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Monotonicity of a mean related to polygamma functions with an application

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Abstract

Let $\psi_n = (-1)^{n-1}\ \psi^{(n)}$ $(n = 0, 1, 2, \ldots)$, where $\psi^{(n)}$ denotes the psi and polygamma functions. We prove that for $n \geq 0$ and two different real numbers a and b, the function

$$x \mapsto \psi_n^{-1}\left(\frac{\int_a^b \psi_n(x+t)\,dt}{b-a}\right) - x$$

is strictly increasing from $(-\min(a,b), \infty)$ onto $(\min(a,b), (a+b)/2)$, which generalizes a well-known result. As an application, the complete monotonicity for a ratio of gamma functions is improved.

1 Introduction

The classical Euler's gamma and psi (or called digamma) functions are defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\,dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

respectively. Furthermore, the derivatives $\psi', \psi'', \ldots, \psi^{(i)}$ for $i = 1, 2, \ldots,$ are called polygamma functions.

For convenience, we denote $\psi_n(x) = (-1)^{n-1}\psi^{(n)}(x)$. It is well known that $\psi_n(x)$ is strictly complete monotonic on $(0, \infty)$; namely, $(-1)^{n-1}\psi^{(n)}(x) > 0$ for $x > 0$ and $n \in \mathbb{N}$. Note that for the following integral and series representations (see [1], Sections 6.3, 6.4):

$$\psi_0(x) = -\psi(x) = \gamma + \int_0^\infty \frac{e^{-xt} - e^{-t}}{1 - e^{-t}}\,dt = \gamma + \frac{1}{x} - \sum_{k=1}^\infty \frac{x}{k(x+k)}, \tag{1.1}$$

$$\psi_n(x) = (-1)^{n-1}\psi^{(n)}(x) = \int_0^\infty \frac{t^n}{1 - e^{-t}} e^{-xt}\,dt = n! \sum_{k=0}^\infty \frac{1}{(x+k)^{n+1}}, \tag{1.2}$$

it is easy to see that $\psi_n(0^+) = \infty$ for $n \geq 0$, $\psi_n(\infty) = 0$ for $n \geq 1$, and $\psi_0(\infty) = -\infty$. Moreover, $\psi_n' = -\psi_{(n+1)}(x) < 0$.

Let $f : I \to \mathbb{R}$ be strictly monotone and $a, b \in I$. Then the so-called integral f-mean of a and b is defined in [2] by

$$I_f(a,b)=f^{-1}\left(\frac{\int_a^b f(x)\,dx}{b-a}\right)\quad \text{if } a\neq b,\quad \text{and}\quad I_f(a,a)=a.$$

For $f=\psi$, Elezović and Pečarić [2], Theorem 6, proved an interesting result as follows.

Theorem EP *For $x,a,b>0$, the digamma function ψ has the following properties*:

(i) $I_{\psi'}(a,b)\leq I_\psi(a,b)$; *namely*,

$$\left(\psi'\right)^{-1}\left(\frac{\int_a^b \psi'(x)\,dx}{b-a}\right)\leq \psi^{-1}\left(\frac{\int_a^b \psi(x)\,dx}{b-a}\right).$$

(ii) $x\mapsto I_\psi(x+a,x+b)-x$ *is increasing concave, and*

$$\lim_{x\to\infty}\left[I_\psi(x+a,x+b)-x\right]=\frac{a+b}{2}.$$

Remark 1.1 It should be noted that, for $a,b\in I$, if $A(a,b)$ is a mean of a and b, then for $x+a,x+b\in I$ the function $x\mapsto A(x+a,x+b)-x$ is still a mean of a and b, which is due to the following relations:

$$\begin{aligned}\min(a,b) &= \min(x+a,x+b)-x\leq A(x+a,x+b)-x\\ &\leq \max(x+a,x+b)-x=\max(a,b).\end{aligned}$$

Further, Batir [3], Theorem 2.7, gave a nice double inequality for $I_{\psi_n}(a,b)$ as follows.

Theorem B *Let a and b be distinct positive real numbers and n be a positive integer. Then we have*

$$(-1)^n\psi^{(n+1)}\left(\frac{a+b}{2}\right)<(-1)^n\frac{\psi^{(n)}(a)-\psi^{(n)}(b)}{a-b}<(-1)^n\psi^{(n+1)}\bigl(S_{-(n+1)}(a,b)\bigr),$$

or, equivalently,

$$S_{-(n+1)}(a,b)<I_{\psi_{n+1}}(a,b)=\psi_{n+1}^{-1}\left(\frac{\int_a^b \psi_{n+1}(t)\,dt}{b-a}\right)<\frac{a+b}{2},$$

where

$$S_p(a,b)=\begin{cases}\left(\frac{a^p-b^p}{p(a-b)}\right)^{1/(p-1)}, & \text{if } p\neq 0,1,\\ \frac{a-b}{\ln a-\ln b}, & \text{if } p=0,\\ e^{-1}\left(\frac{a^a}{b^b}\right)^{1/(a-b)}, & \text{if } p=1,\end{cases}\tag{1.3}$$

is the generalized logarithmic mean of a and b.

An improvement of Theorem B was given in [4], Theorem 1, and [5], Theorem 1, by Qi as follows.

Theorem Q1 *For real numbers $a,b>0$ with $a\neq b$ and an integer $n\geq 0$, the inequality*

$$(-1)^n\psi^{(n)}\bigl(S_p(a,b)\bigr)<(-1)^n\frac{\int_a^b \psi^{(n)}(t)\,dt}{b-a}\leq(-1)^n\psi^{(n)}\bigl(S_q(a,b)\bigr)$$

or

$$S_p(a,b) < I_{\psi_n}(a,b) = \psi_n^{-1}\left(\frac{\int_a^b \psi_n(t)\,dt}{b-a}\right) \le S_q(a,b)$$

holds if $p \le -n$ *and* $q \ge -n+1$, *where* $S_p(a,b)$ *is given in* (1.3).

Motivated by the results just mentioned, the main aim of this paper is to continue the study of some further properties of the mean $I_{\psi_n}(a,b)$ and $I_{\psi_n}(x+a,x+b)-x$. More precisely, we have the following.

Theorem 1.2 *For* $a,b>0$ *with* $a\neq b$, *the sequence* $\{I_{\psi_n}(a,b)\}_{n\ge 0}$ *is strictly decreasing, and*

$$\lim_{n\to\infty} I_{\psi_n}(a,b) = \min(a,b).$$

Theorem 1.3 *Let a and b be distinct real numbers, and* $n \ge 0$ *be an integer. If* ψ_n^{-1} *is strictly decreasing with respected to x, then the function* $x \mapsto A_{\psi_n}(x)$ *with*

$$A_{\psi_n}(x) = I_{\psi_n}(x+a,x+b) - x = \psi_n^{-1}\left(\frac{\int_a^b \psi_n(x+t)\,dt}{b-a}\right) - x \tag{1.4}$$

is strictly increasing from $(-\min(a,b),\infty)$ *onto* $(\min(a,b),(a+b)/2)$.

As a direct consequence, noting that ψ_n^{-1} is strictly decreasing, by Theorem 1.3 we have the following.

Corollary 1.4 *Let a and b be distinct real numbers and* $n \ge 0$ *be an integer. Then for* $x > -\min(a,b)$ *we have*

$$\psi_n\left(x+\frac{a+b}{2}\right) < \frac{\int_a^b \psi_n(x+t)\,dt}{b-a} < \psi_n\big(x+\min(a,b)\big),$$

where $\min(a,b)$ *and* $(a+b)/2$ *are the best constants. In particular, note that* $\psi_0 = -\psi$, *the double inequality*

$$\psi\big(x+\min(a,b)\big) < \frac{\int_a^b \psi(x+t)\,dt}{b-a} < \psi\left(x+\frac{a+b}{2}\right)$$

or

$$\exp\psi\big(x+\min(a,b)\big) < \left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} < \exp\psi\left(x+\frac{a+b}{2}\right) \tag{1.5}$$

holds for $x > -\min(a,b)$ *with the best constants* $\min(a,b)$ *and* $(a+b)/2$.

Suppose that $a,b>0$ with $a\neq b$ in Theorem 1.3. Utilizing the strictly increasing property of $x\mapsto A_{\psi_n}(x)$ on $(0,\infty)$, we have $A_{\psi_n}(0) < A_{\psi_n}(x) < A_{\psi_n}(\infty)$; namely,

$$I_{\psi_n}(a,b) = \psi_n^{-1}\left(\frac{\int_a^b \psi_n(t)\,dt}{b-a}\right) < \psi_n^{-1}\left(\frac{\int_a^b \psi_n(x+t)\,dt}{b-a}\right) - x < \frac{a+b}{2}.$$

Therefore, we conclude the following.

Corollary 1.5 *Let $a, b > 0$ with $a \neq b$ and $n \geq 0$ be an integer. Then for $x > 0$ we have*

$$\psi_n\left(x + \frac{a+b}{2}\right) < \frac{\int_a^b \psi_n(x+t)\,dt}{b-a} < \psi_n\bigl(x + I_{\psi_n}(a,b)\bigr),$$

where $I_{\psi_n}(a,b)$ and $(a+b)/2$ are the best constants. Particularly, noting that $\psi_0 = -\psi$, the double inequality

$$\psi\bigl(x + I_\psi(a,b)\bigr) < \frac{\int_a^b \psi(x+t)\,dt}{b-a} < \psi\left(x + \frac{a+b}{2}\right)$$

or

$$\exp\psi\bigl(x + I_\psi(a,b)\bigr) < \left[\frac{\Gamma(x+b)}{\Gamma(x+a)}\right]^{1/(b-a)} < \exp\psi\left(x + \frac{a+b}{2}\right) \tag{1.6}$$

holds for $x > 0$ with the best constants $I_\psi(a,b)$ and $(a+b)/2$.

We would think it worth noticing that the double inequality (1.6) was first proved in [6] by Elezović *et al.*

Remark 1.6 The second Kershaw double inequality [7] states that

$$\exp\bigl[(1-s)\psi(x+\sqrt{s})\bigr] < \frac{\Gamma(x+1)}{\Gamma(x+s)} < \exp\left[(1-s)\psi\left(x + \frac{s+1}{2}\right)\right] \tag{1.7}$$

for $s \in (0,1)$ and $x \geq 0$. Some of the refinements, extensions, and generalizations of the double inequality (1.7) can be found in Qi's review paper [8] and the references therein. It seems that our double inequality (1.5) may be the best second Kershaw type inequality, since the ranges of a and b in (1.5) are arbitrary real numbers, and the lower and upper bounds are sharp.

As an application of Theorem 1.3, we use it to prove a necessary and sufficient condition for the functions $x \mapsto F_{a,b,c}(x)$ defined by (3.1) and $x \mapsto 1/F_{a,b,c}(x)$ to be logarithmically monotonic on $(-\rho, \infty)$ with $\rho = \min(a,b,c)$, which improves a well-known result.

2 Proofs of main results

This section we devote to the proof of our main results. First of all, let us give the following assertion, which is an improvement of Theorem 4 in [2].

Lemma 2.1 *Let $f \in C^{(2)}(I)$. If f is strictly monotone, then the mean function*

$$A_f(x) = I_f(a+x, b+x) - x = f^{-1}\left(\frac{\int_a^b f(x+t)\,dt}{b-a}\right) - x \tag{2.1}$$

is strictly increasing (decreasing) according to f''/f' being strictly increasing (decreasing).

Proof By the Jensen inequality we have

$$f'\left(f^{-1}\left(\frac{\int_a^b f(x+t)\,dt}{b-a}\right)\right) < (>) \frac{\int_a^b f'(x+t)\,dt}{b-a} \tag{2.2}$$

if $f' \circ f^{-1}$ is strictly convex (concave).

Differentiation yields

$$\frac{df'(f^{-1}(x))}{dx} = f''\big(f^{-1}(x)\big)\frac{d(f^{-1}(x))}{dx} = \frac{f''(f^{-1}(x))}{f'(f^{-1}(x))} = \frac{f''(u)}{f'(u)},$$

where $u = f^{-1}(x)$. This shows that $f' \circ f^{-1}$ is strictly convex if and only if both f and f''/f' are either increasing or decreasing, and concave if and only if one of f and f''/f' is increasing, while the other is decreasing.

Case 1: Both f and f''/f' are increasing. Then $f' > 0$ and $f' \circ f^{-1}$ is convex, and it follows from (2.2) that

$$\frac{dA_f(x)}{dx} = \frac{\int_a^b f'(x+t)\,dt}{b-a} \Big/ f'\left(f^{-1}\left(\frac{\int_a^b f(x+t)\,dt}{b-a}\right)\right) - 1 > 0.$$

Case 2: f is decreasing and f''/f' is increasing. Then $f' < 0$ and $f' \circ f^{-1}$ is concave and by (2.2) we also have $dA_f(t)/dt > 0$.

Case 3: Both f and f''/f' are decreasing. Then $f' < 0$ and $f' \circ f^{-1}$ is convex. Similarly, we have $dA_f(t)/dt < 0$.

Case 4: f is increasing and f''/f' is decreasing. Then $f' > 0$ and $f' \circ f^{-1}$ is concave. Obviously, we see that $dA_f(t)/dt < 0$.

To sum up, if f''/f' is increasing (decreasing), then so is A_f, which completes the proof. □

The following lemma is useful for our main proof, which is a generalization of Lemma 1.4 in [3] and Lemma 4 in [9].

Lemma 2.2 *Let $A : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a differentiable one-order homogeneous mean. Then, for all $x+t, y+t \in (0,\infty)$, we have*

$$\lim_{t\to\infty}\big(A(x+t,y+t)-t\big) = px + (1-p)y, \tag{2.3}$$

where $p = A_x(1,1) \in [0,1]$. In particular, if $A(x,y)$ is symmetric with respect to x and y, then

$$\lim_{t\to\infty}\big(A(x+t,y+t)-t\big) = \frac{x+y}{2}. \tag{2.4}$$

Proof Using homogeneity of $A(x,y)$ and the L'Hospital rule yield

$$\begin{aligned}\lim_{t\to\infty}\big(A(x+t,y+t)-t\big) &= \lim_{t\to\infty}\frac{A(t^{-1}x+1,t^{-1}y+1)-1}{t^{-1}}\\ &\overset{t^{-1}=u}{=} \lim_{u\to 0}\frac{A(ux+1,uy+1)-1}{u}\\ &= \lim_{u\to 0}\frac{\partial A(ux+1,uy+1)}{\partial u} = xA_x(1,1)+yA_y(1,1).\end{aligned}$$

In addition, it follows from [10] that

$$A_x(x,x), A_y(x,x) \in [0,1] \quad \text{and} \quad A_x(x,x) + A_y(x,x) = 1. \tag{2.5}$$

Putting the above together, we get (2.3).

In particular, if A is symmetric, that is, $A(x,y) = A(y,x)$, then we clearly see that $A_x(x,y) = A_y(y,x)$, and so $A_x(x,x) = A_y(x,x)$. It follows from (2.5) that $A_x(x,x) = A_y(x,x) = 1/2$, and then (2.4) holds. The proof is complete. □

Lemma 2.3 *Let $\psi_n = (-1)^{n-1}\psi^{(n)}$ for $n \in \mathbb{N}$. Then all the following statements are true, and mutually equivalent.*

(i) *the sequence $\{\psi_{n+1}/\psi_n\}_{n\in\mathbb{N}}$ is strictly increasing*;
(ii) *the function $x \mapsto \psi_{n+1}(x)/\psi_n(x)$ is strictly decreasing on $(0,\infty)$*;
(iii) *the function $x \mapsto \psi_n(x)$ is log-convex on $(0,\infty)$*.

Proof (i) It suffices to prove $\psi_{n+2}/\psi_{n+1} > \psi_{n+1}/\psi_n$ for $n \in \mathbb{N}$, which is equivalent to $\psi_{n+2}\psi_n - \psi_{n+1}^2 > 0$. By virtue of the integral representation given in (1.2), we get

$$\begin{aligned}\psi_{n+2}\psi_n - \psi_{n+1}^2 &= \int_0^\infty \frac{t^{n+2}}{1-e^{-t}}e^{-xt}\,dt \int_0^\infty \frac{t^n}{1-e^{-t}}e^{-xt}\,dt - \left(\int_0^\infty \frac{t^{n+1}}{1-e^{-t}}e^{-xt}\,dt\right)^2 \\ &= \frac{1}{2}\int_0^\infty\int_0^\infty \frac{t^n s^n (t-s)^2}{(1-e^{-t})(1-e^{-s})}e^{-x(t+s)}\,dt\,ds > 0,\end{aligned}$$

which proves assertion (i).

(ii) Note that $\psi_n' = -\psi_{n+1}$, we have

$$\left(\frac{\psi_{n+1}}{\psi_n}\right)' = \frac{\psi_{n+1}'\psi_n - \psi_{n+1}\psi_n'}{\psi_n^2} = \frac{-\psi_{n+2}\psi_n + \psi_{n+1}^2}{\psi_n^2} < 0,$$

which implies that the second assertion is true.

(iii) Differentiation gives

$$(\ln \psi_n)' = \frac{\psi_n'}{\psi_n} = -\frac{\psi_{n+1}}{\psi_n}, \qquad (\ln \psi_n)'' = -\left(\frac{\psi_{n+1}}{\psi_n}\right)' > 0,$$

which completes the proof. □

Now we are in a position to prove our main results.

Proof of Theorem 1.2 We first prove that the sequence $\{I_{\psi_n}(a,b)\}_{n\geq 0}$ is strictly decreasing, which means that for $n \geq 0$ the inequality

$$\psi_n^{-1}\left(\frac{\int_a^b \psi_n(x)\,dx}{b-a}\right) > \psi_{n+1}^{-1}\left(\frac{\int_a^b \psi_{n+1}(x)\,dx}{b-a}\right) \tag{2.6}$$

holds for $a, b > 0$ with $a \neq b$. By the Jensen inequality, it suffices to check that $\psi_{n+1} \circ \psi_n^{-1}$ is convex on $(0,\infty)$. In fact, by Lemma 2.3 we have

$$\frac{d}{dx}\psi_{n+1}\bigl(\psi_n^{-1}(x)\bigr) = \frac{\psi_{n+1}'(\psi_n^{-1}(x))}{\psi_n'(\psi_n^{-1}(x))} = \frac{\psi_{n+2}(\psi_n^{-1}(x))}{\psi_{n+1}(\psi_n^{-1}(x))},$$

$$\frac{d^2}{dx^2}\psi_{n+1}\bigl(\psi_n^{-1}(x)\bigr) = \left(\frac{\psi_{n+2}(u)}{\psi_{n+1}(u)}\right)'\frac{1}{\psi_n'(u)} = -\left(\frac{\psi_{n+2}(u)}{\psi_{n+1}(u)}\right)'\frac{1}{\psi_{n+1}(u)} > 0,$$

where $u = \psi_n^{-1}(x)$. This means that $\psi_{n+1} \circ \psi_n^{-1}$ is convex, which proves inequality (2.6).

Taking $p = -n$ and $q = -n + 1$ in Theorem Q1 gives

$$S_{-n}(a,b) < \psi_n^{-1}\left(\frac{\int_a^b \psi_n(t)\,dt}{b-a}\right) < S_{-n+1}(a,b). \tag{2.7}$$

Considering that $\lim_{p\to-\infty} S_p(a,b) = \min(a,b)$ in [11], then we get

$$\lim_{n\to\infty}\psi_n^{-1}\left(\frac{\int_a^b \psi_n(t)\,dt}{b-a}\right) = \min(a,b),$$

which completes the proof. □

Proof of Theorem 1.3 To prove $x \mapsto A_{\psi_n}(x)$ is strictly increasing on $(-\min(a,b),\infty)$, by Lemma 2.1 it suffices to check that ψ_n''/ψ_n' is strictly increasing on $(0,\infty)$. In fact, since $\psi_n' = -\psi_{n+1}$ we see that $\psi_n''/\psi_n' = -\psi_{n+2}/\psi_{n+1}$ is strictly increasing on $(0,\infty)$ by the second assertion of Lemma 2.3. Thus, the increasing property of A_{ψ_n} follows.

As mentioned in the introduction, we see that $\psi_n(0^+) = \infty$ for $n \ge 0$, and so $\psi_n^{-1}(\infty) = 0$. Note that the symmetry of a and b, without loss of generality we may assume that $b > a$. Then we have

$$\lim_{x\to-a^+}\frac{\int_a^b \psi_n(x+t)\,dt}{b-a} = \lim_{x\to-a^+}\frac{(-1)^{n-1}(\psi^{(n-1)}(x+b) - \psi^{(n-1)}(x+a))}{b-a} = \infty,$$

which implies

$$\begin{aligned}\lim_{x\to-a^+} A_{\psi_n}(x) &= \lim_{x\to-a^+}\psi_n^{-1}\left(\frac{\int_a^b \psi_n(x+t)\,dt}{b-a}\right) - \lim_{x\to-a^+} x \\ &= \psi_n^{-1}\left(\lim_{x\to-a^+}\frac{\int_a^b \psi_n(x+t)\,dt}{b-a}\right) + a = \psi_n^{-1}(\infty) + a = a.\end{aligned}$$

To obtain $\lim_{x\to\infty} A_{\psi_n}(x) = (a+b)/2$, we use (2.7) to derive that

$$S_{-n}(x+b,x+a) - x < \psi_n^{-1}\left(\frac{\int_a^b \psi_n(x+t)\,dt}{b-a}\right) - x < S_{-n+1}(x+b,x+a) - x.$$

Note that the generalized logarithmic mean $S_p(x,y)$ is homogeneous and symmetric, it follows from Lemma 2.2 that

$$\lim_{x\to\infty}\bigl(S_p(x+b,x+a) - x\bigr) = \frac{a+b}{2}.$$

Therefore, we conclude that $\lim_{x\to\infty} A_{\psi_n}(x) = (a+b)/2$, which completes the proof. □

3 An application

A function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and $(-1)^n(f(x))^{(n)} \geq 0$ for $x \in I$ and $n \geq 0$ (see [12]). A positive function f is called logarithmically completely monotonic on an interval I if f has derivatives of all orders on I and its logarithm $\ln f$ satisfies $(-1)^n(\ln f(x))^{(n)} \geq 0$ for all $n \in \mathbb{N}$ on I (see [13]). For convenience, we denote the sets of the completely monotonic functions and the logarithmically completely monotonic functions on I by $\mathcal{C}[I]$ and $\mathcal{L}[I]$, respectively. Qi in [14], Theorem 1, [15], Theorem 1, investigated the logarithmically complete monotonicity of the functions

$$x \mapsto F_{a,b,c}(x) = \begin{cases} (\frac{\Gamma(x+b)}{\Gamma(x+a)})^{1/(a-b)} e^{\psi(x+c)}, & \text{if } a \neq b, \\ e^{\psi(x+c)-\psi(x+a)}, & \text{if } a = b, \end{cases} \tag{3.1}$$

and $x \mapsto 1/F_{a,b,c}(x)$. Furthermore, he concluded the following result.

Theorem Q2 *Let a, b, and c be real numbers and $\rho = \min(a,b,c)$. If $\theta(t)$ is an implicit function defined by*

$$e^t - t = e^{\theta(t)} - \theta(t)$$

on $(-\infty, \infty)$, then $\theta(t)$ is decreasing and $t\theta(t) < 0$ for $\theta(t) \neq t$. Moreover:

(1) $F_{a,b,c}(x) \in \mathcal{L}[(-\rho, \infty)]$ *if*

$$\begin{aligned}(a,b,c) \in {} & \{c \geq a, c \geq b\} \cup \bigl\{c \geq a, 0 \geq c - b \geq \theta(c-a)\bigr\} \\ & \cup \bigl\{c \leq a, c - b \geq \theta(c-a)\bigr\} \backslash \{a = b = c\}.\end{aligned}$$

(2) $1/F_{a,b,c}(x) \in \mathcal{L}[(-\rho, \infty)]$ *if*

$$\begin{aligned}(a,b,c) \in {} & \{c \leq a, c \leq b\} \cup \bigl\{c \geq a, c - b \leq \theta(c-a)\bigr\} \\ & \cup \bigl\{c \leq a, 0 \leq c - b \leq \theta(c-a)\bigr\} \backslash \{a = b = c\}.\end{aligned}$$

Later, Qi and Guo in [16], Theorem 1, [17], Theorem 1, proved another result concerning the logarithmically complete monotonicity of the functions $x \mapsto F_{a,b,c}(x)$ and $x \mapsto 1/F_{a,b,c}(x)$ for $x > -\min(a,b,c)$, where $c = c(a,b)$ is a constant depending on a and b. More precisely, they showed the following.

Theorem QG *Let a and b be two real numbers with $a \neq b$ and $c(a,b)$ be a constant depending on a and b.*

(1) *If $c(a,b) \leq \min(a,b)$, then $1/F_{a,b,c}(x) \in \mathcal{L}[(-c(a,b), \infty)]$.*

(2) $F_{a,b,c}(x) \in \mathcal{L}[(-\min(a,b), \infty)]$ *if and only if $c(a,b) \geq (a+b)/2$.*

We would like to remark that the result in Theorem Q2 is rather interesting but somewhat complicated. Theorem QG shows that c is a constant depending on a and b, and $c(a,b) \leq \min(a,b)$ is only sufficient for $1/F_{a,b,c}(x) \in \mathcal{L}[(-c(a,b), \infty)]$. Here, we apply Theorem 1.3 to deduce that c is a constant independent of a and b, and $c \leq \min(a,b)$ is also necessary for $1/F_{a,b,c}(x) \in \mathcal{L}[(-c(a,b), \infty)]$. This improved result can be restated as follows.

Theorem 3.1 *Let a, b, and c be real numbers, and $\rho = \min(a,b,c)$. Then $1/F_{a,b,c}(x) \in \mathcal{L}[(-\rho,\infty)]$ if and only if $c \le \min(a,b)$, while $F_{a,b,c}(x) \in \mathcal{L}[(-\rho,\infty)]$ if and only if $c \ge (a+b)/2$.*

Proof For $a \neq b$, we have

$$\ln F_{a,b,c}(x) = \psi(x+c) - \frac{\ln\Gamma(x+b) - \ln\Gamma(x+a)}{b-a} = \psi(x+c) - \frac{\int_a^b \psi(x+t)\,dt}{b-a}$$

and

$$\begin{aligned}(-1)^n \bigl(\ln F_{a,b,c}(x)\bigr)^{(n)} &= (-1)^n \psi^{(n)}(x+c) - \frac{(-1)^n \int_a^b \psi^{(n)}(x+t)\,dt}{b-a} \\ &= \frac{\int_a^b \psi_n(x+t)\,dt}{b-a} - \psi_n(x+c) \\ &= \frac{(b-a)^{-1}\int_a^b \psi_n(x+t)\,dt - \psi_n(x+c)}{\psi_n^{-1}((b-a)^{-1}\int_a^b \psi_n(x+t)\,dt) - (x+c)} \bigl(A_{\psi_n}(x) - c\bigr),\end{aligned}$$

where $\psi_n = (-1)^{n-1}\psi^{(n)}$ and $A_{\psi_n}(x)$ is defined by (1.4).

Since $\psi_n' = -\psi^{(n+1)} < 0$, $(\psi_n^{-1})' < 0$, which means that ψ_n^{-1} is strictly decreasing on $(0,\infty)$. This yields

$$\frac{(b-a)^{-1}\int_a^b \psi_n(x+t)\,dt - \psi_n(x+c)}{\psi_n^{-1}((b-a)^{-1}\int_a^b \psi_n(x+t)\,dt) - \psi_n^{-1}(\psi_n(x+c))} < 0$$

for $x \in (-\rho,\infty)$. Therefore, we have

$$\operatorname{sgn}\bigl((-1)^n \bigl(\ln F_{a,b,c}(x)\bigr)^{(n)}\bigr) = \operatorname{sgn}\bigl(c - A_{\psi_n}(x)\bigr).$$

Theorem 1.3 tells that $x \mapsto A_{\psi_n}(x) = I_{\psi_n}(x+a, x+b) - x$ is strictly increasing from $(-\min(a,b),\infty)$ onto $(\min(a,b),(a+b)/2)$, which implies that

$$\operatorname{sgn}\bigl(c - A_{\psi_n}(x)\bigr) \le 0 \quad\Longleftrightarrow\quad c \le \inf A_{\psi_n}(x) = \min(a,b)$$

and

$$\operatorname{sgn}\bigl(c - A_{\psi_n}(x)\bigr) \ge 0 \quad\Longleftrightarrow\quad c \ge \sup A_{\psi_n}(x) = \frac{a+b}{2}.$$

It is obvious that these are also true for $a = b$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

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