbb{N}_0$. It is worth noticing that, among others, the composition operator, multiplication operator, differentiation operator, as well as all the

products of composition, multiplication, and differentiation operators in (1), can be obtained from the operators $T^0_{\psi_1,\psi_2,\varphi}$ and $T^1_{\psi_1,\psi_2,\varphi}$ by fixing ψ_1 and ψ_2. More specifically, we have $C_\varphi = T^0_{1,0,\varphi}$, $M^0_\psi = T^0_{\psi,0,z}$, $D = T^1_{1,0,z} = T^0_{0,1,z}$, $M_\psi C_\varphi = T^0_{\psi,0,\varphi}$, $C_\varphi D = T^1_{1,0,\varphi} = T^0_{0,1,\varphi}$, $C_\varphi M_\psi = T^0_{\psi\circ\varphi,0,\varphi}$, $DC_\varphi = T^0_{0,\varphi',\varphi} = T^1_{\varphi',0,\varphi}$, $M_\psi D = T^0_{0,\psi,z} = T^1_{\psi,0,z}$, $DM_\psi = T^0_{\psi',\psi,z}$, $M_\psi C_\varphi D = T^0_{0,\psi,\varphi} = T^1_{\psi,0,\varphi}$, $M_\psi DC_\varphi f = T^0_{0,\psi\varphi',\varphi} = T^1_{\psi\varphi',0,\varphi}$, $C_\varphi M_\psi D = T^0_{0,\psi\circ\varphi,\varphi} = T^1_{\psi\circ\varphi,0,\varphi}$, $DM_\psi C_\varphi = T^0_{\psi',\psi\varphi,\varphi}$, $C_\varphi DM_\psi = T^0_{\psi'\circ\varphi,\psi\circ\varphi,\varphi}$, $DC_\varphi M_\psi = T^0_{(\psi'\circ\varphi)\varphi',(\psi\circ\varphi)\varphi',\varphi}$. Note also that, for $\psi_2 \equiv 0$, we obtain the *weighted differentiation composition operator*, which was studied, for example, in [17–19, 28, 30].

Our aim here is to characterize the boundedness and compactness of the operator $T^n_{\psi_1,\psi_2,\varphi}$ from the $F(p,q,s)$ and $F_0(p,q,s)$ spaces to Bloch-type spaces. The paper can be regarded as a continuation of our line of investigations in [6, 7, 9, 10, 15–24, 34, 35].

Throughout this paper, constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means that $B \lesssim A \lesssim B$, where $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$. We also use the standard convention $\prod_{j=k}^{k-1} a_j = 1$.

2 Boundedness and compactness of $T^n_{\psi_1,\psi_2,\varphi}$

In this section, we prove our main results. Namely, we characterize the boundedness and compactness of the operator $T^n_{\psi_1,\psi_2,\varphi}$ from the $F(p,q,s)$ and $F_0(p,q,s)$ spaces to Bloch-type spaces. We also give some upper and lower bounds for the norm of $T^n_{\psi_1,\psi_2,\varphi}$: $F(p,q,s)$ (or $F_0(p,q,s)) \to B_\mu$.

For this purpose, we need several lemmas. The next lemma can be found in [4].

Lemma 1 *Let* $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, *and* $f \in F(p,q,s)$. *Then*

$$\|f\|_{\mathcal{B}^{\frac{2+q}{p}}} \lesssim \|f\|_{F(p,q,s)}.$$

Moreover, if $f \in F_0(p,q,s)$, *then* $f \in \mathcal{B}_0^{\frac{2+q}{p}}$.

The following folklore point-evaluation result can be found, for example, in [2].

Lemma 2 *Let* $\alpha > 0$ *and* $f \in \mathcal{B}^\alpha$. *Then*

$$|f(z)| \lesssim \begin{cases} \|f\|_{\mathcal{B}^\alpha}, & 0 < \alpha < 1, \\ \|f\|_{\mathcal{B}^\alpha} \ln \frac{2}{1-|z|^2}, & \alpha = 1, \\ \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}}, & \alpha > 1, \end{cases}$$

and

$$|f^{(n)}(z)| \lesssim \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+n-1}}$$

for each $n \in \mathbb{N}$.

The following lemma can easily be obtained by combining the inequalities in Lemmas 1 and 2.

Lemma 3 *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, and $f \in F(p,q,s)$. Then*

$$\left|f(z)\right| \lesssim \begin{cases} \|f\|_{F(p,q,s)}, & p > q + 2, \\ \|f\|_{F(p,q,s)} \ln \frac{2}{1-|z|^2}, & p = 2 + q, \\ \dfrac{\|f\|_{F(p,q,s)}}{(1-|z|^2)^{\frac{2+q}{p}-1}}, & p < 2 + q, \end{cases}$$

and

$$\left|f^{(n)}(z)\right| \lesssim \frac{\|f\|_{F(p,q,s)}}{(1-|z|^2)^{\frac{2+q}{p}+n-1}}, \quad n \in \mathbb{N}.$$

The following lemma gives us important test functions belonging to the $F_0(p,q,s)$ space.

Lemma 4 *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, and*

$$f_w(z) = \frac{1 - |w|^2}{(1 - \overline{w}z)^{\frac{2+q}{p}}}, \quad z, w \in \mathbb{D}.$$

Then $\sup_{w \in \mathbb{D}} \|f_w\|_{F(p,q,s)} < \infty$ and $f_w \in F_0(p,q,s)$ for every $w \in \mathbb{D}$.

Proof First, $\sup_{w \in \mathbb{D}} \|f_w\|_{F(p,q,s)} < \infty$ was proved in [36]. Second, $f_w \in F_0(p,q,s)$ for every $w \in \mathbb{D}$ is possibly a known statement too; however, we have not managed to find it in the literature, so we give a proof. Let

$$I_f(a) = \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1 - |z|^2\right)^q \left(1 - \left|\eta_a(z)\right|^2\right)^s dA(z).$$

If $s > 1$, then there is $\varepsilon \in (0, s)$ such that $2 + q - \varepsilon > 0$. Then by Proposition 1.4.10 in [37] with $n = 1$ and some elementary inequalities, since $s + \varepsilon - 2 > -1$ and $s - \varepsilon > 0$, we have that, for $|a|$ close to 1,

$$\begin{aligned}
I_{f_w}(a) &= \left(\frac{2+q}{p}\right)^p |w|^p \int_{\mathbb{D}} \frac{(1-|w|^2)^p}{|1-\overline{w}z|^{2+q+p}} \left(1-|z|^2\right)^q \left(1-\left|\eta_a(z)\right|^2\right)^s dA(z) \\
&\lesssim \int_{\mathbb{D}} \frac{(1-|a|^2)^s (1-|z|^2)^{q+s}}{|1-\overline{w}z|^{2+q} |1-\overline{a}z|^{2s}} dA(z) \\
&\lesssim \frac{(1-|a|^2)^s}{(1-|w|)^\varepsilon} \int_{\mathbb{D}} \frac{(1-|z|^2)^{s+\varepsilon-2}}{|1-\overline{a}z|^{2s}} dA(z) \\
&\lesssim \frac{(1-|a|^2)^\varepsilon}{(1-|w|)^\varepsilon} \to 0 \quad \text{as } |a| \to 1^-,
\end{aligned} \tag{3}$$

from which the statement follows in this case.

Now assume that $0 < s \leq 1$. Let u and u' be chosen such that

$$\max\left\{1, \frac{1}{q+2}\right\} < u < \frac{1}{1-s} \quad \text{and} \quad \frac{1}{u} + \frac{1}{u'} = 1,$$

and let $v = q + \frac{2}{u'}$ (for $s = 1$, we set $1/(1-s) = +\infty$).

Let $\varepsilon \in (0, s)$ be such that

$$(\nu - \varepsilon)u > -1 \quad \text{and} \quad (q + s + \varepsilon - \nu)u' > -1 \tag{4}$$

(it is not difficult to see that such $\varepsilon > 0$ exists). Then, using the Hölder inequality, Proposition 1.4.10 in [37] with $n = 1$, (4), and the assumption $s - \varepsilon > 0$, we have

$$
\begin{aligned}
I_{f_w}(a) &\lesssim \int_{\mathbb{D}} \frac{(1-|w|^2)^p (1-|a|^2)^s (1-|z|^2)^{q+s}}{|1-\overline{w}z|^{2+q+p}|1-\overline{a}z|^{2s}} \, dA(z) \\
&= \int_{\mathbb{D}} \frac{(1-|w|^2)^p (1-|a|^2)^s (1-|z|^2)^{(q+s+\varepsilon-\nu)+\nu-\varepsilon}}{|1-\overline{w}z|^{2+q+p}|1-\overline{a}z|^{2s}} \, dA(z) \\
&\leq (1-|w|^2)^p \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{(\nu-\varepsilon)u}}{|1-\overline{w}z|^{(2+q+p)u}} \, dA(z) \right)^{1/u} \\
&\quad \times (1-|a|^2)^s \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{(q+s+\varepsilon-\nu)u'}}{|1-\overline{a}z|^{2su'}} \, dA(z) \right)^{1/u'} \\
&\lesssim \frac{(1-|a|^2)^\varepsilon}{(1-|w|)^\varepsilon} \to 0, \quad \text{as } |a| \to 1^-,
\end{aligned}
$$

from which the statement follows in this case. □

Our first result gives some characterizations for the boundedness of the operator $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$.

Theorem 1 *Let* $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, $n \in \mathbb{N}_0$, μ *be a typical weight, and* $\varphi \in S(\mathbb{D})$. *Then the following statements are true.*

(i) *If* $n \in \mathbb{N}$, *or* $n = 0$ *and* $p < 2 + q$, *then* $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ *is bounded if and only if*

(a) $M_1 := \sup_{z \in \mathbb{D}} \dfrac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} < \infty$,

(b) $M_2 := \sup_{z \in \mathbb{D}} \dfrac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} < \infty$, *and*

(c) $M_3 := \sup_{z \in \mathbb{D}} \dfrac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} < \infty$.

Moreover, the following asymptotic relations hold:

$$
\begin{aligned}
M_1 + M_2 + M_3 &\lesssim \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F(p,q,s) \, (or \, F_0(p,q,s)) \to \mathcal{B}_\mu} \\
&\lesssim M_1 + M_2 + M_3 + \frac{|\psi_1(0)|}{(1-|\varphi(0)|^2)^{\frac{2+q}{p}+n-1}} + \frac{|\psi_2(0)|}{(1-|\varphi(0)|^2)^{\frac{2+q}{p}+n}}.
\end{aligned} \tag{5}
$$

(ii) *If* $p > 2 + q$, *then* $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ *is bounded if and only if* $\psi_1 \in \mathcal{B}_\mu$,

(d) $M_4 := \sup_{z \in \mathbb{D}} \dfrac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}} < \infty$, *and*

(e) $M_5 := \sup_{z \in \mathbb{D}} \dfrac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+1}} < \infty$.

Moreover, the following asymptotic relations hold:

$$b_\mu(\psi_1) + M_4 + M_5 \lesssim \left\| T^0_{\psi_1,\psi_2,\varphi} \right\|_{F(p,q,s) \ (or \ F_0(p,q,s)) \to \mathcal{B}_\mu}$$

$$\lesssim b_\mu(\psi_1) + M_4 + M_5 + \left| \psi_1(0) \right| + \frac{|\psi_2(0)|}{(1 - |\varphi(0)|^2)^{\frac{2+q}{p}}}. \tag{6}$$

(iii) *If $p = 2 + q$ and $s > 1$, then $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s) \ (or \ F_0(p,q,s)) \to \mathcal{B}_\mu$ is bounded if and only if*

(f) $M_6 := \sup_{z \in \mathbb{D}} \mu(z)|\psi_1'(z)| \ln(\frac{2}{1-|\varphi(z)|^2}) < \infty,$

(g) $M_7 := \sup_{z \in \mathbb{D}} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{1-|\varphi(z)|^2} < \infty,$ *and*

(h) $M_8 := \sup_{z \in \mathbb{D}} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2} < \infty.$

Moreover, the following asymptotic relations hold:

$$M_6 + M_7 + M_8 \lesssim \left\| T^0_{\psi_1,\psi_2,\varphi} \right\|_{F(p,q,s) \ (or \ F_0(p,q,s)) \to \mathcal{B}_\mu}$$

$$\lesssim M_6 + M_7 + M_8 + \left| \psi_1(0) \right| \ln\left(\frac{2}{1-|\varphi(0)|^2} \right) + \frac{|\psi_2(0)|}{1-|\varphi(0)|^2}. \tag{7}$$

Proof (i) Suppose that $n \in \mathbb{N}$, or $n = 0$ and $p < 2 + q$, and that conditions (a), (b), and (c) hold. Then by Lemma 3, for arbitrary $z \in \mathbb{D}$ and $f \in F(p,q,s)$, we have

$$\mu(z)\left| \left(T^n_{\psi_1,\psi_2,\varphi}f \right)'(z) \right| = \mu(z)\left| \psi_1'(z)f^{(n)}(\varphi(z)) + \left(\psi_1(z)\varphi'(z) + \psi_2'(z) \right)f^{(n+1)}(\varphi(z)) \right.$$

$$\left. + \psi_2(z)\varphi'(z)f^{(n+2)}(\varphi(z)) \right|$$

$$\lesssim \mu(z)\left(\frac{|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} + \frac{|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n}} \right.$$

$$\left. + \frac{|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} \right) \|f\|_{F(p,q,s)}. \tag{8}$$

On the other hand, we have

$$\left| \left(T^n_{\psi_1,\psi_2,\varphi}f \right)(0) \right| = \left| \psi_1(0)f^{(n)}(\varphi(0)) + \psi_2(0)f^{(n+1)}(\varphi(0)) \right|$$

$$\lesssim \left(\frac{|\psi_1(0)|}{(1 - |\varphi(0)|^2)^{\frac{2+q}{p}+n-1}} + \frac{|\psi_2(0)|}{(1 - |\varphi(0)|^2)^{\frac{2+q}{p}+n}} \right) \|f\|_{F(p,q,s)}. \tag{9}$$

Hence, from (8) and (9) we obtain that $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s) \ (or \ F_0(p,q,s)) \to \mathcal{B}_\mu$ is bounded and

$$\left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F(p,q,s) \ (or \ F_0(p,q,s)) \to \mathcal{B}_\mu} \lesssim M_1 + M_2 + M_3$$

$$+ \frac{|\psi_1(0)|}{(1 - |\varphi(0)|^2)^{\frac{2+q}{p}+n-1}} + \frac{|\psi_2(0)|}{(1 - |\varphi(0)|^2)^{\frac{2+q}{p}+n}}. \tag{10}$$

Conversely, suppose that $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s) \ (or \ F_0(p,q,s)) \to \mathcal{B}_\mu$ is bounded. First note that if $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is bounded, then $T^n_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_\mu$ is bounded, and

$$\left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s) \to \mathcal{B}_\mu} \le \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F(p,q,s) \to \mathcal{B}_\mu}. \tag{11}$$

Hence, if $T^n_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_\mu$ is bounded, then we have

$$\left\| T^n_{\psi_1,\psi_2,\varphi} f \right\|_{\mathcal{B}_\mu} \lesssim \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s) \to \mathcal{B}_\mu} \|f\|_{F(p,q,s)} \tag{12}$$

for every $f \in F_0(p,q,s)$.

Taking $f(z) = z^n/n! \in F_0(p,q,s)$ in (12), we obtain that

$$\sup_{z \in \mathbb{D}} \mu(z) \left| \psi'_1(z) \right| \lesssim \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s) \to \mathcal{B}_\mu}. \tag{13}$$

Taking $f(z) = z^{n+1}/(n+1)! \in F_0(p,q,s)$ in (12) and using (13) and the fact that $|\varphi(z)| < 1$, we have

$$\sup_{z \in \mathbb{D}} \mu(z) \left| \psi_1(z)\varphi'(z) + \psi'_2(z) \right| \lesssim \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s) \to \mathcal{B}_\mu} + \sup_{z \in \mathbb{D}} \mu(z) \left| \psi'_1(z) \right|$$

$$\lesssim \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s) \to \mathcal{B}_\mu}. \tag{14}$$

Taking $f(z) = z^{n+2}/(n+2)! \in F_0(p,q,s)$ in (12) and using (13), (14), and the fact that $|\varphi(z)| < 1$, we easily get

$$\sup_{z \in \mathbb{D}} \mu(z) \left| \psi_2(z)\varphi'(z) \right| \lesssim \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s) \to \mathcal{B}_\mu}. \tag{15}$$

For $\zeta \in \mathbb{D}$ and $n \in \mathbb{N}_0$, consider the family of functions

$$f_\zeta(z) = \prod_{j=1}^n \left(\frac{2+q}{p} + j \right)\left(\frac{2+q}{p} + j + 1 \right) \frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}}}$$

$$- 2 \prod_{j=1}^n \left(\frac{2+q}{p} + j - 1 \right)\left(\frac{2+q}{p} + j + 1 \right) \frac{(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+1}}$$

$$+ \prod_{j=1}^n \left(\frac{2+q}{p} + j - 1 \right)\left(\frac{2+q}{p} + j \right) \frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+2}}.$$

Then by Lemma 4 it is easy to see that $\sup_{\zeta \in \mathbb{D}} \|f_\zeta\|_{F(p,q,s)} \lesssim 1$ and that $f_\zeta \in F_0(p,q,s)$ for every $\zeta \in \mathbb{D}$.

We also have

$$f_\zeta^{(n)}(z) = \prod_{j=1}^n \left(\frac{2+q}{p} + j - 1 \right)\left(\frac{2+q}{p} + j \right)\left(\frac{2+q}{p} + j + 1 \right)\left(\overline{\varphi(\zeta)} \right)^n$$

$$\times \left(\frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n}} - 2 \frac{(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+1}} + \frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+2}} \right),$$

$$f_\zeta^{(n+1)}(z) = \prod_{j=1}^n \left(\frac{2+q}{p} + j - 1 \right)\left(\frac{2+q}{p} + j \right)\left(\frac{2+q}{p} + j + 1 \right)\left(\overline{\varphi(\zeta)} \right)^{n+1}$$

$$\times \left(\left(\frac{2+q}{p} + n \right) \frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+1}} \right.$$

$$- 2\left(\frac{2+q}{p} + n + 1\right)\frac{(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p} + n + 2}}$$

$$+ \left(\frac{2+q}{p} + n + 2\right)\frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p} + n + 3}}\Bigg),$$

and

$$f_\zeta^{(n+2)}(z) = \prod_{j=1}^{n}\left(\frac{2+q}{p} + j - 1\right)\left(\frac{2+q}{p} + j\right)\left(\frac{2+q}{p} + j + 1\right)\big(\overline{\varphi(\zeta)}\big)^{n+2}$$

$$\times \Bigg(\left(\left(\frac{2+q}{p} + n\right)\left(\frac{2+q}{p} + n + 1\right)\frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p} + n + 2}}\right.$$

$$- 2\left(\frac{2+q}{p} + n + 1\right)\left(\frac{2+q}{p} + n + 2\right)\frac{(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p} + n + 3}}$$

$$+ \left(\frac{2+q}{p} + n + 2\right)\left(\frac{2+q}{p} + n + 3\right)\frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p} + n + 4}}\Bigg).$$

Hence,

$$f_\zeta^{(n)}\big(\varphi(\zeta)\big) = 0, \qquad f_\zeta^{(n+1)}\big(\varphi(\zeta)\big) = 0,$$

and

$$f_\zeta^{(n+2)}\big(\varphi(\zeta)\big) = \prod_{j=1}^{n}\left(\frac{2+q}{p} + j - 1\right)\left(\frac{2+q}{p} + j\right)\left(\frac{2+q}{p} + j + 1\right)\frac{2\big(\overline{\varphi(\zeta)}\big)^{n+2}}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n + 1}},$$

which, along with the boundedness of the operator, implies that

$$\big\| T^n_{\psi_1, \psi_2, \varphi}\big\|_{F_0(p,q,s) \to \mathcal{B}_\mu} \gtrsim \big\| T^n_{\psi_1, \psi_2, \varphi} f_\zeta\big\|_{\mathcal{B}_\mu}$$

$$\gtrsim \mu(\zeta)\big|\psi_1'(\zeta)f_\zeta^{(n)}\big(\varphi(\zeta)\big) + \big(\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)\big)f_\zeta^{(n+1)}\big(\varphi(\zeta)\big)$$

$$+ \psi_2(\zeta)\varphi'(\zeta)f_\zeta^{(n+2)}\big(\varphi(\zeta)\big)\big|$$

$$= C_1(p,q,n)\mu(\zeta)\frac{|\psi_2(\zeta)\varphi'(\zeta)(\varphi(\zeta))^{n+2}|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n + 1}}, \tag{16}$$

where

$$C_1(p,q,n) = 2\prod_{j=1}^{n}\left(\frac{2+q}{p} + j - 1\right)\left(\frac{2+q}{p} + j\right)\left(\frac{2+q}{p} + j + 1\right).$$

From (15) and (16), for $\delta \in (0,1)$, we obtain

$$\sup_{\zeta \in \mathbb{D}} \frac{\mu(\zeta)|\psi_2(\zeta)\varphi'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n + 1}}$$

$$\leq \sup_{|\varphi(\zeta)| > \delta} \frac{\mu(\zeta)|\psi_2(\zeta)\varphi'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n + 1}} + \sup_{|\varphi(\zeta)| \leq \delta} \frac{\mu(\zeta)|\psi_2(\zeta)\varphi'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n + 1}}$$

$$\leq \frac{1}{\delta^{n+2}} \sup_{|\varphi(\zeta)|>\delta} \frac{\mu(\zeta)|\psi_2(\zeta)\varphi'(\zeta)(\varphi(\zeta))^{n+2}|}{(1-|\varphi(\zeta)|^2)^{\frac{2+q}{p}+n+1}}$$

$$+ \frac{1}{(1-\delta^2)^{\frac{2+q}{p}+n+1}} \sup_{|\varphi(\zeta)|\leq\delta} \mu(\zeta)|\psi_2(\zeta)\varphi'(\zeta)|$$

$$\lesssim \left(\frac{1}{\delta^{n+2}} + \frac{1}{(1-\delta^2)^{\frac{2+q}{p}+n+1}} \right) \| T_{\psi_1,\psi_2,\varphi}^n \|_{F_0(p,q,s)\to\mathcal{B}_\mu}.$$

From this it follows that (c) holds and, moreover,

$$M_3 \lesssim \| T_{\psi_1,\psi_2,\varphi}^n \|_{F_0(p,q,s)\to\mathcal{B}_\mu}. \tag{17}$$

For $\zeta \in \mathbb{D}$ and $n \in \mathbb{N}_0$, consider the family of functions

$$g_\zeta(z) = \left(\frac{2+q}{p} + n + 2 \right) \prod_{j=1}^n \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{1-|\varphi(\zeta)|^2}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}}}$$

$$- \left(2\frac{(2+q)}{p} + 2n + 3 \right) \prod_{j=1}^n \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{(1-|\varphi(\zeta)|^2)^2}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+1}}$$

$$+ \left(\frac{2+q}{p} + n + 1 \right) \prod_{j=1}^n \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \frac{(1-|\varphi(\zeta)|^3)^3}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+2}}.$$

Then by Lemma 4 it is easy to see that $\sup_{\zeta\in\mathbb{D}} \|g_\zeta\|_{F(p,q,s)} \lesssim 1$ and that $g_\zeta \in F_0(p,q,s)$ for every $\zeta \in \mathbb{D}$.

We also have

$$g_\zeta^{(n)}(z) = \prod_{j=1}^n \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) (\overline{\varphi(\zeta)})^n$$

$$\times \left(\left(\frac{2+q}{p} + n + 2 \right) \frac{1-|\varphi(\zeta)|^2}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n}} \right.$$

$$- \left(2\frac{(2+q)}{p} + 2n + 3 \right) \frac{(1-|\varphi(\zeta)|^2)^2}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+1}}$$

$$\left. + \left(\frac{2+q}{p} + n + 1 \right) \frac{(1-|\varphi(\zeta)|^3)^3}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+2}} \right),$$

$$g_\zeta^{(n+1)}(z) = \prod_{j=1}^n \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) (\overline{\varphi(\zeta)})^{n+1}$$

$$\times \left(\left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 2 \right) \frac{1-|\varphi(\zeta)|^2}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+1}} \right.$$

$$- \left(\frac{2+q}{p} + n + 1 \right) \left(2\frac{(2+q)}{p} + 2n + 3 \right) \frac{(1-|\varphi(\zeta)|^2)^2}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+2}}$$

$$\left. + \left(\frac{2+q}{p} + n + 2 \right) \left(\frac{2+q}{p} + n + 1 \right) \frac{(1-|\varphi(\zeta)|^3)^3}{(1-\overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+3}} \right),$$

and

$$g_\zeta^{(n+2)}(z) = \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \left(\overline{\varphi(\zeta)} \right)^{n+2}$$

$$\times \left(\frac{2+q}{p} + n + 1 \right) \left(\frac{2+q}{p} + n + 2 \right) \left(\left(\frac{2+q}{p} + n \right) \frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+2}} \right.$$

$$- \left(2\frac{(2+q)}{p} + 2n + 3 \right) \frac{(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+3}}$$

$$+ \left. \left(\frac{2+q}{p} + n + 3 \right) \frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+4}} \right).$$

Hence,

$$g_\zeta^{(n)}\big(\varphi(\zeta)\big) = 0, \qquad g_\zeta^{(n+2)}\big(\varphi(\zeta)\big) = 0,$$

and

$$g_\zeta^{(n+1)}\big(\varphi(\zeta)\big) = -\prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{(\overline{\varphi(\zeta)})^{n+1}}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n}},$$

which, along with the boundedness of the operator $T_{\psi_1,\psi_2,\varphi}^n : F_0(p,q,s) \to \mathcal{B}_\mu$, implies that

$$\big\| T_{\psi_1,\psi_2,\varphi}^n \big\|_{F_0(p,q,s) \to \mathcal{B}_\mu}$$

$$\gtrsim \big\| T_{\psi_1,\psi_2,\varphi}^n g_\zeta \big\|_{\mathcal{B}_\mu}$$

$$\gtrsim \mu(\zeta) \big| \psi_1'(\zeta) g_\zeta^{(n)}\big(\varphi(\zeta)\big) + \big(\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)\big) g^{(n+1)}\big(\varphi(\zeta)\big)$$

$$+ \psi_2(\zeta)\varphi'(\zeta) g^{(n+2)}\big(\varphi(\zeta)\big) \big|$$

$$= C_2(p,q,n)\mu(\zeta) \frac{|\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)||\varphi(\zeta)|^{n+1}}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n}}, \tag{18}$$

where

$$C_2(p,q,n) = \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right).$$

From (14) and (18), for $\delta \in (0,1)$, we obtain

$$\sup_{\zeta \in \mathbb{D}} \frac{\mu(\zeta)|\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n}}$$

$$\leq \sup_{|\varphi(\zeta)|>\delta} \frac{\mu(\zeta)|\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n}} + \sup_{|\varphi(\zeta)|\leq\delta} \frac{\mu(\zeta)|\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n}}$$

$$\leq \frac{1}{\delta^{n+1}} \sup_{|\varphi(\zeta)|>\delta} \frac{\mu(\zeta)|\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)||\varphi(\zeta)|^{n+1}}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n}}$$

$$+ \frac{1}{(1-\delta^2)^{\frac{2+q}{p}+n}} \sup_{|\varphi(\zeta)| \leq \delta} \mu(\zeta) |\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)|$$

$$\lesssim \left(\frac{1}{\delta^{n+1}} + \frac{1}{(1-\delta^2)^{\frac{2+q}{p}+n}} \right) \| T_{\psi_1,\psi_2,\varphi}^n \|_{F_0(p,q,s) \to \mathcal{B}_\mu}.$$

From this it follows that (b) holds and, moreover,

$$M_2 \lesssim \| T_{\psi_1,\psi_2,\varphi}^n \|_{F_0(p,q,s) \to \mathcal{B}_\mu}. \tag{19}$$

Finally, for $\zeta \in \mathbb{D}$ and $n \in \mathbb{N}_0$, consider the family of functions

$$h_\zeta(z) = \left(\frac{2+q}{p} + n + 1 \right) \left(\frac{2+q}{p} + n + 2 \right)$$

$$\times \prod_{j=1}^{n} \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}}}$$

$$- 2 \left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 2 \right)$$

$$\times \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+1}}$$

$$+ \left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 1 \right)$$

$$\times \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+2}}.$$

Then, by Lemma 4 it is easy to see that $\sup_{\zeta \in \mathbb{D}} \|h_\zeta\|_{F(p,q,s)} \lesssim 1$ and that $h_\zeta \in F_0(p,q,s)$ for every $\zeta \in \mathbb{D}$. We also have

$$h_\zeta^{(n)}(z) = \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \left(\overline{\varphi(\zeta)} \right)^n$$

$$\times \left(\left(\frac{2+q}{p} + n + 1 \right) \left(\frac{2+q}{p} + n + 2 \right) \frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n}} \right.$$

$$- 2 \left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 2 \right) \frac{(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+1}}$$

$$+ \left. \left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 1 \right) \frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+2}} \right),$$

$$h_\zeta^{(n+1)}(z) = \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \left(\overline{\varphi(\zeta)} \right)^{n+1}$$

$$\times \left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 1 \right) \left(\frac{2+q}{p} + n + 2 \right)$$

$$\times \left(\frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+1}} - \frac{2(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+2}} + \frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p}+n+3}} \right),$$

and

$$h_\zeta^{(n+2)}(z) = \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \overline{(\varphi(\zeta))}^{n+2}$$

$$\times \left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 1 \right) \left(\frac{2+q}{p} + n + 2 \right)$$

$$\times \left(\left(\frac{2+q}{p} + n + 1 \right) \frac{1 - |\varphi(\zeta)|^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p} + n + 2}} \right.$$

$$- 2 \left(\frac{2+q}{p} + n + 2 \right) \frac{(1 - |\varphi(\zeta)|^2)^2}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p} + n + 3}}$$

$$\left. + \left(\frac{2+q}{p} + n + 3 \right) \frac{(1 - |\varphi(\zeta)|^3)^3}{(1 - \overline{\varphi(\zeta)}z)^{\frac{2+q}{p} + n + 4}} \right).$$

Hence,

$$h_\zeta^{(n+1)}(\varphi(\zeta)) = 0, \qquad h_\zeta^{(n+2)}(\varphi(\zeta)) = 0,$$

and

$$h_\zeta^{(n)}(\varphi(\zeta)) = C_3(p,q,n) \frac{(\overline{\varphi(\zeta)})^n}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n - 1}},$$

where

$$C_3(p,q,n) = 2 \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right).$$

Thus, using the boundedness of the operator, we have

$$\left\| T_{\psi_1,\psi_2,\varphi}^n \right\|_{F_0(p,q,s) \to \mathcal{B}_\mu} \gtrsim \left\| T_{\psi_1,\psi_2,\varphi}^n h_\zeta \right\|_{\mathcal{B}_\mu}$$

$$\gtrsim \mu(\zeta) \left| \psi_1'(\zeta) h_\zeta^{(n)}(\varphi(\zeta)) + (\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)) h_\zeta^{(n+1)}(\varphi(\zeta)) \right.$$

$$\left. + \psi_2(\zeta)\varphi'(\zeta) h_\zeta^{(n+2)}(\varphi(\zeta)) \right|$$

$$\geq C_3(p,q,n)\mu(\zeta) \frac{|\psi_1'(\zeta)(\varphi(\zeta))^n|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n - 1}}. \tag{20}$$

From (13) and (20), for $\delta \in (0,1)$, we obtain

$$\sup_{\zeta \in \mathbb{D}} \frac{\mu(\zeta)|\psi_1'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n - 1}}$$

$$\leq \sup_{|\varphi(\zeta)| > \delta} \frac{\mu(\zeta)|\psi_1'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n - 1}} + \sup_{|\varphi(\zeta)| \leq \delta} \frac{\mu(\zeta)|\psi_1'(\zeta)|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n - 1}}$$

$$\leq \frac{1}{\delta^n} \sup_{|\varphi(\zeta)| > \delta} \frac{\mu(\zeta)|\psi_1'(\zeta)(\varphi(\zeta))^n|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p} + n - 1}} + \frac{1}{(1 - \delta^2)^{\frac{2+q}{p} + n - 1}} \sup_{|\varphi(\zeta)| \leq \delta} \mu(\zeta)|\psi_1'(\zeta)|$$

$$\lesssim \left(\frac{1}{\delta^n} + \frac{1}{(1 - \delta^2)^{\frac{2+q}{p} + n - 1}} \right) \left\| T_{\psi_1,\psi_2,\varphi}^n \right\|_{F_0(p,q,s) \to \mathcal{B}_\mu}.$$

From this it follows that (a) holds and, moreover,

$$M_1 \lesssim \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s)\to\mathcal{B}_\mu}. \tag{21}$$

From (17), (19), and (21) we have

$$M_1 + M_2 + M_3 \lesssim \left\| T^n_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s)\to\mathcal{B}_\mu}. \tag{22}$$

Hence, from (10), (11), and (22) we have that (5) holds, as desired.

(ii) Assume that $n = 0$, $p > 2 + q$, $\psi_1 \in \mathcal{B}_\mu$, and that conditions (d) and (e) hold. If $f \in F(p,q,s)$, then by Lemma 3 we have

$$
\begin{aligned}
\mu(z)\big|\big(T^0_{\psi_1,\psi_2,\varphi}f\big)'(z)\big| &\leq \mu(z)\big|\psi_1'(z)\big|\big|f\big(\varphi(z)\big)\big| + \mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f'\big(\varphi(z)\big)\big| \\
&\quad + \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f''\big(\varphi(z)\big)\big| \\
&\lesssim \Bigg(\mu(z)\big|\psi_1'(z)\big| + \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}}} \\
&\quad + \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+1}} \Bigg) \|f\|_{F(p,q,s)} \\
&\lesssim \big(b_\mu(\psi_1) + M_4 + M_5\big)\|f\|_{F(p,q,s)}
\end{aligned}
\tag{23}
$$

and

$$
\begin{aligned}
\big|\big(T^0_{\psi_1,\psi_2,\varphi}f\big)(0)\big| &= \big|\psi_1(0)f\big(\varphi(0)\big) + \psi_2(0)f'\big(\varphi(0)\big)\big| \\
&\lesssim \Bigg(\big|\psi_1(0)\big| + \frac{|\psi_2(0)|}{(1 - |\varphi(0)|^2)^{\frac{2+q}{p}}} \Bigg)\|f\|_{F(p,q,s)}.
\end{aligned}
\tag{24}
$$

From (23) and (24) we see that $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is bounded and that

$$\big\| T^0_{\psi_1,\psi_2,\varphi} \big\|_{F(p,q,s) \ (or \ F_0(p,q,s))\to\mathcal{B}_\mu} \lesssim b_\mu(\psi_1) + M_4 + M_5 + \big|\psi_1(0)\big| + \frac{|\psi_2(0)|}{(1 - |\varphi(0)|^2)^{\frac{2+q}{p}}}. \tag{25}$$

Conversely, if $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is bounded, then $T^0_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_\mu$ is bounded, (11) and (12) hold, and by using the function $f(z) \equiv 1 \in F_0(p,q,s)$ therein we obtain that $\psi_1 \in \mathcal{B}_\mu$ and that (13) holds. From the proof of (i) for $n = 0$, we see that (d) and (e) hold, as well as the asymptotic relations $M_4 \lesssim \|T^0_{\psi_1,\psi_2,\varphi}\|_{F_0(p,q,s)\to\mathcal{B}_\mu}$ and $M_5 \lesssim \|T^0_{\psi_1,\psi_2,\varphi}\|_{F_0(p,q,s)\to\mathcal{B}_\mu}$. Hence,

$$b_\mu(\psi_1) + M_4 + M_5 \lesssim \big\| T^0_{\psi_1,\psi_2,\varphi} \big\|_{F_0(p,q,s)\to\mathcal{B}_\mu},$$

from which, along with (11) and (25), we see that (6) holds.

(iii) Assume that $n = 0$, $p = 2 + q$, $s > 1$, and that conditions (f), (g), and (h) hold. Assume that $f \in F(p,q,s)$. Then by Lemma 3 we have

$$
\begin{aligned}
\mu(z)\big|\big(T^0_{\psi_1,\psi_2,\varphi}f\big)'(z)\big| &\leq \mu(z)\big|\psi_1'(z)\big|\big|f\big(\varphi(z)\big)\big| + \mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2(z)\big|\big|f'\big(\varphi(z)\big)\big| \\
&\quad + \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f''\big(\varphi(z)\big)\big|
\end{aligned}
$$

$$\lesssim \left(\mu(z)\big|\psi_1'(z)\big| \ln\left(\frac{2}{1-|\varphi(z)|^2} \right) + \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{1-|\varphi(z)|^2} \right.$$

$$\left. + \frac{\mu(z)|\varphi'(z)\psi_2(z)|}{(1-|\varphi(z)|^2)^2} \right) \|f\|_{F(p,q,s)}$$

$$\lesssim (M_6 + M_7 + M_8)\|f\|_{F(p,q,s)} \tag{26}$$

and

$$\big|\big(T^0_{\psi_1,\psi_2,\varphi}f\big)(0)\big| = \big|\psi_1(0)f\big(\varphi(0)\big) + \psi_2(0)f'\big(\varphi(0)\big)\big|$$

$$\lesssim \left(\big|\psi_1(0)\big| \ln\left(\frac{2}{1-|\varphi(0)|^2} \right) + \frac{|\psi_2(0)|}{1-|\varphi(0)|^2} \right) \|f\|_{F(p,q,s)}. \tag{27}$$

From (26) and (27) we see that $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)$) $\to \mathcal{B}_\mu$ is bounded and that

$$\big\| T^0_{\psi_1,\psi_2,\varphi} \big\|_{F(p,q,s)\to\mathcal{B}_\mu} \lesssim M_6 + M_7 + M_8 + \big|\psi_1(0)\big| \ln\left(\frac{2}{1-|\varphi(0)|^2} \right) + \frac{|\psi_2(0)|}{1-|\varphi(0)|^2}. \tag{28}$$

Conversely, suppose that $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is bounded. Then $T^0_{\psi_1,\psi_2,\varphi} : F_0(p, q,s) \to \mathcal{B}_\mu$ is bounded too. For $\zeta \in \mathbb{D}$, let

$$f_\zeta(z) = \ln\left(\frac{2}{1-\overline{\varphi(\zeta)}z} \right).$$

Then

$$f_\zeta'(z) = \frac{\overline{\varphi(\zeta)}}{1-\overline{\varphi(\zeta)}z} \quad \text{and} \quad f_\zeta''(z) = \frac{(\overline{\varphi(\zeta)})^2}{(1-\overline{\varphi(\zeta)}z)^2}.$$

Easy calculation shows that $f_\zeta \in \mathcal{B}_0^1$, $\zeta \in \mathbb{D}$, and that there is $C > 0$ such that $\sup_{\zeta\in\mathbb{D}} \|f_\zeta\|_{\mathcal{B}} \le C$. Since $s > 1$, we have $f_\zeta \in F_0(p,q,s)$ and $\sup_{\zeta\in\mathbb{D}} \|f_\zeta\|_{F(p,q,s)} \le C$.

Therefore,

$$\big\| T^0_{\psi_1,\psi_2,\varphi} \big\|_{F_0(p,q,s)\to\mathcal{B}_\mu} \gtrsim \big\| T^0_{\psi_1,\psi_2,\varphi}f_\zeta \big\|_{\mathcal{B}_\mu} \ge \mu(\zeta)\big|\big(T^0_{\psi_1,\psi_2,\varphi}f_\zeta\big)'(\zeta)\big|$$

$$\ge \mu(\zeta)\big|\psi_1'(\zeta)f\big(\varphi(\zeta)\big) + \big(\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)\big)f'\big(\varphi(\zeta)\big)$$

$$+ \psi_2(\zeta)\varphi'(\zeta)f''\big(\varphi(\zeta)\big)\big|$$

$$= \mu(\zeta)\bigg|\psi_1'(\zeta)\ln\left(\frac{2}{1-|\varphi(\zeta)|^2} \right) + \psi_2(\zeta)\varphi'(\zeta)\frac{(\overline{\varphi(\zeta)})^2}{(1-|\varphi(\zeta)|^2)^2}$$

$$+ \big(\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)\big)\frac{\overline{\varphi(\zeta)}}{1-|\varphi(\zeta)|^2}\bigg| \tag{29}$$

for every $\zeta \in \mathbb{D}$, from which, along with the fact that $|\varphi(\zeta)| < 1$, $\zeta \in \mathbb{D}$, we get

$$\mu(\zeta)\big|\psi_1'(\zeta)\big| \ln\left(\frac{2}{1-|\varphi(\zeta)|^2} \right)$$

$$\lesssim \big\| T^0_{\psi_1,\psi_2,\varphi} \big\|_{F_0(p,q,s)\to\mathcal{B}_\mu} + \frac{|\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta)|}{1-|\varphi(\zeta)|^2} + \frac{|\psi_2(\zeta)\varphi'(\zeta)|}{(1-|\varphi(\zeta)|^2)^2}. \tag{30}$$

From the proof of (i) we see that (g) and (h) hold and that

$$M_7 \lesssim \left\| T^0_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s)\to\mathcal{B}_\mu} \quad \text{and} \quad M_8 \lesssim \left\| T^0_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s)\to\mathcal{B}_\mu}.$$

Taking the supremum over $\mathbb{D}$ in (30) and then using the last asymptotic relations, we see that

$$M_6 \lesssim \left\| T^0_{\psi_1,\psi_2,\varphi} \right\|_{F_0(p,q,s)\to\mathcal{B}_\mu}.$$

Hence, we have $M_6 + M_7 + M_8 \lesssim \| T^0_{\psi_1,\psi_2,\varphi} \|_{F_0(p,q,s)\to\mathcal{B}_\mu}$, which, along with (11) and (28), yields (7), completing the proof of the theorem. □

The next lemma is proved by using standard Schwartz's arguments in [38].

Lemma 5 *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, $n \in \mathbb{N}_0$, μ be a typical weight, and $\varphi \in S(\mathbb{D})$. Then $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)$) $\to \mathcal{B}_\mu$ is compact if and only if for every bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $F(p,q,s)$ (or $F_0(p,q,s)$) that converges to zero on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\| T^n_{\psi_1,\psi_2,\varphi} f_k \|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$.*

Theorem 2 *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, $n \in \mathbb{N}_0$, μ be a typical weight, and $\varphi \in S(\mathbb{D})$. Then the following statements are true.*

(i) If $n \in \mathbb{N}$, or $n = 0$ and $p < 2 + q$, then $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)$) $\to \mathcal{B}_\mu$ is compact if and only if $\psi_1 \in \mathcal{B}_\mu$, $\psi_1\varphi' + \psi_2' \in H^\infty_\mu$, $\psi_2\varphi' \in H^\infty_\mu$, and

(a) $\lim_{|\varphi(z)|\to 1} \dfrac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} = 0$,

(b) $\lim_{|\varphi(z)|\to 1} \dfrac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} = 0$, *and*

(c) $\lim_{|\varphi(z)|\to 1} \dfrac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} = 0$.

(ii) If $p > 2 + q$, then $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)$) $\to \mathcal{B}_\mu$ is compact if and only if $\psi_1 \in \mathcal{B}_\mu$, $\psi_1\varphi' + \psi_2' \in H^\infty_\mu$, $\psi_2\varphi' \in H^\infty_\mu$, and

(d) $\lim_{|\varphi(z)|\to 1} \dfrac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}} = 0$ *and*

(e) $\lim_{|\varphi(z)|\to 1} \dfrac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+1}} = 0$.

(iii) If $p = 2 + q$ and $s > 1$, then $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)$) $\to \mathcal{B}_\mu$ is compact if and only if $\psi_1 \in \mathcal{B}$, $\psi_1\varphi' + \psi_2' \in H^\infty_\mu$, $\psi_2\varphi' \in H^\infty_\mu$, and

(f) $\lim_{|\varphi(z)|\to 1} \mu(z)|\psi_1'(z)| \ln(\frac{2}{1-|\varphi(z)|^2}) = 0$,

(g) $\lim_{|\varphi(z)|\to 1} \dfrac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2} = 0$, *and*

(h) $\lim_{|\varphi(z)|\to 1} \dfrac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = 0$.

Proof (i) Suppose that $n \in \mathbb{N}$, or $n = 0$ and $p < 2 + q$, and $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)$) $\to \mathcal{B}_\mu$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a

sequence does not exist, then (a)-(c) obviously hold). Let

$$
f_k(z) = \prod_{j=1}^{n} \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}}}
$$
$$
- 2 \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{(1 - |\varphi(z_k)|^2)^2}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}+1}}
$$
$$
+ \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \frac{(1 - |\varphi(z_k)|^3)^3}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}+2}},
$$

$$
g_k(z) = \left(\frac{2+q}{p} + n + 2 \right) \prod_{j=1}^{n} \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}}}
$$
$$
- \left(2\frac{(2+q)}{p} + 2n + 3 \right) \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{(1 - |\varphi(z_k)|^2)^2}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}+1}}
$$
$$
+ \left(\frac{2+q}{p} + n + 1 \right) \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \frac{(1 - |\varphi(z_k)|^3)^3}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}+2}},
$$

and

$$
h_k(z) = \left(\frac{2+q}{p} + n + 1 \right) \left(\frac{2+q}{p} + n + 2 \right)
$$
$$
\times \prod_{j=1}^{n} \left(\frac{2+q}{p} + j \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}}}
$$
$$
- 2 \left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 2 \right)
$$
$$
\times \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j + 1 \right) \frac{(1 - |\varphi(z_k)|^2)^2}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}+1}}
$$
$$
+ \left(\frac{2+q}{p} + n \right) \left(\frac{2+q}{p} + n + 1 \right)
$$
$$
\times \prod_{j=1}^{n} \left(\frac{2+q}{p} + j - 1 \right) \left(\frac{2+q}{p} + j \right) \frac{(1 - |\varphi(z_k)|^3)^3}{(1 - \overline{\varphi(z_k)}z)^{\frac{2+q}{p}+2}}.
$$

From the proof of Theorem 1 we know that $(f_k)_{k \in \mathbb{N}}$, $(g_k)_{k \in \mathbb{N}}$, and $(h_k)_{k \in \mathbb{N}}$ are norm bounded sequences in $F_0(p, q, s)$, and it is easy to see that they converge to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Hence, by Lemma 5 we have

$$
\lim_{k \to \infty} \left\| T^n_{\psi_1, \psi_2, \varphi} f_k \right\|_{\mathcal{B}_\mu} = 0, \qquad \lim_{k \to \infty} \left\| T^n_{\psi_1, \psi_2, \varphi} g_k \right\|_{\mathcal{B}_\mu}, \quad \text{and}
$$
$$
\lim_{k \to \infty} \left\| T^n_{\psi_1, \psi_2, \varphi} h_k \right\|_{\mathcal{B}_\mu} = 0.
\tag{31}
$$

From (16), (18), and (20) it follows that

$$
\mu(\zeta) \frac{|\psi_2(\zeta)\varphi'(\zeta)(\varphi(\zeta))^{n+2}|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n+1}} \lesssim \left\| T^n_{\psi_1, \psi_2, \varphi} f_k \right\|_{\mathcal{B}_\mu},
\tag{32}
$$

$$\mu(\zeta)\frac{|(\psi_1(\zeta)\varphi'(\zeta) + \psi_2'(\zeta))(\varphi(\zeta))^{n+1}|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n}} \lesssim \|T^n_{\psi_1,\psi_2,\varphi}g_k\|_{\mathcal{B}_\mu}, \tag{33}$$

and

$$\mu(\zeta)\frac{|\psi_1'(\zeta)(\varphi(\zeta))^n|}{(1 - |\varphi(\zeta)|^2)^{\frac{2+q}{p}+n-1}} \lesssim \|T^n_{\psi_1,\psi_2,\varphi}h_k\|_{\mathcal{B}_\mu}. \tag{34}$$

Letting $k \to \infty$ in (32)-(34) and employing (31), we get (a), (b), and (c).

Conversely, assume that $\psi_1 \in \mathcal{B}_\mu$, $\psi_1\varphi' + \psi_2' \in H^\infty_\mu$, $\psi_2\varphi' \in H^\infty_\mu$, and that (a), (b), and (c) hold. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $F(p,q,s)$ (or $F_0(p,q,s)$) such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ and $\|f_k\|_{F(p,q,s)} \lesssim 1$. Then $f_k^{(n)}, f_k^{(n+1)}$, and $f_k^{(n+2)}$ converge to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$.

Since (a), (b), and (c) hold, for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} < \varepsilon, \qquad \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n}} < \varepsilon,$$

and

$$\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} < \varepsilon$$

when $\delta < |\varphi(z)| < 1$.

Since $\psi_1 \in \mathcal{B}_\mu$, $\psi_1\varphi' + \psi_2' \in H^\infty_\mu$, and $\psi_2\varphi' \in H^\infty_\mu$, we have $N_1 = \sup_{z\in\mathbb{D}} \mu(z)|\psi_1'(z)| < \infty$, $N_2 = \sup_{z\in\mathbb{D}} \mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)| < \infty$, and $N_3 = \sup_{z\in\mathbb{D}} \mu(z)|\psi_2(z)\varphi'(z)| < \infty$. This, together with Lemma 3, yields

$$\|T^n_{\psi_1,\psi_2,\varphi}f_k\|_{\mathcal{B}_\mu}$$

$$\leq \sup_{z\in\mathbb{D}} \mu(z)\big|\big(T^n_{\psi_1,\psi_2,\varphi}f_k\big)'(z)\big|$$

$$\leq \sup_{z\in\mathbb{D}} \mu(z)\big|\psi_1'(z)f_k^{(n)}(\varphi(z))\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|(\psi_1(z)\varphi'(z) + \psi_2'(z))f_k^{(n+1)}(\varphi(z))\big|$$

$$+ \sup_{z\in\mathbb{D}} \mu(z)\big|\psi_2(z)\varphi'(z)f_k^{(n+2)}(\varphi(z))\big|$$

$$\leq \sup_{|\varphi(z)|\leq\delta} \mu(z)\big|\psi_1'(z)\big|\big|f_k^{(n)}(\varphi(z))\big| + \sup_{|\varphi(z)|>\delta} \mu(z)\big|\psi_1'(z)\big|\big|f_k^{(n)}(\varphi(z))\big|$$

$$+ \sup_{|\varphi(z)|\leq\delta} \mu(z)\big|(\psi_1(z)\varphi'(z) + \psi_2'(z))\big|\big|f_k^{(n+1)}(\varphi(z))\big|$$

$$+ \sup_{|\varphi(z)|>\delta} \mu(z)\big|(\psi_1(z)\varphi'(z) + \psi_2'(z))\big|\big|f_k^{(n+1)}(\varphi(z))\big|$$

$$+ \sup_{|\varphi(z)|\leq\delta} \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f_k^{(n+2)}(\varphi(z))\big| + \sup_{|\varphi(z)|>\delta} \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f_k^{(n+2)}(\varphi(z))\big|$$

$$\lesssim N_1 \sup_{|\varphi(z)|\leq\delta} \big|f_k^{(n)}(\varphi(z))\big| + \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}}$$

$$+ N_2 \sup_{|\varphi(z)|\leq\delta} \big|f_k^{(n+1)}(\varphi(z))\big| + \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n}}$$

$$+ N_3 \sup_{|\varphi(z)| \leq \delta} \left| f_k^{(n+2)}(\varphi(z)) \right| + \sup_{|\varphi(z)| > \delta} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p} + n + 1}}$$

$$\leq N_1 \sup_{|\zeta| \leq \delta} \left| f_k^{(n)}(\zeta) \right| + N_2 \sup_{|\zeta| \leq \delta} \left| f_k^{(n+1)}(\zeta) \right| + N_3 \sup_{|\zeta| \leq \delta} \left| f_k^{(n+2)}(\zeta) \right| + 3\varepsilon.$$

From the arbitrariness of ε and the fact that $f_k^{(n)}$, $f_k^{(n+1)}$, and $f_k^{(n+2)}$ converges to zero uniformly on compact subsets of $\mathbb{D}$ we have $\lim_{k \to \infty} \| T_{\psi_1,\psi_2,\varphi}^n f_k \|_{\mathcal{B}_\mu} = 0$, and so by Lemma 5, $T_{\psi_1,\psi_2,\varphi}^n : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact.

(ii) If $T_{\psi_1,\psi_2,\varphi}^0 : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact, then it is bounded and from Theorem 1 we see that $\psi_1 \in \mathcal{B}_\mu$, $\psi_1 \varphi' + \psi_2' \in H_\mu^\infty$, and $\psi_2 \varphi' \in H_\mu^\infty$. Relations (d) and (e) follow from the proof of (i) with $n = 0$.

Now assume that $p > 2 + q$, $\psi_1 \in \mathcal{B}_\mu$, $\psi_2 \varphi' \in H_\mu^\infty$, $\psi_1 \varphi' + \psi_2' \in H_\mu^\infty$, and that (d) and (e) hold. Let $(f_k)_{k \in \mathbb{N}}$ be a bounded sequence in $F(p,q,s)$ (or $F_0(p,q,s)$) that converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. To show that $T_{\psi_1,\psi_2,\varphi}^0 : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact, we need to show that, for each such sequence, $\| T_{\psi_1,\psi_2,\varphi}^0 f_k \|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$.

From (d) and (e) we have that for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}}} < \varepsilon \quad \text{and} \quad \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p} + 1}} < \varepsilon$$

when $\delta < |\varphi(z)| < 1$.

We have

$$\| T_{\psi_1,\psi_2,\varphi}^0 f_k \|_{\mathcal{B}_\mu}$$

$$\leq \sup_{z \in \mathbb{D}} \mu(z) \left| \left(T_{\psi_1,\psi_2,\varphi}^0 f_k \right)'(z) \right|$$

$$\leq \sup_{z \in \mathbb{D}} \mu(z) \left| \psi_1'(z) f_k(\varphi(z)) \right| + \sup_{z \in \mathbb{D}} \mu(z) \left| (\psi_1(z)\varphi'(z) + \psi_2'(z)) f_k'(\varphi(z)) \right|$$

$$\quad + \sup_{z \in \mathbb{D}} \mu(z) \left| \psi_2(z)\varphi'(z) f_k''(\varphi(z)) \right|$$

$$\leq N_1 \sup_{z \in \mathbb{D}} \left| f_k(\varphi(z)) \right| + \sup_{|\varphi(z)| \leq \delta} \mu(z) \left| (\psi_1(z)\varphi'(z) + \psi_2'(z)) \right| \left| f_k'(\varphi(z)) \right|$$

$$\quad + \sup_{|\varphi(z)| > \delta} \mu(z) \left| (\psi_1(z)\varphi'(z) + \psi_2'(z)) \right| \left| f_k'(\varphi(z)) \right|$$

$$\quad + \sup_{|\varphi(z)| \leq \delta} \mu(z) \left| \psi_2(z)\varphi'(z) \right| \left| f_k''(\varphi(z)) \right| + \sup_{|\varphi(z)| > \delta} \mu(z) \left| \psi_2(z)\varphi'(z) \right| \left| f_k''(\varphi(z)) \right|$$

$$\lesssim N_1 \sup_{z \in \mathbb{D}} \left| f_k(z) \right| + N_2 \sup_{|\varphi(z)| \leq \delta} \left| f_k'(\varphi(z)) \right| + \sup_{|\varphi(z)| > \delta} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}}}$$

$$\quad + N_3 \sup_{|\varphi(z)| \leq \delta} \left| f_k''(\varphi(z)) \right| + \sup_{|\varphi(z)| > \delta} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p} + 1}}$$

$$\lesssim N_1 \sup_{z \in \mathbb{D}} \left| f_k(z) \right| + N_2 \sup_{|\zeta| \leq \delta} \left| f_k'(\zeta) \right| + N_3 \sup_{|\zeta| \leq \delta} \left| f_k'(\zeta) \right| + 2\varepsilon,$$

where N_j, $j = \overline{1,3}$, are as in (i).

From this and from the fact that f_k' and f_k'' converge to zero on compacts of $\mathbb{D}$ as $k \to \infty$, by Lemma 5.2 in [39] we have $\sup_{z \in \mathbb{D}} |f_k(z)| \to 0$ as $k \to \infty$, and by the arbi-

trariness of ε it follows that $\lim_{k\to\infty}\|T^0_{\psi_1,\psi_2,\varphi}f_k\|_{\mathcal{B}_\mu} = 0$. Hence, by Lemma 5, $T^0_{\psi_1,\psi_2,\varphi}$: $F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact.

(iii) If $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact, then it is bounded, and from Theorem 1 we see that $\psi_1 \in \mathcal{B}_\mu$, $\psi_1\varphi' + \psi_2' \in H^\infty_\mu$, and $\psi_2\varphi' \in H^\infty_\mu$. Relations (g) and (h) follow from the proof in (i) with $n = 0$ and $p = 2 + q$.

Assume that $(z_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ for which $|\varphi(z_k)| \to 1$ as $k \to \infty$ (if such a sequence does not exist, then (f), (g), and (h) obviously hold). Consider the sequence of functions

$$g_k(z) = \frac{20}{\mu_k^2}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^3 - \frac{30}{\mu_k^3}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^4 + \frac{12}{\mu_k^4}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^5,$$

where

$$\mu_k = \ln\frac{2}{1-|\varphi(z_k)|^2}, \quad k \in \mathbb{N}.$$

Then

$$g_k'(z) = \left(\frac{60}{\mu_k^2}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^2 - \frac{120}{\mu_k^3}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^3\right.$$
$$\left. + \frac{60}{\mu_k^4}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^4\right)\frac{\overline{\varphi(z_k)}}{1-\overline{\varphi(z_k)}z},$$
$$g_k''(z) = \left(\frac{60}{\mu_k^2}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^2 - \frac{120}{\mu_k^3}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^3\right.$$
$$+ \frac{60}{\mu_k^4}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^4\right)\frac{(\overline{\varphi(z_k)})^2}{(1-\overline{\varphi(z_k)}z)^2}$$
$$+ \left(\frac{120}{(\mu_k)^2}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right) - \frac{360}{\mu_k^3}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^2\right.$$
$$\left. + \frac{240}{\mu_k^4}\left(\ln\frac{2}{1-\overline{\varphi(z_k)}z}\right)^3\right)\frac{(\overline{\varphi(z_k)})^2}{(1-\overline{\varphi(z_k)}z)^2}.$$

By some calculation we have $g_k \in \mathcal{B}_0$, $k \in \mathbb{N}$, and $\sup_{k\in\mathbb{N}}\|g_k\|_{\mathcal{B}} \le C < \infty$. Since $s > 1$, we have $g_k \in F_0(p,q,s)$ and $\sup_{k\in\mathbb{N}}\|g_k\|_{F(p,q,s)} < \infty$. It is easy to see that g_k converges to zero uniformly on compact subsets of $\mathbb{D}$ and that

$$g_k'\big(\varphi(z_k)\big) = 0, \qquad g_k''\big(\varphi(z_k)\big) = 0, \quad \text{and} \quad g_k\big(\varphi(z_k)\big) = 2\ln\left(\frac{2}{1-|\varphi(z_k)|^2}\right).$$

So if $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact, then $\|T^0_{\psi_1,\psi_2,\varphi}g_k\|_{\mathcal{B}_\mu} \to 0$ as $k \to \infty$, and, consequently,

$$\mu(z_k)\big|\psi_1'(z_k)\big|\ln\left(\frac{2}{1-|\varphi(z_k)|^2}\right) \lesssim \big\|T^0_{\psi_1,\psi_2,\varphi}g_k\big\|_{\mathcal{B}_\mu} \to 0 \tag{35}$$

as $k \to \infty$, so (f) holds.

Now assume that $p = 2 + q$, $\psi_1 \in \mathcal{B}_\mu$, $\psi_2\varphi' \in H_\mu^\infty$, $\psi_1\varphi' + \psi_2' \in H_\mu^\infty$, and that (f), (g), and (h) hold. Let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in $F(p,q,s)$ (or $F_0(p,q,s)$) that converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$.

From (f), (g), and (h) we have that for every $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\mu(z)\big|\psi_1'(z)\big|\ln\left(\frac{2}{1-|\varphi(z)|^2}\right) < \varepsilon,$$

$$\frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{1 - |\varphi(z)|^2} < \varepsilon, \quad \text{and} \quad \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} < \varepsilon$$

when $\delta < |\varphi(z)| < 1$.

Using the inequalities, we have

$$\big\| T_{\psi_1,\psi_2,\varphi}^0 f_k \big\|_{\mathcal{B}_\mu}$$

$$\leq \sup_{z\in\mathbb{D}} \mu(z)\big|\big(T_{\psi_1,\psi_2,\varphi}^0 f_k\big)'(z)\big|$$

$$\leq \sup_{z\in\mathbb{D}} \mu(z)\big|\psi_1'(z)f_k\big(\varphi(z)\big)\big| + \sup_{z\in\mathbb{D}} \mu(z)\big|\big(\psi_1(z)\varphi'(z) + \psi_2'(z)\big)f_k'\big(\varphi(z)\big)\big|$$

$$\quad + \sup_{z\in\mathbb{D}} \mu(z)\big|\psi_2(z)\varphi'(z)f_k''\big(\varphi(z)\big)\big|$$

$$\leq \sup_{|\varphi(z)|\leq\delta} \mu(z)\big|\psi_1'(z)f_k\big(\varphi(z)\big)\big| + \sup_{|\varphi(z)|>\delta} \mu(z)\big|\psi_1'(z)f_k\big(\varphi(z)\big)\big|$$

$$\quad + \sup_{|\varphi(z)|\leq\delta} \mu(z)\big|\big(\psi_1(z)\varphi'(z) + \psi_2'(z)\big)\big|\big|f_k'\big(\varphi(z)\big)\big|$$

$$\quad + \sup_{|\varphi(z)|>\delta} \mu(z)\big|\big(\psi_1(z)\varphi'(z) + \psi_2'(z)\big)\big|\big|f_k'\big(\varphi(z)\big)\big|$$

$$\quad + \sup_{|\varphi(z)|\leq\delta} \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f_k''\big(\varphi(z)\big)\big| + \sup_{|\varphi(z)|>\delta} \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f_k''\big(\varphi(z)\big)\big|$$

$$\lesssim N_1 \sup_{|\varphi(z)|\leq\delta} \big|f_k\big(\varphi(z)\big)\big| + \sup_{|\varphi(z)|>\delta} \mu(z)\big|\psi_1'(z)\big|\ln\frac{2}{1-|\varphi(z)|^2}$$

$$\quad + N_2 \sup_{|\varphi(z)|\leq\delta} \big|f_k'\big(\varphi(z)\big)\big| + \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{1-|\varphi(z)|^2}$$

$$\quad + N_3 \sup_{|\varphi(z)|\leq\delta} \big|f_k''\big(\varphi(z)\big)\big| + \sup_{|\varphi(z)|>\delta} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}$$

$$\leq N_1 \sup_{\zeta\in\mathbb{D}} \big|f_k(\zeta)\big| + N_2 \sup_{|\zeta|\leq\delta} \big|f_k'(\zeta)\big| + N_3 \sup_{|\zeta|\leq\delta} \big|f_k''(\zeta)\big| + 3\varepsilon,$$

where N_j, $j = \overline{1,3}$, are as in (i).

From this, since f_k, f_k', and f_k'' converge to zero on compacts of $\mathbb{D}$ as $k \to \infty$, by the arbitrariness of ε it follows that $\lim_{k\to\infty} \| T_{\psi_1,\psi_2,\varphi}^0 f_k \|_{\mathcal{B}_\mu} = 0$. Hence, by Lemma 5, $T_{\psi_1,\psi_2,\varphi}^0 :$ $F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact. $\qquad\square$

Theorem 3 *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, $n \in \mathbb{N}_0$, μ be a typical weight, and $\varphi \in S(\mathbb{D})$. Then the following statements are true.*

    (i) *If $n \in \mathbb{N}$, or $n = 0$ and $p < 2 + q$, then $T_{\psi_1,\psi_2,\varphi}^n : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded if and only if $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H_{\mu,0}^\infty$, $\psi_2\varphi' \in H_{\mu,0}^\infty$, and conditions (a)-(c) of Theorem 1 hold.*

(ii) *If $p > 2 + q$, then $T^0_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded if and only if $\psi_1 \in \mathcal{B}_{\mu,0}$,*
$\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2\varphi' \in H^\infty_{\mu,0}$, *and conditions* (d) *and* (e) *of Theorem* 1 *hold.*

(iii) *If $p = 2 + q$ and $s > 1$, then $T^0_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded if and only if*
$\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2\varphi' \in H^\infty_{\mu,0}$, *and conditions* (f)-(h) *of Theorem* 1 *hold.*

Proof (i) First, assume that $T^n_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded. Taking the test functions $p_k(z) = z^k/k!$ for $k \in \{n, n+1, n+2\}$, we easily get that $\psi_1 \in \mathcal{B}_{\mu,0}$ and that the functions $\psi_1\varphi' + \psi_2'$ and $\psi_2\varphi'$ are in $H^\infty_{\mu,0}$. Moreover, as in the proof of Theorem 1, we can easily show that conditions (a)-(c) hold.

Conversely, suppose that $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2\varphi' \in H^\infty_{\mu,0}$, and that conditions (a)-(c) of Theorem 1 hold. Then by Theorem 1(i) we know that the operator $T^n_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_\mu$ is bounded. Hence, to show that the operator $T^n_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded, it suffices to show that $T^n_{\psi_1,\psi_2,\varphi} f \in \mathcal{B}_{\mu,0}$ for every $f \in F_0(p,q,s)$. Take any $\varepsilon > 0$. Let $f \in F_0(p,q,s)$. Then by Lemma 1, $f \in \mathcal{B}_0^{(q+2)/p}$. From this by using Propositions 7 and 8 in [2] we have that there is $\delta_1 \in (0,1)$ such that, for any $z \in \mathbb{D}$ such that $|z| > \delta_1$,

$$\left|f^{(j)}(z)\right| < \frac{\varepsilon}{(1 - |z|^2)^{\frac{2+q}{p}+j-1}}$$

for $j \in \{n, n+1, n+2\}$, where $n \in \mathbb{N}$ or $n = 0$ and $p < 2 + q$.

From this, using conditions (a)-(c) of Theorem 1, respectively, we have that

$$\mu(z)\left|\psi_1'(z)f^{(n)}\big(\varphi(z)\big)\right| < \varepsilon \frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} \le M_1\varepsilon, \tag{36}$$

$$\mu(z)\left|\psi_1(z)\varphi'(z) + \psi_2'(z)\right|\left|f^{(n+1)}\big(\varphi(z)\big)\right| < \varepsilon \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n}} \le M_2\varepsilon, \tag{37}$$

and

$$\mu(z)\left|\psi_2(z)\varphi'(z)\right|\left|f^{(n+2)}\big(\varphi(z)\big)\right| < \varepsilon \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} \le M_3\varepsilon \tag{38}$$

when $|\varphi(z)| > \delta_1$, where

$$M_1 := \sup_{|\varphi(z)|>\delta_1} \frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}}, \qquad M_2 := \sup_{|\varphi(z)|>\delta_1} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n}},$$

$$M_3 := \sup_{|\varphi(z)|>\delta_1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n+1}}$$

(note that these quantities are finite due to conditions (a)-(c)).

On the other hand, since $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, and $\psi_2\varphi' \in H^\infty_{\mu,0}$, we have that there is $\delta_2 \in (0,1)$ such that

$$\mu(z)\left|\psi_1'(z)\right| < \varepsilon\left(1 - \delta_1^2\right)^{\frac{2+q}{p}+n-1},$$

$$\mu(z)\left|\psi_1(z)\varphi'(z) + \psi_2'(z)\right| < \varepsilon\left(1 - \delta_1^2\right)^{\frac{2+q}{p}+n},$$

and

$$\mu(z)\big|\psi_2(z)\varphi'(z)\big| < \varepsilon\big(1 - \delta_1^2\big)^{\frac{2+q}{p}+n+1}$$

whenever $|z| > \delta_2$.

From this, by Lemma 3 we have that, for $|\varphi(z)| \le \delta_1$ and $|z| > \delta_2$,

$$\mu(z)\big|\psi_1'(z)f^{(n)}\big(\varphi(z)\big)\big| \lesssim \|f\|_{F(p,q,s)} \frac{\mu(z)|\psi_1'(z)|}{(1-\delta_1^2)^{\frac{2+q}{p}+n-1}} \lesssim \varepsilon\|f\|_{F(p,q,s)}, \tag{39}$$

$$\mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f^{(n+1)}\big(\varphi(z)\big)\big| \lesssim \|f\|_{F(p,q,s)} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1-\delta_1^2)^{\frac{2+q}{p}+n}}$$

$$\lesssim \varepsilon\|f\|_{F(p,q,s)}, \tag{40}$$

and

$$\mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f^{(n+2)}\big(\varphi(z)\big)\big| \lesssim \|f\|_{F(p,q,s)} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-\delta_1^2)^{\frac{2+q}{p}+n+1}} \lesssim \varepsilon\|f\|_{F(p,q,s)}. \tag{41}$$

Combining (36) and (39), (37) and (40), and (38) and (41), respectively, we have that, whenever $|z| > \delta_2$,

$$\mu(z)\big|\psi_1'(z)f^{(n)}\big(\varphi(z)\big)\big| \lesssim \big\{M_1, \|f\|_{F(p,q,s)}\big\}\varepsilon, \tag{42}$$

$$\mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f^{(n+1)}\big(\varphi(z)\big)\big| \lesssim \big\{M_2, \|f\|_{F(p,q,s)}\big\}\varepsilon, \tag{43}$$

and

$$\mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f^{(n+2)}\big(\varphi(z)\big)\big| \lesssim \big\{M_3, \|f\|_{F(p,q,s)}\big\}\varepsilon. \tag{44}$$

Hence,

$$\mu(z)\big|\big(T_{\psi_1,\psi_2,\varphi}^n f\big)'(z)\big| \le \mu(z)\big|\psi_1'(z)f^{(n)}\big(\varphi(z)\big)\big| + \mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f^{(n+1)}\big(\varphi(z)\big)\big|$$

$$+ \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f^{(n+2)}\big(\varphi(z)\big)\big|$$

$$\lesssim \big\{M_1, M_2, M_3, \|f\|_{F(p,q,s)}\big\}\varepsilon,$$

from which it follows that $T_{\psi_1,\psi_2,\varphi}^n f \in \mathcal{B}_{\mu,0}$ for every $f \in F_0(p,q,s)$ and, consequently, the boundedness of the operator $T_{\psi_1,\psi_2,\varphi}^n : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$.

(ii) If $T_{\psi_1,\psi_2,\varphi}^0 : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded, then as in the proof of (i), we get that $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H_{\mu,0}^\infty$, $\psi_2\varphi' \in H_{\mu,0}^\infty$, and that conditions (d) and (e) of Theorem 1 hold.

Conversely, suppose that $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H_{\mu,0}^\infty$, $\psi_2\varphi' \in H_{\mu,0}^\infty$, and that conditions (d) and (e) of Theorem 1 hold. Then by Theorem 1(ii) we know that the operator $T_{\psi_1,\psi_2,\varphi}^0 : F_0(p,q,s) \to \mathcal{B}_\mu$ is bounded. Hence, as in (i), to show that the operator $T_{\psi_1,\psi_2,\varphi}^0 : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded, it suffices to show that $T_{\psi_1,\psi_2,\varphi}^0 f \in \mathcal{B}_{\mu,0}$ for every $f \in F_0(p,q,s)$.

Take any $\varepsilon > 0$. Then $\psi_1 \in \mathcal{B}_{\mu,0}$ implies that there is some $\delta_1 \in (0,1)$ such that

$$\mu(z)\big|\psi_1(z)\big| < \varepsilon \tag{45}$$

for $|z| > \delta_1$. Let $f \in F_0(p,q,s)$. Then by Lemma 1, $f \in \mathcal{B}_0^{(q+2)/p}$. From this, by Proposition 8 in [2] it follows that there is $\delta_2 \in (0,1)$ such that, for any $z \in \mathbb{D}$ such that $|z| > \delta_2$,

$$\big|f^{(j)}(z)\big| < \frac{\varepsilon}{(1-|z|^2)^{\frac{2+q}{p}+j-1}}$$

for $j \in \{1,2\}$.

Thus, using conditions (d) and (e) of Theorem 1 respectively, we have that

$$\mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f'\big(\varphi(z)\big)\big| < \varepsilon \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}} \le N_1 \varepsilon \tag{46}$$

and

$$\mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f''\big(\varphi(z)\big)\big| < \varepsilon \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+1}} \le N_2 \varepsilon \tag{47}$$

when $|\varphi(z)| > \delta_2$, where

$$N_1 := \sup_{|\varphi(z)|>\delta_2} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}} \quad \text{and} \quad N_2 := \sup_{|\varphi(z)|>\delta_2} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+1}}$$

(note that these quantities are finite due to conditions (d) and (e) of Theorem 1).

On the other hand, since $\psi_1\varphi' + \psi_2' \in H_{\mu,0}^\infty$ and $\psi_2\varphi' \in H_{\mu,0}^\infty$, we have that there is $\delta_3 \in (0,1)$ such that

$$\mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big| < \varepsilon\big(1-\delta_2^2\big)^{\frac{2+q}{p}}$$

and

$$\mu(z)\big|\psi_2(z)\varphi'(z)\big| < \varepsilon\big(1-\delta_2^2\big)^{\frac{2+q}{p}+1}$$

whenever $|z| > \delta_3$.

Thus, for $|\varphi(z)| \le \delta_2$ and $|z| > \delta_3$, we have that

$$\mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f'\big(\varphi(z)\big)\big| \lesssim \|f\|_{F(p,q,s)} \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1-\delta_2^2)^{\frac{2+q}{p}}}$$

$$\lesssim \varepsilon\|f\|_{F(p,q,s)} \tag{48}$$

and

$$\mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f''\big(\varphi(z)\big)\big| \lesssim \|f\|_{F(p,q,s)} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-\delta_2^2)^{\frac{2+q}{p}+1}} \lesssim \varepsilon\|f\|_{F(p,q,s)}. \tag{49}$$

Combining (46) and (48) and, respectively, (47) and (49), we have that, whenever $|z| > \delta_3$,

$$\mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f'\big(\varphi(z)\big)\big| \lesssim \big\{N_1, \|f\|_{F(p,q,s)}\big\}\varepsilon \tag{50}$$

and

$$\mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f''\big(\varphi(z)\big)\big| \lesssim \big\{N_2, \|f\|_{F(p,q,s)}\big\}\varepsilon. \tag{51}$$

From (45), (50), and (51), by Lemma 3 we have that

$$\mu(z)\big|\big(T^0_{\psi_1,\psi_2,\varphi}f\big)'(z)\big| \le \mu(z)\big|\psi_1'(z)f\big(\varphi(z)\big)\big| + \mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f'\big(\varphi(z)\big)\big|$$
$$+ \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f''\big(\varphi(z)\big)\big|$$
$$\lesssim \big\{N_1, N_2, \|f\|_{F(p,q,s)}\big\}\varepsilon$$

when $|z| > \max\{\delta_1, \delta_3\}$. Hence, $T^0_{\psi_1,\psi_2,\varphi}f \in \mathcal{B}_{\mu,0}$ for every $f \in F_0(p,q,s)$, from which the boundedness of $T^0_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ follows.

(iii) If $T^0_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded, then as in the proof of (i), we get that $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2\varphi' \in H^\infty_{\mu,0}$, and that conditions (f), (g), and (h) of Theorem 1 hold.

Conversely, suppose that $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2\varphi' \in H^\infty_{\mu,0}$, and conditions (f), (g), and (h) of Theorem 1 hold. Then by Theorem 1(iii) we know that the operator $T^0_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_\mu$ is bounded. As in the previous two cases, to show that the operator $T^0_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded, it suffices to show that $T^0_{\psi_1,\psi_2,\varphi}f \in \mathcal{B}_{\mu,0}$ for every $f \in F_0(p,q,s)$.

Take any $\varepsilon > 0$. Let $f \in F_0(p,q,s)$. Then by Lemma 1, $f \in \mathcal{B}_0$. From this, by [2], Proposition 8, and [8], Lemma 3, for the case $n = 1$ with f' replaced by f, it easily follows that there is $\delta_1 \in (0,1)$ such that, for any $z \in \mathbb{D}$ such that $|z| > \delta_1$,

$$\big|f(z)\big| < \varepsilon \ln \frac{2}{1 - |z|^2}$$

and

$$\big|f^{(j)}(z)\big| < \frac{\varepsilon}{(1 - |z|^2)^j}$$

for $j \in \{1, 2\}$.

From this, using conditions (f), (g), and (h) of Theorem 1, respectively, we have that

$$\mu(z)\big|\psi_1'(z)f\big(\varphi(z)\big)\big| < \varepsilon\mu(z)\big|\psi_1'(z)\big|\ln\frac{2}{1 - |\varphi(z)|^2} \le L_1\varepsilon, \tag{52}$$

$$\mu(z)\big|\psi_1(z)\varphi'(z) + \psi_2'(z)\big|\big|f'\big(\varphi(z)\big)\big| < \varepsilon\frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{1 - |\varphi(z)|^2} \le L_2\varepsilon, \tag{53}$$

and

$$\mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f''\big(\varphi(z)\big)\big| < \varepsilon\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^2} \le L_3\varepsilon \tag{54}$$

when $|\varphi(z)| > \delta_1$, where

$$L_1 := \sup_{|\varphi(z)| > \delta_1} \mu(z) |\psi_1'(z)| \ln \frac{2}{1 - |\varphi(z)|^2}, \qquad L_2 := \sup_{|\varphi(z)| > \delta_1} \frac{\mu(z) |\psi_1(z)\varphi'(z) + \psi_2'(z)|}{1 - |\varphi(z)|^2},$$

$$L_3 := \sup_{|\varphi(z)| > \delta_1} \frac{\mu(z) |\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^2}$$

(note that these quantities are finite due to conditions (f), (g), and (h) of Theorem 1).

On the other hand, since $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H_{\mu,0}^\infty$, and $\psi_2\varphi' \in H_{\mu,0}^\infty$, we have that there is $\delta_2 \in (0,1)$ such that

$$\mu(z) |\psi_1'(z)| < \varepsilon / \ln \frac{2}{1 - \delta_1^2}, \qquad \mu(z) |\psi_1(z)\varphi'(z) + \psi_2'(z)| < \varepsilon(1 - \delta_1^2),$$

and

$$\mu(z) |\psi_2(z)\varphi'(z)| < \varepsilon(1 - \delta_1^2)^2$$

whenever $|z| > \delta_2$.

Thus, for $|\varphi(z)| \le \delta_1$ and $|z| > \delta_2$, we have that

$$\mu(z) |\psi_1'(z)| |f(\varphi(z))| \lesssim \|f\|_{F(p,q,s)} \mu(z) |\psi_1'(z)| \ln \frac{2}{1 - \delta_1^2} < \varepsilon \|f\|_{F(p,q,s)}, \tag{55}$$

$$\mu(z) |\psi_1(z)\varphi'(z) + \psi_2'(z)| |f'(\varphi(z))| \lesssim \|f\|_{F(p,q,s)} \frac{\mu(z) |\psi_1(z)\varphi'(z) + \psi_2'(z)|}{1 - \delta_1^2}$$

$$< \varepsilon \|f\|_{F(p,q,s)}, \tag{56}$$

and

$$\mu(z) |\psi_2(z)\varphi'(z)| |f''(\varphi(z))| \lesssim \|f\|_{F(p,q,s)} \frac{\mu(z) |\psi_2(z)\varphi'(z)|}{(1 - \delta_1^2)^2} < \varepsilon \|f\|_{F(p,q,s)}. \tag{57}$$

Combining (52) and (55), (53) and (56), and (54) and (57), respectively, we have that, whenever $|z| > \delta_2$,

$$\mu(z) |\psi_1'(z)| |f(\varphi(z))| \lesssim \{L_1, \|f\|_{F(p,q,s)}\} \varepsilon, \tag{58}$$

$$\mu(z) |\psi_1(z)\varphi'(z) + \psi_2'(z)| |f'(\varphi(z))| \lesssim \{L_2, \|f\|_{F(p,q,s)}\} \varepsilon, \tag{59}$$

and

$$\mu(z) |\psi_2(z)\varphi'(z)| |f''(\varphi(z))| \lesssim \{L_3, \|f\|_{F(p,q,s)}\} \varepsilon. \tag{60}$$

Thus, from (58), (59), and (60) we have that

$$\mu(z) |(T_{\psi_1,\psi_2,\varphi}^0 f)'(z)| \le \mu(z) |\psi_1'(z) f(\varphi(z))| + \mu(z) |\psi_1(z)\varphi'(z) + \psi_2(z)| |f'(\varphi(z))|$$

$$+ \mu(z) |\psi_2(z)\varphi'(z)| |f''(\varphi(z))|$$

$$\lesssim \{L_1, L_2, L_3, \|f\|_{F(p,q,s)}\} \varepsilon$$

when $|z| > \delta_2$. Hence, $T^0_{\psi_1,\psi_2,\varphi} f \in \mathcal{B}_{\mu,0}$ for every $f \in F_0(p,q,s)$, from which the boundedness of $T^0_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 4 *Let $0 < p, s < \infty$, $-2 < q < \infty$, $q + s > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, $n \in \mathbb{N}_0$, μ be a typical weight, and $\varphi \in S(\mathbb{D})$.*

(i) If $n \in \mathbb{N}$, or $n = 0$ and $p < 2 + q$, and $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded, then $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2 \varphi' \in H^\infty_{\mu,0}$, and conditions (a)-(c) *of Theorem 1 hold.*

Also, if $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2 \varphi' \in H^\infty_{\mu,0}$, and conditions (a)-(c) *of Theorem 2 hold, then $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded.*

(ii) If $2 + q < p$ and $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded, then $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2 \varphi' \in H^\infty_{\mu,0}$, and conditions (d) *and* (f) *of Theorem 1 hold.*

Also, if $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2 \varphi' \in H^\infty_{\mu,0}$, and conditions (d) *and* (f) *of Theorem 2 hold, then $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded.*

(iii) If $2 + q = p$, $s > 1$, and $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded, then $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2 \varphi' \in H^\infty_{\mu,0}$, and conditions (f)-(h) *of Theorem 1 hold.*

Also, if $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2 \varphi' \in H^\infty_{\mu,0}$, and conditions (f)-(h) *of Theorem 2 hold, then $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded.*

Proof (i) Suppose that $n \in \mathbb{N}$, or $n = 0$ and $p < 2 + q$, and that $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_{\mu,0}$ is bounded. Then $T^n_{\psi_1,\psi_2,\varphi} : F_0(p,q,s) \to \mathcal{B}_{\mu,0}$ is also bounded, and so by (i) of Theorem 3, $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2 \varphi' \in H^\infty_{\mu,0}$, and conditions (a)-(c) of Theorem 1 hold.

Conversely, suppose that $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, $\psi_2 \varphi' \in H^\infty_{\mu,0}$, and conditions (a)-(c) of Theorem 2 hold. Then by Theorem 2(i) we have that $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_\mu$ is compact. Hence, to show the boundedness of $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s) \to \mathcal{B}_{\mu,0}$, it suffices to show that $T^n_{\psi_1,\psi_2,\varphi} f \in \mathcal{B}_{\mu,0}$ for every $f \in F(p,q,s)$. Take any $\varepsilon > 0$. By conditions (a)-(c) of Theorem 2 there is $\delta_1 \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} < \varepsilon, \tag{61}$$

$$\frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n}} < \varepsilon, \tag{62}$$

and

$$\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} < \varepsilon \tag{63}$$

when $|\varphi(z)| > \delta_1$.

On the other hand, since $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1 \varphi' + \psi_2' \in H^\infty_{\mu,0}$, and $\psi_2 \varphi' \in H^\infty_{\mu,0}$, we have that there is $\delta_2 \in (0,1)$ such that

$$\mu(z)|\psi_1'(z)| < \varepsilon \left(1 - \delta_1^2\right)^{\frac{2+q}{p}+n-1}, \qquad \mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)| < \varepsilon \left(1 - \delta_1^2\right)^{\frac{2+q}{p}+n},$$

and

$$\mu(z)|\psi_2(z)\varphi'(z)| < \varepsilon \left(1 - \delta_1^2\right)^{\frac{2+q}{p}+n+1}$$

whenever $|z| > \delta_2$.

Thus, for $|\varphi(z)| \le \delta_1$ and $|z| > \delta_2$, we have that

$$\frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} \le \frac{\mu(z)|\psi_1'(z)|}{(1-\delta_1^2)^{\frac{2+q}{p}+n-1}} < \varepsilon, \tag{64}$$

$$\frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} \le \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-\delta_1^2)^{\frac{2+q}{p}+n}} < \varepsilon, \tag{65}$$

and

$$\frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} \le \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-\delta_1^2)^{\frac{2+q}{p}+n+1}} < \varepsilon. \tag{66}$$

Combining (61) and (64), (62) and (65), and (63) and (66), respectively, we have that, whenever $|z| > \delta_2$,

$$\mu(z)\big|\big(T_{\psi_1,\psi_2,\varphi}^n f\big)'(z)\big| \le \mu(z)\big|\psi_1'(z)f^{(n)}\big(\varphi(z)\big)\big| + \mu(z)\big|\psi_1(z)\varphi'(z)+\psi_2(z)\big|\big|f^{(n+1)}\big(\varphi(z)\big)\big|$$
$$+ \mu(z)\big|\psi_2(z)\varphi'(z)\big|\big|f^{(n+2)}\big(\varphi(z)\big)\big|$$
$$\lesssim \|f\|_{F(p,q,s)}\varepsilon,$$

from which it follows that $T_{\psi_1,\psi_2,\varphi}^n f \in \mathcal{B}_{\mu,0}$ for every $f \in F(p,q,s)$, as desired.

(ii) and (iii) These statements are proved similarly to (i). Hence, we omit the details. □

The following lemma was essentially proved in [40].

Lemma 6 *Let μ be a typical weight. Then a closed set K in $\mathcal{B}_{\mu,0}$ is compact if and only if K is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)\big|f'(z)\big| = 0.$$

Theorem 5 *Let $0 < p, s < \infty, -2 < q < \infty, q+s > -1$, $\psi_1, \psi_2 \in H(\mathbb{D})$, $n \in \mathbb{N}_0$, μ be a typical weight, and $\varphi \in S(\mathbb{D})$. Then the following statements hold.*

(i) If $n \in \mathbb{N}$, or $n = 0$ and $p < 2+q$, then $T_{\psi_1,\psi_2,\varphi}^n : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact if and only if

(a) $\lim_{|z|\to 1} \frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} = 0,$

(b) $\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} = 0,$ *and*

(c) $\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} = 0.$

(ii) If $p > 2+q$, then $T_{\psi_1,\psi_2,\varphi}^0 : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact if and only if $\psi_1 \in \mathcal{B}_{\mu,0}$ and

(d) $\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}}} = 0$ *and*

(e) $\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+1}} = 0.$

(iii) If $p = 2+q$ and $s > 1$, then $T_{\psi_1,\psi_2,\varphi}^0 : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact if and only if

(f) $\lim_{|z|\to 1} \mu(z)|\psi_1'(z)| \ln(\frac{2}{1-|\varphi(z)|^2}) = 0,$

(g) $\lim_{|z|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{1-|\varphi(z)|^2} = 0$, *and*

(h) $\lim_{|z|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2} = 0$.

Proof (i) Suppose that $n \in \mathbb{N}$, or $n = 0$ and $p < 2 + q$, and that (a), (b), and (c) hold. Then by Theorem 1 the operator $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is bounded, $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, and $\psi_2\varphi' \in H^\infty_{\mu,0}$, from which it follows that $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is bounded, and, consequently, its image of the unit ball in $F(p,q,s)$.

Hence, due to Lemma 6, to show that $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact, we only need to show that

$$\lim_{|z|\to 1} \sup_{\|f\|_{F(p,q,s)}\le 1} \mu(z)\left|\left(T^n_{\psi_1,\psi_2,\varphi}f\right)'(z)\right| = 0. \tag{67}$$

From (8) we have

$$\mu(z)\left|\left(T^n_{\psi_1,\psi_2,\varphi}f\right)'(z)\right| \lesssim \left(\frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} + \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} \right.$$
$$\left. + \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} \right) \|f\|_{F(p,q,s)}. \tag{68}$$

Taking the supremum in (68) over all $f \in F(p,q,s)$ (or $F_0(p,q,s)$) such that $\|f\|_{F(p,q,s)} \le 1$, then letting $|z| \to 1$, and using conditions (a)-(c), we see that (67) holds, so $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact.

Conversely, suppose that $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact. Then $T^n_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact, so by Theorem 1 we have

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} = 0, \tag{69}$$

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n}} = 0, \tag{70}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{2+q}{p}+n+1}} = 0. \tag{71}$$

On the other hand, using the test functions $f(z) = z^k/k!$, $k \in \{n, n+1, n+2\}$, which belong to $F_0(p,q,s)$, we have

$$\lim_{|z|\to 1} \mu(z)\left|\psi_1'(z)\right| = 0, \tag{72}$$

$$\lim_{|z|\to 1} \mu(z)\left|\psi_1(z)\varphi'(z)+\psi_2'(z)\right| = 0, \tag{73}$$

$$\lim_{|z|\to 1} \mu(z)\left|\psi_2(z)\varphi'(z)\right| = 0. \tag{74}$$

From (69) we have that for any $\varepsilon > 0$, there exists $t \in (0,1)$ such that

$$\frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} < \varepsilon \tag{75}$$

when $t < |\varphi(z)| < 1$. From (72) it follows that there exists $r \in (0,1)$ such that

$$\mu(z)\big|\psi_1'(z)\big| < \varepsilon \big(1 - t^2\big)^{\frac{2+q}{p}+n-1} \tag{76}$$

when $r < |z| < 1$.

Therefore, when $r < |z| < 1$ and $t < |\varphi(z)| < 1$, we have

$$\frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} < \varepsilon,$$

whereas when $r < |z| < 1$ and $|\varphi(z)| \leq t$, using (76), we have

$$\frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} < \varepsilon \frac{(1 - t^2)^{\frac{2+q}{p}+n-1}}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} \leq \varepsilon.$$

Since ε is an arbitrary positive number, we get

$$\lim_{|z|\to 1} \frac{\mu(z)|\psi_1'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}+n-1}} = 0,$$

that is, condition (a) holds.

Similarly, we prove that (70) and (73) imply (b) and that (71) and (74) imply (c).

(ii) Assume that $\psi_1 \in \mathcal{B}_{\mu,0}$ and that (d) and (e) hold. Then by Theorem 1(ii), $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is bounded, $\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, and $\psi_2\varphi' \in H^\infty_{\mu,0}$. From all these facts it follows that $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is bounded, and, consequently, so is its image of the unit ball in $F(p,q,s)$.

Suppose that $f \in F(p,q,s)$ (or $F_0(p,q,s)$) is such that $\|f\|_{F(p,q,s)} \leq 1$. Then by Lemma 3 we have

$$\mu(z)\big|\big(T^0_{\psi_1,\psi_2,\varphi}f\big)'(z)\big| \lesssim \mu(z)\big|\psi_1'(z)\big| + \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{(1 - |\varphi(z)|^2)^{\frac{2+q}{p}}} + \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\frac{1+q}{p}+1}},$$

from which, along with the assumptions $\psi \in \mathcal{B}_{\mu,0}$, (d), and (e), we have

$$\lim_{|z|\to 1} \sup\big\{\mu(z)\big|\big(T^0_{\psi_1,\psi_2,\varphi}f\big)'(z)\big| : f \in F(p,q,s) \ \big(\text{or } F_0(p,q,s)\big), \|f\|_{F(p,q,s)} \leq 1\big\} = 0,$$

and so by Lemma 6 we get that $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact.

Conversely, if $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact, then for $f(z) \equiv 1$, we obtain that $T^0_{\psi_1,\psi_2,\varphi}1 = \psi_1 \in \mathcal{B}_0$. Relations (d) and (e) follow from the proof of (i) with $n = 0$.

(iii) Assume that (f), (g), and (h) hold. Then by Theorem 1(iii) the operator $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is bounded, $\psi_1 \in \mathcal{B}_{\mu,0}$, $\psi_1\varphi' + \psi_2' \in H^\infty_{\mu,0}$, and $\psi_2\varphi' \in H^\infty_{\mu,0}$.

Thus, the operator $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is bounded, and so is its image of the unit ball in $F(p,q,s)$.

By Lemma 3 we have that for every $f \in F(p,q,s)$ (or $F_0(p,q,s)$) such that $\|f\|_{F(p,q,s)} \le 1$, the following asymptotic relation holds:

$$\mu(z)\left|\left(T^0_{\psi_1,\psi_2,\varphi}f\right)'(z)\right| \lesssim \mu(z)\left|\psi_1'(z)\right|\left(\ln\frac{2}{1-|\varphi(z)|^2}\right) + \frac{\mu(z)|\psi_1(z)\varphi'(z) + \psi_2'(z)|}{1-|\varphi(z)|^2}$$
$$+ \frac{\mu(z)|\psi_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2}. \tag{77}$$

Taking the supremum in (77) over all $f \in F(p,q,s)$ (or $F_0(p,q,s)$) such that $\|f\|_{F(p,q,s)} \le 1$, then letting $|z| \to 1$, and using Lemma 6, we obtain that $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact.

If $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_{\mu,0}$ is compact, then $T^0_{\psi_1,\psi_2,\varphi} : F(p,q,s)$ (or $F_0(p,q,s)) \to \mathcal{B}_\mu$ is compact too. Hence, by Theorem 2(iii) we have that

$$\lim_{|\varphi(z)|\to 1} \mu(z)\left|\psi_1'(z)\right|\ln\left(\frac{2}{1-|\varphi(z)|^2}\right) = 0,$$

which means that, for every $\varepsilon > 0$, there is $t \in (0,1)$ such that

$$\mu(z)\left|\psi_1'(z)\right|\ln\frac{2}{1-|\varphi(z)|^2} < \varepsilon \tag{78}$$

when $t < |\varphi(z)| < 1$.

On the other hand, $\psi_1 \in \mathcal{B}_{\mu,0}$, which implies that there is $r \in (0,1)$ such that

$$\mu(z)\left|\psi_1'(z)\right| < \varepsilon/\ln\frac{2}{1-t^2} \tag{79}$$

when $r < |z| < 1$. Hence, (78) holds when $r < |z| < 1$ and $t < |\varphi(z)| < 1$, whereas form (79) we see that (78) holds when $r < |z| < 1$ and $|\varphi(z)| \le t$, so that (f) holds.

Relations (g) and (h) follow from the proof of (i) with $n = 0$ and $p = 2 + q$. □

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