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Approximation for a generalization of Bernstein operators

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Abstract
In this paper, we give a direct approximation theorem, inverse theorem, and equivalent theorem for a generalization of Bernstein operators in the space $L_p[0,1]$ $(1 \le p \le \infty)$.

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1 Introduction

The Sikkema-Bézier-type generalization of Bernstein-Kantorovich operators is given by

$$\begin{aligned} S_{n,\alpha}(f,x) &= S_{n,\alpha}(f,s_n,x) \\ &= \sum_{k=0}^{n} \bigl[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)\bigr](n+s_n+1) \int_{\frac{k}{n+s_n+1}}^{\frac{k+1}{n+s_n+1}} f(u)\,du, \quad n = 1,2,\ldots, \end{aligned}$$

where $J_{n,k}(x) = \sum_{j=k}^{n} p_{n,j}(x)$ are Bézier basic functions, $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $\alpha \ge 1$, and s_n is a bounded sequence of natural numbers. If $\alpha = 1$ and $s_n = 0$, then $S_{n,\alpha}(f,x)$ are just the well-known Bernstein-Kantorovich operators [1]

$$B_n(f,x) = (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\,du, \quad n = 1,2,\ldots.$$

Bézier-type operators were introduced by Chang [2], later many results were given in [3–8], and more recent approximation results can be found in [9]. Most of these results are on the rate of convergence of some Bézier-type operators for functions of bounded variation, whereas in the present paper, we give direct, inverse, and equivalent approximation theorems for a generalization of Bézier-type operators in L_p spaces. We showed [5] that for Bézier-type operators, the second-order modulus cannot be used, so here we shall use the first-order modulus too. For Sikkema-type operators, we can also see many investigations (see [10]). Next, we state the central approximation theorem for $S_{n,\alpha}(f,x)$ in the spaces $L_p[0,1]$ $(1 \le p \le \infty)$, which will be proved in Sections 2 and 3.

Theorem 1.1 *For* $f \in L_p[0,1]$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x(1-x)}$, *and* $0 < \beta < 1$, *we have*

$$\left\|S_{n,\alpha}(f,x) - f(x)\right\|_p = O\left(\left(\frac{1}{\sqrt{n}}\right)^{\beta}\right) \quad \Leftrightarrow \quad \omega_\varphi(f,t)_p = O\left(t^\beta\right). \tag{1.1}$$

In this theorem, we use the first-order modulus defined by

$$\omega_\varphi(f,t)_p = \sup_{0<h\le t}\left\|f\left(x + \frac{h\varphi(x)}{2}\right) - f\left(x - \frac{h\varphi(x)}{2}\right)\right\|_p,$$

which is equivalent to the K-functionals

$$K_\varphi(f,t)_p = \inf_{g\in W_p}\left\{\|f-g\|_p + t\left\|\varphi g'\right\|_p\right\}$$

and

$$\overline{K}_\varphi(f,t)_p = \inf_{g\in W_p}\left\{\|f-g\|_p + t\left\|\varphi g'\right\|_p + t^2\left\|g'\right\|_p\right\},$$

where $W_p = \{f | f \in A.C._{\text{loc}}, \|f'\|_p < \infty\}$. It is well known that [1]

$$\omega_\varphi(f,t)_p \sim K_\varphi(f,t)_p \sim \overline{K}_\varphi(f,t)_p, \tag{1.2}$$

where $a \sim b$ means that there exists $C > 0$ such that $C^{-1}a \le b \le Ca$.

Throughout this paper, C denotes a constant independent of n and x, but it is not necessarily the same in different cases.

Remark In [8], we obtained a pointwise approximation for $S_{n,\alpha}(f,x)$. In Theorem 1 of [8], $\lambda = 1$, which is the case where $p = \infty$ in (1.1). So in the present paper, by the Riesz-Thorin theorem, we shall only need to prove the case where $p = 1$.

2 Direct theorem

To prove the direct theorem, we need the following convergence property of Bernstein-Kantorovich-Bézier operators defined by

$$B_{n,\alpha}(f,x) = \sum_{k=0}^{n}\left[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x)\right](n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)\,du, \quad n = 1,2,\ldots.$$

Lemma 2.1 *For* $f \in L_p[0,1]$ $(1 \le p \le \infty)$, *we have*

$$\left\|B_{n,\alpha}(f,x) - f(x)\right\|_p \le C\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right)_p. \tag{2.1}$$

Proof For $p = 1$, we will have to split estimate (2.1) into estimates on two domains, that is, $x \in E_n^c = [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1]$ and $x \in E_n = (\frac{1}{n}, 1 - \frac{1}{n})$.

First, we choose $g = g_n$ such that

$$\|f-g\|_1 + \frac{1}{\sqrt{n}}\left\|\varphi g'\right\|_1 \le C\omega_\varphi\left(f, \frac{1}{\sqrt{n}}\right)_1. \tag{2.2}$$

For $x \in E_n^c$, we have

$$\begin{aligned}\left|B_{n,\alpha}(g,x)-g(x)\right| &\leq \alpha \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left|\int_x^t g'(u)\,du\right|dt \\ &\leq \alpha \sum_{k=0}^{n} p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\left(\varphi^{-1}(x)+\varphi^{-1}(t)\right)dt\int_0^1\left|\varphi(u)g'(u)\right|du \\ &\leq \alpha\left\|\varphi g'\right\|_1\left(\varphi^{-1}(x)+\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\varphi^{-1}(t)\,dt\right).\end{aligned}$$

Noting that

$$\int_{I_k}\varphi^{-1}(t)\,dt \leq \int_{I_k}\left(\frac{1}{\sqrt{t}}+\frac{1}{\sqrt{1-t}}\right)dt \leq \frac{4}{\sqrt{n+1}},$$

we get

$$\left|B_{n,\alpha}(g,x)-g(x)\right| \leq C\left\|\varphi g'\right\|_1\left(\varphi^{-1}(x)+\sqrt{n}\right). \tag{2.3}$$

Since

$$\begin{aligned}&\int_{E_n^c}\varphi^{-1}(x)\,dx \leq 2\int_0^{\frac{1}{n}}\frac{1}{\sqrt{x}}\,dx+2\int_{1-\frac{1}{n}}^1\frac{1}{\sqrt{1-x}}\,dx \leq \frac{8}{\sqrt{n}}, \\ &\int_{E_n^c}\sqrt{n}\,dx = \frac{2}{\sqrt{n}},\end{aligned}$$

we obtain

$$\int_{E_n^c}\left|B_{n,\alpha}(g,x)-g(x)\right|dx \leq C\frac{1}{\sqrt{n}}\left\|\varphi g'\right\|_1. \tag{2.4}$$

For $x \in E_n$, let $|\int_x^{\frac{k^*}{n+1}}\varphi(u)|g'(u)|\,du| = \max_{j=k,k+1}|\int_x^{\frac{j}{n+1}}\varphi(u)|g'(u)|\,du|$, where k^* is either k or $k+1$. Then we have

$$\begin{aligned}&\int_{E_n}\left|B_{n,\alpha}(g,x)-g(x)\right|dx \\ &\quad\leq \alpha\int_{E_n}\sum_{k=0}^{n}p_{n,k}(x)(n+1)\int_{I_k}\left(\varphi^{-1}(x)+\varphi^{-1}(t)\right)dt\left|\int_x^{\frac{k^*}{n+1}}\varphi(u)\left|g'(u)\right|du\right|dx \\ &\quad\leq \alpha\int_{E_n}\sum_{k=0}^{n}p_{n,k}(x)\left(\varphi^{-1}(x)+\sqrt{\frac{n+1}{k+1}}\right)\left|\int_x^{\frac{k^*}{n+1}}\varphi(u)\left|g'(u)\right|du\right|dx =: \alpha R_1+\alpha R_2.\end{aligned} \tag{2.5}$$

To estimate R_1 and R_2, we follow [3], pp. 146-147, with a similar method. We now define

$$D(l,n,x) = \left\{k : l\varphi(x)n^{-\frac{1}{2}} \leq \left|\frac{k}{n}-x\right| < (l+1)\varphi(x)n^{-\frac{1}{2}}\right\}.$$

Rewrite R_1 as follows:

$$R_1 = \int_{E_n} \varphi^{-1}(x) \sum_{l=0}^{\infty} \sum_{k\in D(l,n,x)} p_{n,k}(x) \left| \int_x^{\frac{k^*}{n+1}} \varphi(u) \left| g'(u) \right| du \right| dx.$$

Similarly to [3], (9.6.11), for $x \in E_n$, we have

$$\sum_{k\in D(l,n,x)} p_{n,k}(x) \le \frac{C}{(l+1)^4}.$$

We now define

$$F(l,x) = \left\{ v : v \in [0,1], |v-x| \le (l+1)\varphi(x)n^{-\frac{1}{2}} + \frac{1}{n} \right\},$$
$$G(l,v) = \left\{ x : x \in E_n, v \in F(l,x) \right\}$$

and by the procedure of [3], p.147, obtain

$$\begin{aligned} R_1 &\le C \sum_{l=0}^{\infty} \frac{1}{(l+1)^4} \int_{E_n} \varphi^{-1}(x) \int_{F(l,x)} \varphi(v) \left| g'(v) \right| dv\, dx \\ &\le C \sum_{l=0}^{\infty} \frac{1}{(l+1)^4} \int_0^1 \varphi(v) \left| g'(v) \right| \int_{G(l,v)} \varphi^{-1}(x)\, dx\, dv \le C \frac{1}{\sqrt{n}} \left\| \varphi g' \right\|_1. \end{aligned} \tag{2.6}$$

On the other hand, for R_2, we have

$$\begin{aligned} \sum_{k\in D(l,n,x)} p_{n,k}(x) \sqrt{\frac{n+1}{k+1}} &\le \left(\sum_{k\in D(l,n,x)} p_{n,k}(x) \frac{n+1}{k+1} \right)^{\frac{1}{2}} = \left(\sum_{k\in D(l,n,x)} p_{n+1,k+1}(x) \frac{1}{x} \right)^{\frac{1}{2}} \\ &= \varphi^{-1}(x) \left(\frac{C}{(1+l)^3} \right)^{\frac{1}{2}} \le \frac{C}{(1+l)^4} \varphi^{-1}(x). \end{aligned}$$

Similarly, we get

$$R_2 \le C \frac{1}{\sqrt{n}} \left\| \varphi g' \right\|_1. \tag{2.7}$$

Hence, by (2.5)-(2.7) we have

$$\int_{E_n} \left| B_{n,\alpha}(g,x) - g(x) \right| dx \le \frac{C}{\sqrt{n}} \left\| \varphi g' \right\|_1. \tag{2.8}$$

Using (2.4) and (2.8), we complete the proof of Lemma 2.1. □

Theorem 2.2 *For $f \in L_1[0,1]$, we have*

$$\left\| S_{n,\alpha}(f,x) - f(x) \right\|_1 \le C\omega_{\varphi}\left(f, \frac{1}{\sqrt{n}} \right)_1. \tag{2.9}$$

Proof By Lemma 2.1 we have

$$\begin{aligned}
&\left\| S_{n,\alpha}(f,x) - f(x) \right\|_1 \\
&\quad \le \left\| S_{n,\alpha}(f,x) - B_{n,\alpha}(f,x) \right\|_1 + \left\| B_{n,\alpha}(f,x) - f(x) \right\|_1 \\
&\quad \le \int_0^1 \sum_{k=0}^{n} \left[J_{n,k}^{\alpha}(x) - J_{n,k+1}^{\alpha}(x) \right](n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} \left| f\left(\frac{n+1}{n+s_n+1} u \right) - f(u) \right| du\, dx \\
&\qquad + C\omega_{\varphi}\left(f, \frac{1}{\sqrt{n}} \right)_1 \\
&\quad \le C\left(\omega_1\left(f, \frac{1}{n} \right)_1 + \omega_{\varphi}\left(f, \frac{1}{\sqrt{n}} \right)_1 \right),
\end{aligned}$$

and, by (3.1.5) in [1],

$$\omega_1\left(f, \frac{1}{n} \right)_1 \le C\omega_{\varphi}\left(f, \frac{1}{\sqrt{n}} \right)_1.$$

The proof is complete. □

3 Inverse theorem

Lemma 3.1 *For $f \in L_1[0,1]$, $\varphi(x) = \sqrt{x(1-x)}$, and $\delta_n(x) = \varphi(x) + \frac{1}{\sqrt{n}}$, we have*

$$\left\| \delta_n(x) S'_{n,\alpha}(f,x) \right\|_1 \le C\sqrt{n}\|f\|_1, \tag{3.1}$$

and, furthermore, for $f \in W_1$,

$$\left\| \delta_n(x) S'_{n,\alpha}(f,x) \right\|_1 \le C\left\| \delta_n f' \right\|_1. \tag{3.2}$$

Proof First, we prove (3.1), that is,

$$\left\| \delta_n(x) S'_{n,\alpha}(f,x) \right\|_1 \le C\sqrt{n}\|f\|_1. \tag{3.3}$$

Write $a_k(f) = (n+s_n+1)\int_{I_k} f(t)\,dt$, where $I_k = [\frac{k}{n+s_n+1}, \frac{k+1}{n+s_n+1}]$. Noting that $J_{n,n+1}(x) = 0$, we have

$$\begin{aligned}
\left| S'_{n,\alpha}(f,x) \right| &\le \alpha \sum_{k=0}^{n-1} \left| a_k(f) \right| \left| \left[J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \right] J'_{n,k+1}(x) \right. \\
&\quad + \alpha \sum_{k=0}^{n} \left| a_k(f) \right| J_{n,k}^{\alpha-1}(x) \left| p'_{n,k}(x) \right| \\
&=: \alpha(J_1 + J_2).
\end{aligned} \tag{3.4}$$

Then

$$\begin{aligned}
\int_0^1 \left| \delta_n(x) S'_{n,\alpha}(f,x) \right| dx &\le \alpha \int_0^1 \delta_n(x)(J_1 + J_2)\, dx \\
&= \alpha\left(\int_{E_n^c} + \int_{E_n} \right) \delta_n(x)(J_1 + J_2)\, dx.
\end{aligned} \tag{3.5}$$

Next, we estimate the four parts in (3.5):

$$\int_{E_n^c} \delta_n(x) J_2\,dx \le \int_{E_n^c} \delta_n(x) \sum_{k=1}^{n} \big|a_k(f)\big| n\big(p_{n-1,k-1}(x) + p_{n-1,k}(x)\big)\,dx + \int_{E_n^c} \delta_n(x)\big|a_0(f)\big| np_{n-1,0}(x)\,dx.$$

For $x \in E_n^c$, $\delta_n(x) \le \frac{2}{\sqrt{n}}$, and since $\int_0^1 p_{n-1,k}(x)\,dx = \frac{1}{n}$, we get

$$\begin{aligned}\int_{E_n^c} \delta_n(x) J_2\,dx &\le \frac{4}{\sqrt{n}} \sum_{k=1}^{n} (n + s_n + 1) \int_{I_k} \big|f(t)\big|\,dt + \frac{2(n + s_n + 1)}{\sqrt{n}} \int_0^{\frac{1}{n+s_n+1}} \big|f(t)\big|\,dt \\ &\le C\sqrt{n}\|f\|_1. \end{aligned} \tag{3.6}$$

Since $J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \le 1$ and $J'_{n,k+1}(x) = np_{n-1,k}(x)$, it is easy to see that

$$\int_{E_n^c} \delta_n(x) J_1\,dx \le C\sqrt{n}\|f\|_1. \tag{3.7}$$

To estimate $\int_{E_n} \delta_n(x) J_2\,dx$, we recall that by [3], p.129, (9.4.15),

$$\int_{E_n} \frac{(\frac{k}{n} - x)^2}{\varphi^2(x)} p_{n,k}(x)\,dx \le Cn^{-2}$$

with $p'_{n,k}(x) = \frac{n}{\varphi^2(x)}(\frac{k}{n} - x)p_{n,k}(x)$, $x \in (0,1)$. By the Hölder inequality we have

$$\begin{aligned}\int_{E_n} \delta_n(x) J_2\,dx &\le 2\sum_{k=0}^{n} \big|a_k(f)\big| \int_{E_n} \varphi(x) \cdot \frac{n}{\varphi^2(x)} \left|\frac{k}{n} - x\right| p_{n,k}(x)\,dx \\ &\le 2n \sum_{k=0}^{n} \big|a_k(f)\big| (n+1)^{-\frac{1}{2}} \left(\int_{E_n} \frac{(\frac{k}{n} - x)^2}{\varphi^2(x)} p_{n,k}(x)\,dx \right)^{\frac{1}{2}} \\ &\le C\sqrt{n} \sum_{k=0}^{n} \int_{I_k} \big|f(t)\big|\,dt = C\sqrt{n}\|f\|_1. \end{aligned} \tag{3.8}$$

To estimate $\int_{E_n} \delta_n(x) J_1\,dx$, we will consider two cases, $\alpha \ge 2$ and $1 < \alpha < 2$ ($J_1 = 0$ when $\alpha = 1$).

For $\alpha \ge 2$, we have $J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) \le (\alpha - 1)p_{n,k}(x)$, and we need a result of [4], p.375,

$$p_{n,k}(x) \le \frac{1}{\sqrt{2e}} \frac{1}{\sqrt{nx(1-x)}} \quad \text{for } 0 \le k \le n. \tag{3.9}$$

Since $J'_{n,k}(x) = np_{n-1,k-1}(x) \ge 0$, we have

$$\begin{aligned}\int_{E_n} \delta_n(x) J_1\,dx &\le C \sum_{k=0}^{n-1} \big|a_k(f)\big| \int_{E_n} \varphi(x) p_{n,k}(x) np_{n-1,k}(x)\,dx \\ &\le C \sum_{k=0}^{n} \big|a_k(f)\big| \frac{1}{\sqrt{n}} \le C\sqrt{n}\|f\|_1. \end{aligned} \tag{3.10}$$

For $1 < \alpha < 2$, applying $u(a) - u(b) = u'(\xi)(a - b)$ $(a < \xi < b)$, we get that there exists $\xi_k(x)$, $J_{n,k+1}(x) < \xi_k(x) < J_{n,k}(x)$ such that

$$J_{n,k}^{\alpha-1}(x) - J_{n,k+1}^{\alpha-1}(x) = (\alpha - 1)\big(\xi_k(x)\big)^{\alpha-2} p_{n,k}(x) \le (\alpha - 1) J_{n,k+1}^{\alpha-2}(x) p_{n,k}(x).$$

Hence, we have

$$\begin{aligned}\int_{E_n} \delta_n(x) J_1 \, dx &\le C \int_{E_n} \varphi(x) \sum_{k=0}^{n-1} \big|a_k(f)\big| p_{n,k}(x)(\alpha - 1) J_{n,k+1}^{\alpha-2}(x) J'_{n,k+1}(x) \, dx \\ &=: C \sum_{k=0}^{n-1} \big|a_k(f)\big| J.\end{aligned} \tag{3.11}$$

Since $p_{n,k}(x) = \frac{n(1-x)}{n-k} p_{n-1,k}(x)$ for $k < n$, from (3.9) we can deduce that

$$\begin{aligned}J &\le \int_{E_n} \frac{\alpha - 1}{\sqrt{n}} \cdot \frac{n(1 - x)}{n - k} J_{n,k+1}^{\alpha-2}(x) J'_{n,k+1}(x) \, dx \\ &\le \frac{\sqrt{n}}{n - k} \int_0^1 (1 - x) \, dJ_{n,k+1}^{\alpha-1}(x) = \frac{\sqrt{n}}{n - k} \int_0^1 J_{n,k+1}^{\alpha-1}(x) \, dx \\ &= \frac{\sqrt{n}}{n - k} \left(\int_0^{\frac{k}{n}} J_{n,k+1}^{\alpha-1}(x) \, dx + \int_{\frac{k}{n}}^1 J_{n,k+1}^{\alpha-1}(x) \, dx \right) \\ &=: \frac{\sqrt{n}}{n - k} (L_1 + L_2)\end{aligned} \tag{3.12}$$

and

$$\frac{\sqrt{n}}{n - k} L_2 \le \frac{\sqrt{n}}{n - k} \left(1 - \frac{k}{n}\right) = \frac{1}{\sqrt{n}}. \tag{3.13}$$

In order to estimate $\frac{\sqrt{n}}{n-k} L_1$, choose $l \in \mathbb{N}$ such that $l(\alpha - 1) > 1$. Then, for $k < n$, we have

$$\begin{aligned}J_{n,k+1} &= \sum_{j=k+1}^{n} \frac{(n + l - j) \cdots (n - j + 1)}{(n + l) \cdots (n + 1)} p_{n+l,j}(x)(1 - x)^{-l} \\ &\le \frac{(n + l - k)^l}{(n + 1)^l} (1 - x)^{-l}.\end{aligned}$$

Therefore, for $k \le n - 1$, we get

$$\begin{aligned}\frac{\sqrt{n}}{n - k} L_1 &\le \frac{\sqrt{n}}{n - k} \left(\frac{n + l - k}{n + 1} \right)^{l(\alpha-1)} \int_0^{\frac{k}{n}} (1 - x)^{-l(\alpha-1)} \, dx \\ &\le 2^{l(\alpha-1)} \frac{\sqrt{n}}{n - k} \left[\left(\frac{l}{n + 1} \right)^{l(\alpha-1)} + \left(\frac{n - k}{n + 1} \right)^{l(\alpha-1)} \right] \\ &\quad \cdot \frac{1}{l(\alpha - 1) - 1} \left[\left(1 - \frac{k}{n}\right)^{1-l(\alpha-1)} - 1 \right] \\ &\le C_\alpha \frac{\sqrt{n}}{n - k} \left(\frac{n - k}{n + 1} \right)^{l(\alpha-1)} \left(\frac{n - k}{n} \right)^{1-l(\alpha-1)} \le C \frac{1}{\sqrt{n}}.\end{aligned} \tag{3.14}$$

By (3.11)-(3.14), for $1 < \alpha < 2$, we obtain

$$\int_{E_n} \delta_n(x) J_1 \, dx \le C \sum_{k=0}^{n-1} |a_k(f)| \frac{1}{\sqrt{n}} \le C\sqrt{n} \|f\|_1. \tag{3.15}$$

Estimates (3.5)-(3.8) and (3.15) imply (3.1).

Now we prove (3.2). For $f \in W_p$, noting that $J'_{n,0}(x) = 0$, we have

$$\begin{aligned}
&S'_{n,\alpha}(f,x) \\
&= \alpha \left[\sum_{k=1}^{n} a_k(f) J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) - \sum_{k=1}^{n} a_{k-1}(f) J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) \right] \\
&= \alpha \sum_{k=1}^{n} (n+s_n+1) \left[\int_{\frac{k}{n+s_n+1}}^{\frac{k+1}{n+s_n+1}} f(u)\,du - \int_{\frac{k-1}{n+s_n+1}}^{\frac{k}{n+s_n+1}} f(u)\,du \right] J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) \\
&= \alpha \sum_{k=1}^{n} (n+s_n+1) \int_0^{\frac{1}{n+s_n+1}} \left[f\left(\frac{k}{n+s_n+1}+u\right) - f\left(\frac{k-1}{n+s_n+1}+u\right) \right] du J_{n,k}^{\alpha-1}(x) J'_{n,k}(x) \\
&= \alpha \sum_{k=1}^{n} (n+s_n+1) \int_0^{\frac{1}{n+s_n+1}} \int_0^{\frac{1}{n+s_n+1}} f'\left(\frac{k-1}{n+s_n+1}+u+v\right) dv\,du J_{n,k}^{\alpha-1}(x) J'_{n,k}(x).
\end{aligned}$$

Hence,

$$\begin{aligned}
\left|S'_{n,\alpha}(f,x)\right| &\le \alpha \sum_{k=0}^{n-1} \int_0^{\frac{2}{n+s_n+1}} \left| f'\left(\frac{k}{n+s_n+1}+u\right)\right| du J'_{n,k+1}(x) \\
&= \alpha \left(\int_0^{\frac{2}{n+s_n+1}} \left|f'(u)\right| du J'_{n,1}(x) + \int_0^{\frac{2}{n+s_n+1}} \left| f'\left(\frac{n-1}{n+s_n+1}+u\right)\right| du J'_{n,n}(x) \right. \\
&\quad \left. + \sum_{k=1}^{n-2} \int_0^{\frac{2}{n+s_n+1}} \left| f'\left(\frac{k}{n+s_n+1}+u\right)\right| du J'_{n,k+1}(x) \right) \\
&= \alpha(Q_1 + Q_2 + Q_3).
\end{aligned} \tag{3.16}$$

First, we estimate $\int_0^1 \delta_n(x) Q_3 \, dx$. For $1 \le k \le n-2$ and $0 < u < \frac{2}{n+s_n+1}$, we have $\frac{k}{n}(1-\frac{k}{n}) \le C(\frac{k}{n+s_n+1}+u)(1-\frac{k}{n+s_n+1}-u)$ and (similarly to [1], p.155)

$$\begin{aligned}
&\int_0^{\frac{2}{n+s_n+1}} \left| f'\left(\frac{k}{n+s_n+1}+u\right)\right| du \\
&\le C \int_0^{\frac{2}{n+s_n+1}} \left| \varphi\left(\frac{k}{n+s_n+1}+u\right) f'\left(\frac{k}{n+s_n+1}+u\right)\right| du \varphi^{-1}\left(\frac{k}{n}\right) \\
&\le C \varphi^{-1}\left(\frac{k}{n}\right) \int_{\frac{k}{n+s_n+1}}^{\frac{k+2}{n+s_n+1}} \left|\varphi(u) f'(u)\right| du.
\end{aligned}$$

Therefore,

$$\int_0^1 \delta_n(x) Q_3 \, dx \le Cn \sum_{k=1}^{n-2} \int_{\frac{k}{n+s_n+1}}^{\frac{k+2}{n+s_n+1}} \left|\varphi(u) f'(u)\right| du \int_0^1 \delta_n(x) \varphi^{-1}\left(\frac{k}{n}\right) p_{n-1,k}(x)\,dx.$$

Noting that $\varphi^{-1}(\frac{k}{n}) < \sqrt{n}$ for $0 < k < n-1$, we get

$$\begin{aligned}
&\int_0^1 \left(\varphi(x) + \frac{1}{\sqrt{n}}\right)\varphi^{-1}\left(\frac{k}{n}\right)p_{n-1,k}(x)\,dx \\
&\quad \le \frac{2}{\sqrt{n}}\left(\int_0^1 \left(\varphi^2(x) + \frac{1}{n}\right)\varphi^{-2}\left(\frac{k}{n}\right)p_{n-1,k}(x)\,dx\right)^{\frac{1}{2}} \\
&\quad \le \frac{C}{\sqrt{n}}\left(\int_0^1 \varphi^2(x)\frac{n}{k}\frac{n}{n-k}p_{n-1,k}(x)\,dx + \int_0^1 p_{n-1,k}(x)\,dx\right)^{\frac{1}{2}} \\
&\quad \le \frac{C}{\sqrt{n}}\left(\int_0^1 p_{n+1,k+1}(x)\,dx + \frac{1}{n}\right)^{\frac{1}{2}} \le C\frac{1}{n}.
\end{aligned}$$

Hence, we have

$$\|\delta_n Q_3\|_1 \le C\|\varphi f'\|_1 \le C\|\delta_n f'\|_1. \tag{3.17}$$

Since $\sqrt{n}\delta_n(u) \ge 1$, for Q_1, we write

$$\begin{aligned}
\delta_n(x)Q_1 &= \delta_n(x)\int_0^{\frac{2}{n+s_n+1}} |f'(u)|\,du J'_{n,1}(x) \\
&\le \delta_n(x)n\int_0^{\frac{2}{n+s_n+1}} \sqrt{n}|\delta_n(u)f'(u)|\,du p_{n-1,0}(x) \\
&\le \|\delta_n f'\|_1 \delta_n(x)n^{\frac{3}{2}}p_{n-1,0}(x).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_0^1 \delta_n(x)Q_1\,dx &\le C\|\delta_n f'\|_1\left(\int_{E_n^c} np_{n-1,0}(x)\,dx + \int_{E_n}\varphi(x)n^{\frac{3}{2}}p_{n-1,0}(x)\,dx\right) \\
&\le C\|\delta_n f'\|_1\left[1 + n\left(\int_{E_n}\varphi^2(x)p_{n-1,0}(x)\,dx\right)^{\frac{1}{2}}\right] \\
&= C\|\delta_n f'\|_1\left[1 + n\left(\int_{E_n}\frac{1}{n+1}p_{n+1,1}(x)\,dx\right)^{\frac{1}{2}}\right] \le C\|\delta_n f'\|_1. \qquad (3.18)
\end{aligned}$$

Similarly, we have

$$\int_0^1 \delta_n(x)Q_2\,dx \le C\|\delta_n f'\|_1. \tag{3.19}$$

By (3.16)-(3.19) we obtain

$$\|\delta_n(x)L'_{n\alpha}(f,x)\|_1 \le C\|\delta_n f'\|_1.$$

This is (3.2). The proof of Lemma 3.1 is complete. □

Theorem 3.2 *Let $f \in L_p[0,1]$ $(1 \le p \le \infty)$, $\varphi(x) = \sqrt{x(1-x)}$, and $0 < \beta < 1$. Then*

$$\|S_{n,\alpha}(f,x) - f(x)\|_p = O\left(n^{-\frac{\beta}{2}}\right)$$

implies $\omega_\varphi(f,t)_p = O(t^\beta)$.

Proof By Lemma 3.1, for appropriate g, we have

$$\begin{aligned} K_\varphi(f,t)_p &\le \left\|f - L_{n\alpha}(f)\right\|_p + t\left\|\delta_n L'_{n\alpha}(f)\right\|_p \\ &\le Cn^{-\frac{\beta}{2}} + t\left(\left\|\delta_n L'_{n\alpha}(f-g)\right\|_p + \left\|\delta_n L'_{n\alpha}(g)\right\|_p\right) \\ &\le Cn^{-\frac{\beta}{2}} + Ct\left(\sqrt{n}\|f-g\|_p + \left\|\delta_n g'\right\|_p\right) \\ &\le Cn^{-\frac{\beta}{2}} + Ct\sqrt{n}\left(\|f-g\|_p + \frac{1}{\sqrt{n}}\left\|\varphi g'\right\|_p + \frac{1}{n}\left\|g'\right\|_p\right) \\ &\le C\left(n^{-\frac{\beta}{2}} + \frac{t}{n^{-\frac{1}{2}}}\overline{K}_\varphi\left(f, n^{-\frac{1}{2}}\right)_p\right) \\ &\le C\left(n^{-\frac{\beta}{2}} + \frac{t}{n^{-\frac{1}{2}}}K_\varphi\left(f, n^{-\frac{1}{2}}\right)_p\right), \end{aligned}$$

which by the Berens-Lorentz lemma implies that

$$K_\varphi(f,t)_p = O\left(t^\beta\right). \tag{3.20}$$

From relation (1.2) and (3.20) we see that the proof of Theorem 3.2 is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors conceived of the study, participated its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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