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Dilation-and-modulation systems on the half real line

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Abstract

Translation, dilation, and modulation are fundamental operations in wavelet analysis. Affine frames based on translation-and-dilation operation and Gabor frames based on translation-and-modulation operation have been extensively studied and seen great achievements. But dilation-and-modulation frames have not. This paper addresses a class of dilation-and-modulation systems in $L^2(\mathbb{R}_+)$. We characterize frames, dual frames, and Parseval frames in $L^2(\mathbb{R}_+)$ generated by such systems. Interestingly, it turns out that, for such systems, Parseval frames, orthonormal bases, and orthonormal systems are mutually equivalent to each other, while this is not the case for affine systems and Gabor systems.

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1 Introduction

Before proceeding, we recall some notions and notations. An at most countable sequence $\{e_i\}_{i\in\mathcal{I}}$ in a separable Hilbert space $\mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there exist $0 < C_1 \le C_2 < \infty$ such that

$$C_1\|f\|^2 \le \sum_{i\in\mathcal{I}} \bigl|\langle f, e_i\rangle\bigr|^2 \le C_2\|f\|^2 \quad \text{for } f \in \mathcal{H}, \tag{1.1}$$

where C_1, C_2 are called *frame bounds*; it is called a *Bessel sequence* in $\mathcal{H}$ if the right-hand side inequality in (1.1) holds, where C_2 is called a *Bessel bound*. In particular, $\{e_i\}_{i\in\mathcal{I}}$ is called a *Parseval frame* if $C_1 = C_2 = 1$ in (1.1). Given a frame $\{e_i\}_{i\in\mathcal{I}}$ for $\mathcal{H}$, a sequence $\{\tilde{e}_i\}_{i\in\mathcal{I}}$ is called a *dual* of $\{e_i\}_{i\in\mathcal{I}}$ if it is a frame such that

$$f = \sum_{i\in\mathcal{I}} \langle f, \tilde{e}_i\rangle e_i \quad \text{for } f \in \mathcal{H}. \tag{1.2}$$

It is easy to check that $\{e_i\}_{i\in\mathcal{I}}$ is also a dual of $\{\tilde{e}_i\}_{i\in\mathcal{I}}$ if $\{\tilde{e}_i\}_{i\in\mathcal{I}}$ is a dual of $\{e_i\}_{i\in\mathcal{I}}$. So, in this case, we say $\{e_i\}_{i\in\mathcal{I}}$ and $\{\tilde{e}_i\}_{i\in\mathcal{I}}$ form a pair of dual frames for $\mathcal{H}$. It is well known that $\{e_i\}_{i\in\mathcal{I}}$ and $\{\tilde{e}_i\}_{i\in\mathcal{I}}$ form a pair of dual frames for $\mathcal{H}$ if they are Bessel sequences and satisfy (1.2). The fundamentals of frames can be found in [1–3]. The *Fourier transform* of $c \in l^2(\mathbb{Z})$ is defined by $\hat{c}(\cdot) = \sum_{m\in\mathbb{Z}} c(m)e^{-2\pi i m\cdot}$. For two sequences c and d on $\mathbb{Z}$, the *convolution* $c * d$

is defined by

$$c * d(k) = \sum_{m\in\mathbb{Z}} c(k-m)d(m) \quad \text{for } k \in \mathbb{Z}$$

if it is well defined. The *Kronecker delta* is defined by $\delta_{n,m} = \begin{cases} 1 & \text{if } n = m; \\ 0 & \text{if } n \neq m. \end{cases}$ $l_0(\mathbb{Z})$ denotes the set of finitely supported sequences on $\mathbb{Z}$. We denote by I the identity operator on $l^2(\mathbb{Z})$, and by χ_E its characteristic function for a set E. Write $\mathbb{R}_+ = (0,\infty)$. For a positive number $a > 1$, a function h defined on $\mathbb{R}_+$ is said to be *a-dilation periodic* if $h(a\cdot) = h(\cdot)$ on $\mathbb{R}_+$. For a function f defined on $[1,a)$, we define the function $\tilde{f}$ on $\mathbb{R}_+$ by

$$\tilde{f}(\cdot) = f\left(a^{-l}\cdot\right) \quad \text{on } \left[a^l, a^{l+1}\right) \text{ for } l \in \mathbb{Z},$$

which is called the *a-dilation periodization* of f. Obviously, it is *a-dilation periodic*.

The translation operator T_{x_0}, the modulation operator M_{x_0} with $x_0 \in \mathbb{R}$, and the dilation D_c with $c > 0$ are, respectively, defined by

$$T_{x_0}f(\cdot) = f(\cdot - x_0),$$
$$M_{x_0}f(\cdot) = e^{2\pi i x_0 \cdot} f(\cdot),$$

and

$$D_c f(\cdot) = \sqrt{c} f(c\cdot)$$

for $f \in L^2(\mathbb{R})$. They are the basis of wavelet analysis. Affine systems of the form $\{D_{a^j} T_{bk}\psi : j,k \in \mathbb{Z}\}$ with $\psi \in L^2(\mathbb{R})$ and a, $b > 0$, and Gabor systems of the form $\{E_{mb} T_{na} g : m,n \in \mathbb{Z}\}$ with $g \in L^2(\mathbb{R})$ and a, $b > 0$ have been extensively studied. However, dilation-and-modulation systems of the form

$$\{M_{mb} D_{a^j}\psi : m,j \in \mathbb{Z}\} \quad \text{with } a,b > 0 \tag{1.3}$$

have not been extensively studied. This paper focuses on the following systems that are like (1.3):

$$\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\} \quad \text{with } a > 0, \tag{1.4}$$

where

$$\psi_m(\cdot) = \frac{1}{\sqrt{a-1}} e^{2\pi i \frac{m\cdot}{a-1}} \quad \text{on } [1,a) \text{ for } m \in \mathbb{Z}. \tag{1.5}$$

We will investigate the theory of $L^2(\mathbb{R}_+)$-frames of the form (1.4). It is obvious that $L^2(\mathbb{R}_+)$ can be considered as the Fourier transform of the Hardy space $H^2(\mathbb{R})$ defined by

$$H^2(\mathbb{R}) = \left\{f \in L^2(\mathbb{R}) : \hat{f}(\cdot) = 0 \text{ a.e. on } (-\infty,0)\right\},$$

where the Fourier transform is defined by

$$\hat{f}(\cdot) = \int_{\mathbb{R}} f(x)e^{-2\pi ix\cdot}\,dx \quad \text{for } f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$$

and extended to $L^2(\mathbb{R})$ by the Plancherel theorem. Wavelet frames in $H^2(\mathbb{R})$ of the form $\{D_{a^j}T_m\varphi : j,m \in \mathbb{Z}\}$ were studied in [4, 5], and some variations can be found in [6–11]. By the Plancherel theorem, an $H^2(\mathbb{R})$-frame $\{D_{2^j}T_m\varphi : j,m \in \mathbb{Z}\}$ leads to a $L^2(\mathbb{R}_+)$-frame

$$\left\{e^{-2\pi i2^{-j}m\cdot}\hat{\varphi}\left(2^{-j}\cdot\right) : j,m \in \mathbb{Z}\right\}. \tag{1.6}$$

In (1.6), $e^{-2\pi i2^{-j}m\cdot}$ is $2^j\mathbb{Z}$-periodic with respect to additive operation, and the period depends on the dilation factor 2^j. However, $\widetilde{\psi_m}$ in (1.4) is a-dilation periodic and unrelated to j. Therefore, frames of the form (1.4) are different from ones of the form (1.6) for $L^2(\mathbb{R}_+)$ and of independent interest. They are related to a kind of function-valued frames in [12]. In [13], numerical experiments were made to establish that the nonnegative integer shifts of the Gaussian function formed a Riesz sequence in $L^2(\mathbb{R}_+)$. In [14], a sufficient condition was obtained to determine whether the nonnegative translates form a Riesz sequence on $L^2(\mathbb{R}_+)$.

The rest of this paper is organized as follows. Section 2 is devoted to characterizing frames and dual frames for $L^2(\mathbb{R}_+)$ with the structure of (1.4). Section 3 is devoted to Parseval frames and orthonormal bases for $L^2(\mathbb{R}_+)$ of the form (1.4). It turns out that Parseval frames, orthonormal bases, and orthonormal systems in $L^2(\mathbb{R}_+)$ of the form (1.4) are mutually equivalent to each other. It is worth noting that neither affine systems nor Gabor systems have such a property.

2 Frame and dual frame characterization

This section characterizes $L^2(\mathbb{R}_+)$-frames and dual frames of the form (1.4). For this purpose, we need some notations and lemmas. For $f \in L^2(\mathbb{R}_+)$, we define

$$\mathcal{G}(f,\cdot) = \left(\overline{D_{a^{j+l}}f(\cdot)}\right)_{j,l\in\mathbb{Z}} \tag{2.1}$$

and

$$F(\cdot) = \left(D_{a^l}f(\cdot)\right)_{l\in\mathbb{Z}}. \tag{2.2}$$

Lemma 2.1 *For $m, j \in \mathbb{Z}$, define ψ_m as in* (1.5), *then*

(i) $\{\psi_m : m \in \mathbb{Z}\}$ *is an orthonormal bases for* $L^2([1,a))$;

(ii)

$$\int_{[1,a)} \left|f(x)\right|^2 dx = \sum_{m\in\mathbb{Z}} \left|\int_{[1,a)} f(x)\overline{\psi_m(x)}\,dx\right|^2 \tag{2.3}$$

for $f \in L^1([1,a))$.

Proof (i) By a simple computation, we see that $T_1 D_{(a-1)^{-1}}$ is a unitary operator from $L^2([0,1))$ onto $L^2([1,a))$, and

$$\psi_m(\cdot) = e^{\frac{2\pi im}{a-1}} T_1 D_{(a-1)^{-1}} e^{2\pi im\cdot}$$

for $m \in \mathbb{Z}$. Also observing that $\{e^{2\pi im\cdot} : m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([0,1))$, we have (i).

(ii) By (i), (2.3) holds if $f \in L^2([1,a))$. When $f \in L^1([1,a)) \setminus L^2([1,a))$, the left-hand side of (2.3) is infinity. Now we prove by contradiction that the right-hand side of (2.3) is also infinity. Suppose it is finite. Since $\{\psi_m : m \in \mathbb{Z}\}$ is an orthonormal basis for $L^2([1,a))$, we see that

$$g = \sum_{m\in\mathbb{Z}} \left(\int_{[1,a)} f(x)\overline{\psi_m(x)}\, dx \right) \psi_m$$

belongs to $L^2([1,a))$, and thus to $L^1([1,a))$. It has the same Fourier coefficients as f. So $f = g$ by the uniqueness of Fourier coefficients, and thus $f \in L^2([1,a))$. This is a contradiction. The proof is completed. □

Lemma 2.2 *Let $c, d \in l^2(\mathbb{Z})$. Suppose $c * d \in l^2(\mathbb{Z})$, then*

$$(c * d)^\wedge(\cdot) = \hat{c}(\cdot)\hat{d}(\cdot) \quad \textit{a.e. on } \mathbb{R}.$$

Proof Since $c * d \in l^2(\mathbb{Z})$, we have $(c * d)^\wedge \in L^2([0,1))$, and thus $(c * d)^\wedge \in L^1([0,1))$. Also observing that $\hat{c} \cdot \hat{d} \in L^1([0,1))$, to finish the proof we only need to prove that

$$\int_{[0,1)} \hat{c}(\xi)\hat{d}(\xi)e^{2\pi ik\xi}\, d\xi = c * d(k) \quad \text{for } k \in \mathbb{Z}. \tag{2.4}$$

Define c_n by

$$c_n(k) = \begin{cases} c(k) & \text{if } |k| \leq n; \\ 0 & \text{if } |k| > n \end{cases}$$

for $n \in \mathbb{N}$. Then $c_n \in l^1(\mathbb{Z})$, and

$$\|c_n - c\|_{l^2(\mathbb{Z})} = \|\hat{c}_n - \hat{c}\|_{L^2([0,1))} \to 0 \quad \text{as } n \to \infty. \tag{2.5}$$

Since $c_n \in l^1(\mathbb{Z})$ and $d \in l^2(\mathbb{Z})$, we have $(c_n * d)^\wedge = \hat{c}_n\hat{d} \in L^2([0,1))$, and thus

$$\int_{[0,1)} \hat{c}_n(\xi)\hat{d}(\xi)e^{2\pi ik\xi}\, d\xi = \sum_{l\in\mathbb{Z}} c_n(k-l)d(l) \quad \text{for } k \in \mathbb{Z}. \tag{2.6}$$

Arbitrarily fix $k \in \mathbb{Z}$. Observe that

$$\left| \sum_{l\in\mathbb{Z}} c_n(k-l)d(l) - \sum_{l\in\mathbb{Z}} c(k-l)d(l) \right| \leq \|c_n - c\|_{l^2(\mathbb{Z})}\|d\|_{l^2(\mathbb{Z})} \to 0$$

and

$$\left|\int_{[0,1)} \hat{c}_n(\xi)\hat{d}(\xi)e^{2\pi ik\xi}\,d\xi - \int_{[0,1)} \hat{c}(\xi)\hat{d}(\xi)e^{2\pi ik\xi}\,d\xi\right| \leq \|\hat{c}_n - \hat{c}\|_{L^2([0,1))}\|\hat{d}\|_{L^2([0,1))} \to 0$$

as $n \to \infty$ by (2.5). So, letting $n \to \infty$ in (2.6) we have (2.4). The proof is completed. □

Lemma 2.3 *For $f, \psi \in L^2(\mathbb{R}_+)$, we have*

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|\langle f, \widetilde{\psi_m}D_{a^j}\psi\rangle_{L^2(\mathbb{R}_+)}\right|^2 = \int_1^a \left\|\mathcal{G}(\psi,x)F(x)\right\|^2_{l^2(\mathbb{Z})}\,dx. \tag{2.7}$$

Proof Since $f(\cdot)\overline{\widetilde{\psi_m}(\cdot)D_{a^j}\psi(\cdot)} \in L^1(\mathbb{R}_+)$, we have

$$\begin{aligned}\langle f, \widetilde{\psi_m}D_{a^j}\psi\rangle_{L^2(\mathbb{R}_+)} &= \int_1^a \sum_{l\in\mathbb{Z}} a^l f\left(a^l x\right)\overline{\widetilde{\psi_m}\left(a^l x\right)(D_{a^j}\psi)\left(a^l x\right)}\,dx \\ &= \int_1^a \overline{\psi_m(x)}\sum_{l\in\mathbb{Z}}(D_{a^l}f)(x)\overline{D_{a^{j+l}}\psi(x)}\,dx.\end{aligned}$$

It follows that

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|\langle f, \widetilde{\psi_m}D_{a^j}\psi\rangle_{L^2(\mathbb{R}_+)}\right|^2 = \sum_{j\in\mathbb{Z}}\int_1^a\left|\sum_{l\in\mathbb{Z}}(D_{a^l}f)(x)\overline{(D_{a^{j+l}}\psi)(x)}\right|^2\,dx$$

by Lemma 2.1(ii), and thus (2.7) holds. The proof is completed. □

Theorem 2.1 *Let $\psi \in L^2(\mathbb{R}_+)$. The following are equivalent*:

(i) *$\{\widetilde{\psi_m}D_{a^j}\psi : m, j \in \mathbb{Z}\}$ is a Bessel sequence in $L^2(\mathbb{R}_+)$ with Bessel bound B.*
(ii) *$\mathcal{G}^*\mathcal{G}(\psi,\cdot) \leq BI$ a.e. on $[1,a)$.*
(iii) *$|\sum_{l\in\mathbb{Z}} a^{\frac{l}{2}}\psi(a^l x)e^{-2\pi il\xi}| \leq \sqrt{B}$ for a.e. $(x,\xi) \in [1,a) \times [0,1)$.*

Proof Since

$$\int_1^a \sum_{j\in\mathbb{Z}}\left|D_{a^j}\psi(x)\right|^2\,dx = \int_0^\infty \left|\psi(x)\right|^2\,dx < \infty,$$

we have $\sum_{j\in\mathbb{Z}}|D_{a^j}\psi(\cdot)|^2 < \infty$ a.e. on $[1,a)$. So all rows and columns of $\mathcal{G}(\psi,x)$ belong to $l^2(\mathbb{Z})$ for a.e. $x \in [1,a)$. It follows that $\mathcal{G}^*\mathcal{G}(\psi,\cdot)$ is well defined a.e. on $[1,a)$ by the Cauchy-Schwartz inequality. Thus $\langle\mathcal{G}^*\mathcal{G}(\psi,\cdot)c,c\rangle_{l^2(\mathbb{Z})}$ and $\|\mathcal{G}(\psi,\cdot)c\|^2_{l^2(\mathbb{Z})}$ are well defined, and

$$\left\langle\mathcal{G}^*\mathcal{G}(\psi,\cdot)c,c\right\rangle_{l^2(\mathbb{Z})} = \left\|\mathcal{G}(\psi,\cdot)c\right\|^2_{l^2(\mathbb{Z})} \tag{2.8}$$

a.e. on $[1,a)$ for each $c \in l_0(\mathbb{Z})$. First we show the equivalence between (i) and (ii). Suppose $\mathcal{G}^*\mathcal{G}(\psi,\cdot) \leq BI$ a.e. on $[1,a)$. Arbitrarily fix $f \in L^2(\mathbb{R}_+)$. It is easy to check that

$$\int_1^a \left\|F(x)\right\|^2_{l^2(\mathbb{Z})}\,dx = \|f\|^2_{L^2(\mathbb{R}_+)} < \infty.$$

So $F(x) \in l^2(\mathbb{Z})$, and thus

$$\left\|\mathcal{G}(\psi,x)F(x)\right\|^2_{l^2(\mathbb{Z})} = \left\langle \mathcal{G}^*\mathcal{G}(\psi,x)F(x),F(x)\right\rangle_{l^2(\mathbb{Z})} \leq B\left\|F(x)\right\|^2_{l^2(\mathbb{Z})},$$

for a.e. $x \in [1,a)$. It follows that

$$\int_1^a \left\|\mathcal{G}(\psi,x)F(x)\right\|^2_{l^2(\mathbb{Z})}\,dx \leq B\int_1^a \left\|F(x)\right\|^2_{l^2(\mathbb{Z})}\,dx = B\|f\|^2_{L^2(\mathbb{R}_+)}.$$

Therefore,

$$\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\left|\langle f,\widetilde{\psi}_m D_{a^j}\psi\rangle_{L^2(\mathbb{R}_+)}\right|^2 \leq B\|f\|^2_{L^2(\mathbb{R}_+)},$$

by Lemma 2.3, and thus $\{\widetilde{\psi}_m D_{a^j}\psi : m,j \in \mathbb{Z}\}$ is a Bessel sequence in $L^2(\mathbb{R}_+)$ with Bessel bound B.

Now suppose (i) holds. We prove (ii) by contradiction. Suppose $\mathcal{G}^*\mathcal{G}(\psi,\cdot) > BI$ on some $E \subset [1,a)$ with $|E| > 0$. Take $0 \neq c \in l_0(\mathbb{Z})$ and define $f \in L^2(\mathbb{R}_+)$ by

$$F(\cdot) = \chi_E(\cdot)c \quad \text{on } [1,a).$$

Then f is well defined since

$$\|f\|^2_{L^2(\mathbb{R}_+)} = \int_1^a \left\|F(x)\right\|^2_{l^2(\mathbb{Z})}\,dx = |E|\,\|c\|^2_{l^2(\mathbb{Z})}.$$

Let us check $\int_1^a \|\mathcal{G}(\psi,x)F(x)\|^2_{l^2(\mathbb{Z})}\,dx$:

$$\begin{aligned}\int_1^a \left\|\mathcal{G}(\psi,x)F(x)\right\|^2_{l^2(\mathbb{Z})}\,dx &= \int_1^a \left\langle \mathcal{G}^*\mathcal{G}(\psi,x)F(x),F(x)\right\rangle_{l^2(\mathbb{Z})}\,dx \\ &= \int_E \left\langle \mathcal{G}^*\mathcal{G}(\psi,x)F(x),F(x)\right\rangle_{l^2(\mathbb{Z})}\,dx \\ &> B\int_1^a \left\|F(x)\right\|^2_{l^2(\mathbb{Z})}\,dx \\ &= B\|f\|^2_{L^2(\mathbb{R}_+)}.\end{aligned}$$

So $\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|\langle f,\widetilde{\psi}_m D_{a^j}\psi\rangle_{L^2(\mathbb{R}_+)}|^2 > B\|f\|^2_{L^2(\mathbb{R}_+)}$ by Lemma 2.3, contradicting the fact that $\{\widetilde{\psi}_m D_{a^j}\psi : m,j \in \mathbb{Z}\}$ is a Bessel sequence in $L^2(\mathbb{R}_+)$ with Bessel bound B.

Next we prove the equivalence between (ii) and (iii). By (2.8) and the density of $l_0(\mathbb{Z})$ in $l^2(\mathbb{Z})$, (ii) is equivalent to

$$\left\|\mathcal{G}(\psi,x)c\right\|^2_{l^2(\mathbb{Z})} \leq B\|c\|^2_{l^2(\mathbb{Z})} \quad \text{for } c \in l_0(\mathbb{Z}) \text{ and a.e. } x \in [1,a),$$

which is equivalent to

$$\left\|\overline{\mathcal{G}(\psi,x)}c\right\|^2_{l^2(\mathbb{Z})} \leq B\|c\|^2_{l^2(\mathbb{Z})} \quad \text{for } c \in l_0(\mathbb{Z}) \text{ and a.e. } x \in [1,a). \tag{2.9}$$

For x satisfying (2.9), define $\alpha_x \in l^2(\mathbb{Z})$ by

$$\alpha_x(l) = a^{\frac{l}{2}} \psi\left(a^l x\right).$$

Then (2.9) can be rewritten as

$$\|\alpha_x * d\|^2_{l^2(\mathbb{Z})} \le B\|d\|^2_{l^2(\mathbb{Z})} \quad \text{for } d \in l_0(\mathbb{Z}) \text{ and a.e. } x \in [1,a),$$

equivalently,

$$\int_0^1 \left|\hat{\alpha}_x(\xi)\hat{d}(\xi)\right|^2 d\xi \le B \int_0^1 \left|\hat{d}(\xi)\right|^2 d\xi \quad \text{for } d \in l_0(\mathbb{Z}) \text{ and a.e. } x \in [1,a)$$

by Lemma 2.2, this is also equivalent to

$$\int_0^1 \left|\hat{\alpha}_x(\xi)\right|^2\left|\hat{d}(\xi)\right|^2 d\xi \le B \int_0^1 \left|\hat{d}(\xi)\right|^2 d\xi \quad \text{for } d \in l^2(\mathbb{Z}) \text{ and a.e. } x \in [1,a), \tag{2.10}$$

by the density of trigonometric polynomials in $L^2([0,1))$. It is obvious that (2.10) is equivalent to (iii). The proof is completed. □

By the same procedure as in the proof of Theorem 2.1, we have the following theorem.

Theorem 2.2 *Let $\psi \in L^2(\mathbb{R}_+)$. The following are equivalent*:

(i) $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ *is a frame for $L^2(\mathbb{R}_+)$ with frames bounds A and B*;

(ii) $AI \le \mathcal{G}^*\mathcal{G}(\psi,\cdot) \le BI$ *a.e. on* $[1,a)$;

(iii) $\sqrt{A} \le |\sum_{l\in\mathbb{Z}} a^{\frac{l}{2}}\psi(a^l x)e^{-2\pi i l\xi}| \le \sqrt{B}$ *for a.e.* $(x,\xi) \in [1,a) \times [0,1)$.

Theorem 2.3 *Let $\psi, \varphi \in L^2(\mathbb{R}_+)$. Suppose $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ and $\{\widetilde{\psi_m} D_{a^j}\varphi : m,j \in \mathbb{Z}\}$ are Bessel sequences in $L^2(\mathbb{R}_+)$. The following are equivalent*:

(i) $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ *and* $\{\widetilde{\psi_m} D_{a^j}\varphi : m,j \in \mathbb{Z}\}$ *form a pair of dual frames for* $L^2(\mathbb{R}_+)$.

(ii) $\sum_{j\in\mathbb{Z}} a^j\varphi(a^j\cdot)\overline{\psi(a^{j+l}\cdot)} = \delta_{l,0}$ *a.e. on* $[1,a)$ *for* $l \in \mathbb{Z}$.

(iii) $(\sum_{l\in\mathbb{Z}} a^{\frac{l}{2}}\varphi(a^l x)e^{-2\pi i l\xi})(\sum_{l\in\mathbb{Z}} a^{\frac{l}{2}}\overline{\psi(a^l x)}e^{2\pi i l\xi}) = 1$ *for a.e.* $(x,\xi) \in [1,a) \times [0,1)$.

Proof Since both $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ and $\{\widetilde{\psi_m} D_{a^j}\varphi : m,j \in \mathbb{Z}\}$ are Bessel sequences in $L^2(\mathbb{R}_+)$,

$$\sum_{(m,j)\in\mathbb{Z}^2} \langle f, \widetilde{\psi_m} D_{a^j}\psi\rangle_{L^2(\mathbb{R}_+)} \langle \widetilde{\psi_m} D_{a^j}\varphi, g\rangle_{L^2(\mathbb{R}_+)}$$

is well defined for each f, $g \in L^2(\mathbb{R}_+)$. By Theorem 2.1, $\mathcal{G}(\psi,x)$ and $\mathcal{G}(\varphi,x)$ are bounded linear operators on $l^2(\mathbb{Z})$ for a.e. $x \in [1,a)$. Let us check the (k,l)-entry of $\mathcal{G}^*(\varphi,\cdot)\mathcal{G}(\psi,\cdot)$:

$$\begin{aligned}\left[\mathcal{G}^*(\varphi,\cdot)\mathcal{G}(\psi,\cdot)\right]_{k,l} &= \sum_{j\in\mathbb{Z}} D_{a^{j+k}}\varphi(\cdot)\overline{D_{a^{j+l}}\psi(\cdot)} = \sum_{j\in\mathbb{Z}} a^{\frac{j+k}{2}}\varphi\left(a^{j+k}\cdot\right)a^{\frac{j+l}{2}}\overline{\psi\left(a^{j+l}\cdot\right)} \\ &= a^{\frac{l-k}{2}} \sum_{j\in\mathbb{Z}} a^j\varphi\left(a^j\cdot\right)\overline{\psi\left(a^{j+l-k}\cdot\right)}.\end{aligned}$$

So (ii) is equivalent to

$$\mathcal{G}^*(\varphi,\cdot)\mathcal{G}(\psi,\cdot)=I \quad \text{a.e. on } [1,a). \tag{2.11}$$

By Lemma 2.3 and the polarization identity, we have

$$\sum_{(m,j)\in\mathbb{Z}^2}\langle f,\widetilde{\psi_m}D_{a^j}\psi\rangle_{L^2(\mathbb{R}_+)}\langle\widetilde{\psi_m}D_{a^j}\varphi,g\rangle_{L^2(\mathbb{R}_+)}=\int_1^a\bigl\langle\mathcal{G}^*(\varphi,x)\mathcal{G}(\psi,x)F(x),G(x)\bigr\rangle_{l^2(\mathbb{Z})}\,dx$$

for $f,g\in L^2(\mathbb{R}_+)$. By a simply computation, we also have

$$\langle f,g\rangle_{L^2(\mathbb{R}_+)}=\int_1^a\bigl\langle F(x),G(x)\bigr\rangle_{l^2(\mathbb{Z})}\,dx$$

for f, $g\in L^2(\mathbb{R}_+)$. So (i) holds if and only if

$$\begin{aligned}&\int_1^a\bigl\langle\mathcal{G}^*(\varphi,x)\mathcal{G}(\psi,x)F(x),G(x)\bigr\rangle_{l^2(\mathbb{Z})}\,dx\\&\quad=\int_1^a\bigl\langle F(x),G(x)\bigr\rangle_{l^2(\mathbb{Z})}\,dx\quad\text{for } f,g\in L^2(\mathbb{R}_+).\end{aligned}\tag{2.12}$$

So, to show the equivalence between (i) and (ii), next we only need to prove the equivalence between (2.11) and (2.12). Obviously, (2.11) implies (2.12). Now we prove the converse implication. Suppose (2.12) holds. Let e_n be the vector in $l^2(\mathbb{Z})$ with the nth component being 1 and the others being zero for each $n\in\mathbb{Z}$. Since all entries of $\mathcal{G}^*(\varphi,\cdot)\mathcal{G}(\psi,\cdot)$ are in $L^1[1,a)$, there exists $E_0\subset[1,a)$ with $|E_0|=0$ such that every point of $(1,a)\setminus E_0$ is a Lebesgue point of all entries of $\mathcal{G}^*(\varphi,\cdot)\mathcal{G}(\psi,\cdot)$. Arbitrarily fix $x_0\in(1,a)\setminus E_0$ and n_0, $m_0\in\mathbb{Z}$. Take $\varepsilon>0$ so small that $(x_0-\varepsilon,x_0+\varepsilon)\subset(1,a)$, and take f and g such that

$$F(\cdot)=\chi_{(x_0-\varepsilon,x_0+\varepsilon)}(\cdot)e_{n_0}\quad\text{and}\quad G(\cdot)=\chi_{(x_0-\varepsilon,x_0+\varepsilon)}(\cdot)e_{m_0}$$

a.e. on $[1,a)$. Then from (2.12) we have

$$\frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon}\bigl\langle\mathcal{G}^*(\varphi,x)\mathcal{G}(\psi,x)e_{n_0},e_{m_0}\bigr\rangle_{l^2(\mathbb{Z})}\,dx=\delta_{n_0,m_0}.$$

This leads to

$$\bigl\langle\mathcal{G}^*(\varphi,x_0)\mathcal{G}(\psi,x_0)e_{n_0},e_{m_0}\bigr\rangle_{l^2(\mathbb{Z})}=\delta_{n_0,m_0},$$

by letting $\varepsilon\to0$. By the arbitrariness of x_0, n_0, and m_0, we have

$$\bigl\langle\mathcal{G}^*(\varphi,\cdot)\mathcal{G}(\psi,\cdot)e_n,e_m\bigr\rangle_{l^2(\mathbb{Z})}=\delta_{n,m}$$

on $[1,a)\setminus E_0$ for n, $m\in\mathbb{Z}$. This implies that

$$\mathcal{G}^*(\varphi,\cdot)\mathcal{G}(\psi,\cdot)=I\quad\text{on } (1,a)\setminus E_0.$$

Therefore, we obtain the equivalence between (i) and (ii).

Now we prove the equivalence between (ii) and (iii). Obviously, (ii) is equivalent to

$$\sum_{j\in\mathbb{Z}} a^{\frac{j}{2}}\varphi(a^j\cdot)a^{\frac{j+l}{2}}\overline{\psi(a^{j+l}\cdot)}=\delta_{l,0}\quad\text{a.e. on }[1,a)\text{ for }l\in\mathbb{Z},$$

that is,

$$\sum_{j\in\mathbb{Z}} a^{-\frac{j}{2}}\varphi(a^{-j}\cdot)a^{\frac{l-j}{2}}\overline{\psi(a^{l-j}\cdot)}=\delta_{l,0}\quad\text{a.e. on }[1,a)\text{ for }l\in\mathbb{Z}.\tag{2.13}$$

For x satisfying (2.13), define $\alpha_x,\beta_x\in l^2(\mathbb{Z})$ by

$$\alpha_x(l)=a^{\frac{l}{2}}\overline{\psi(a^lx)}\quad\text{and}\quad\beta_x(l)=a^{-\frac{l}{2}}\varphi(a^{-l}x).$$

Then (2.13) can be rewritten as

$$\alpha_x*\beta_x=\delta_{l,0}\quad\text{for a.e. }x\in[1,a)\text{ and }l\in\mathbb{Z}.$$

This is equivalent to

$$\Big(\sum_{l\in\mathbb{Z}} a^{\frac{l}{2}}\overline{\psi(a^lx)}e^{-2\pi il\xi}\Big)\Big(\sum_{l\in\mathbb{Z}} a^{\frac{l}{2}}\varphi(a^lx)e^{2\pi il\xi}\Big)=1\quad\text{a.e. }(x,\xi)\in[1,a)\times[0,1)$$

by Lemma 2.2, and thus equivalent to (iii) by its $\mathbb{Z}$-periodicity with respect to ξ. The proof is completed. □

3 Parseval frames and orthonormal bases

The following theorem characterizes Parseval frames of the form (1.4). It is worth noting that the theorem shows all such Parseval frames must be orthonormal bases. This is not the case for Parseval wavelet (Gabor) frames.

Theorem 3.1 *Let $\psi\in L^2(\mathbb{R}_+)$. The following are equivalent*:

(i) $\{\widetilde{\psi_m}D_{a^j}\psi:m,j\in\mathbb{Z}\}$ *is a Parseval frames for* $L^2(\mathbb{R}_+)$.

(ii) $\sum_{j\in\mathbb{Z}}a^j\psi(a^j\cdot)\overline{\psi(a^{j+l}\cdot)}=\delta_{l,0}$ *a.e. on* $[1,a)$ *for* $l\in\mathbb{Z}$.

(iii) $|\sum_{l\in\mathbb{Z}}a^{\frac{l}{2}}\psi(a^lx)e^{2\pi il\xi}|=1$ *for a.e.* $(x,\xi)\in[1,a)\times[0,1)$.

(iv) $\{\widetilde{\psi_m}D_{a^j}\psi:m,j\in\mathbb{Z}\}$ *is an orthonormal basis for* $L^2(\mathbb{R}_+)$.

Proof Since (ii) and (iii) are equivalent, and (iii) implies that $\{\widetilde{\psi_m}D_{a^j}\psi:m,j\in\mathbb{Z}\}$ is a Bessel sequence by Theorem 2.1, we can obtain the equivalence among (i), (ii), and (iii) by taking $\varphi=\psi$ in Theorem 2.3. It is obvious that (iv) implies (i). Now we prove the converse implication to finish the proof. Suppose (i) holds. Then (iv) holds if and only if

$$\|\widetilde{\psi_m}D_{a^j}\psi\|_{L^2(\mathbb{R}_+)}=1\quad\text{for }m,j\in\mathbb{Z},$$

equivalently, $\|\psi\|^2_{L^2(\mathbb{R}_+)}=a-1$. By the equivalence between (i) and (ii), we have

$$\sum_{j\in\mathbb{Z}}a^j\big|\psi(a^j\cdot)\big|^2=1\quad\text{a.e. on }[1,a).$$

Integrating this equation over $[1,a)$ leads to

$$\|\psi\|^2_{L^2(\mathbb{R}_+)} = a-1.$$

So (iv) holds. The proof is completed. □

Theorem 3.2 *Let $0 \neq \psi \in L^2(\mathbb{R}_+)$. Then we have*

(i) $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ *is an orthogonal system in $L^2(\mathbb{R}_+)$ if and only if*

$$\sum_{j\in\mathbb{Z}} a^j \psi\left(a^j\cdot\right)\overline{\psi\left(a^{j+l}\cdot\right)} = \frac{\|\psi\|^2_{L^2(\mathbb{R}_+)}}{a-1}\delta_{l,0} \quad \text{a.e. on } [1,a) \text{ for } l \in \mathbb{Z}; \tag{3.1}$$

(ii) $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ *is an orthonormal system for $L^2(\mathbb{R}_+)$ if and only if*

$$\sum_{j\in\mathbb{Z}} a^j \psi\left(a^j\cdot\right)\overline{\psi\left(a^{j+l}\cdot\right)} = \delta_{l,0} \quad \text{a.e. on } [1,a) \text{ for } l \in \mathbb{Z}. \tag{3.2}$$

Proof (i) By a simple computation, we have

$$\langle \widetilde{\psi_{m_1}} D_{a^{j_1}}\psi, \widetilde{\psi_{m_2}} D_{a^{j_2}}\psi\rangle_{L^2(\mathbb{R}_+)} = \frac{1}{\sqrt{a-1}} a^{\frac{j_2-j_1}{2}} \int_1^a \psi_{m_1-m_2}(x) \sum_{j\in\mathbb{Z}} a^j \psi\left(a^j x\right)\overline{\psi\left(a^{j+j_2-j_1}x\right)}\,dx$$

for $(m_1,j_1),(m_2,j_2) \in \mathbb{Z}^2$. It follows that $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ is an orthogonal system in $L^2(\mathbb{R}_+)$ if and only if

$$\int_1^a \psi_m(x) \sum_{j\in\mathbb{Z}} a^j \psi\left(a^j x\right)\overline{\psi\left(a^{j+l}x\right)}\,dx = 0 \quad \text{for } (0,0) \neq (m,l) \in \mathbb{Z}^2. \tag{3.3}$$

Also observing that

$$\|\psi\|^2_{L^2(\mathbb{R}_+)} = \int_1^a \sum_{j\in\mathbb{Z}} a^j \left|\psi\left(a^j x\right)\right|^2 dx, \tag{3.4}$$

we have (i) by Lemma 2.1(i) and the uniqueness theorem of Fourier coefficients.

(ii) $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ is an orthonormal system for $L^2(\mathbb{R}_+)$ if and only if it is an orthogonal system and $\|\psi\|^2_{L^2(\mathbb{R}_+)} = a-1$. So we have (ii) by (i) and (3.4). The proof is completed. □

As an immediate consequence of Theorems 3.1 and 3.2, we have the following corollary.

Corollary 3.1 *Let $0 \neq \psi \in L^2(\mathbb{R}_+)$. Then the following are equivalent:*

(i) $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ *is a Parseval frame for $L^2(\mathbb{R}_+)$.*

(ii) $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ *is an orthonormal basis for $L^2(\mathbb{R}_+)$.*

(iii) $\{\widetilde{\psi_m} D_{a^j}\psi : m,j \in \mathbb{Z}\}$ *is an orthonormal system in $L^2(\mathbb{R}_+)$.*

(iv) $\sum_{j\in\mathbb{Z}} a^j \psi(a^j\cdot)\overline{\psi(a^{j+l}\cdot)} = \delta_{l,0}$ *a.e. on $[1,a)$ for $l \in \mathbb{Z}$.*

(v) $|\sum_{l\in\mathbb{Z}} a^{\frac{l}{2}} \psi(a^l x) e^{2\pi i l\xi}| = 1$ *for a.e. $(x,\xi) \in [1,a) \times [0,1)$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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