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Maximum norm analysis of a nonmatching grids method for a class of variational inequalities with nonlinear source terms

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Abstract

In this paper, we study a nonmatching grid finite element approximation of a class of elliptic variational inequalities with nonlinear source terms in the context of the Schwarz alternating domain decomposition. We show that the approximation converges optimally in the maximum norm, on each subdomain, making use of a Lipschitz continuous dependence with respect to both the boundary condition and the source term.

MSC: 65K15; 65N30; 65N15

Keywords: variational inequalities with nonlinear source terms; Schwarz alternating method; nonmatching grids; finite element; L^∞-error estimate

1 Introduction

This paper deals with the error analysis in the maximum norm, in the context of the nonmatching grids method, of the following variational inequalities with nonlinear source terms: find $u \in K^g$ such that

$$a(u, v-u) + c(u, v-u) \geq \bigl(f(u), v-u\bigr), \quad \forall v \in K^g, \tag{1.1}$$

where $a(u,v) = \int_\Omega (\nabla u \cdot \nabla v)\,dx$ is the bilinear form defined in a bounded domain Ω of $\mathbb{R}^2$ or $\mathbb{R}^3$, c is a positive constant such that

$$c \geq \beta > 0, \tag{1.2}$$

where β is a positive constant, f is a nonlinear Lipschitz functional, K^g is a convex set defined by

$$K^g = \bigl\{v \in H^1(\Omega) \text{ such that } v = g \text{ on } \partial\Omega \text{ and } v \leq \psi \text{ on } \Omega\bigr\} \tag{1.3}$$

with $\psi \in W^{2,\infty}(\Omega)$ such that $\psi \geq 0$ on $\partial\Omega$ is the obstacle function, and $g \in L^\infty(\partial\Omega)$ is the boundary condition.

The concept of the nonmatching finite elements grids used in this paper consists of decomposing the whole domain Ω in two overlapping subdomains and to discretize each subdomain by an independent finite element method. As the two discretizations are independent on the overlap region, the discrete analog of problem (1.1) cannot be defined, and the alternating Schwarz method is then used to resolve the two discrete subproblems arising from these nonmatching finite elements grids.

We refer to [1–7], and the references therein for the analysis of the Schwarz alternating method for elliptic obstacle problems and to the proceedings of the annual domain decomposition conference beginning with [8].

For results on maximum norm error analysis of overlapping nonmatching grids methods for elliptic problems we refer, for example, to [9–14].

In this paper we consider the class of variational inequalities with nonlinear source terms (1.1) [15], where the main objective is to demonstrate that the approximation converges optimally on each subdomain making use of the characterization of the discrete solution as the upper bound of the set of discrete subsolutions [16], a Lipschitz continuous dependence with respect to both the boundary condition and the source term, and the standard finite element L^∞ error estimate for the elliptic obstacle problem [17].

More precisely, if u_i denotes the true solution and $(u^n_{h_i})$ the discrete Schwarz sequence with respect to the triangulation with mesh size h_i on Ω_i, we show that

$$\left\| u_i - u^n_{h_i} \right\|_{L^\infty(\Omega_i)} \leq Ch^2 |\log h|^2, \tag{1.4}$$

where $h = \max(h_1, h_2)$, C is a constant independent of n and h. This result coincides with the optimal convergence order of elliptic variational inequalities of an obstacle type problem [17].

2 Elliptic variational inequalities of obstacle type problem

In this section we begin by laying down some definitions and classical results related to variational inequalities, then we prove a Lipschitz continuous and discrete dependence with respect to both the boundary condition and the source term which will assume a crucial role in the proof of the main result of this paper.

Let Ω be a bounded polyhedral domain of $\mathbb{R}^2$ or $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. We consider the bilinear form

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v)\, dx, \tag{2.1}$$

the linear form

$$(f,v) = \int_\Omega f(x) \cdot v(x)\, dx, \tag{2.2}$$

the right-hand side

$$f \in L^\infty(\Omega), \tag{2.3}$$

the obstacle

$$\psi \in W^{2,\infty}(\Omega) \quad \text{such that} \quad \psi \geq 0 \quad \text{on } \partial\Omega, \tag{2.4}$$

the boundary condition $g \in L^\infty(\partial\Omega)$, and the nonempty convex set

$$K^g = \{v \in H^1(\Omega) \text{ such that } v = g \text{ on } \partial\Omega \text{ and } v \le \psi \text{ on } \Omega\}. \tag{2.5}$$

We consider the variational inequality (VI): find $u \in K^g$ such that

$$a(u, v-u) + c(u, v-u) \ge (f, v-u), \quad \forall v \in K^g, \tag{2.6}$$

where $c \in \mathbb{R}$ and $c > 0$ such that

$$c \ge \beta > 0, \tag{2.7}$$

where β is a positive constant. Let τ_h be a triangulation of Ω with mesh size h, V_h be the space of finite elements consisting of continuous piecewise linear functions v vanishing on $\partial\Omega$, and ϕ_s, $s = 1, 2, \ldots, m(h)$ be the basis functions of V_h.

The discrete counterpart of (2.6) consists of finding $u_h \in K_h^g$ such that

$$a(u_h, v-u_h) + c(u_h, v-u_h) \ge (f, v-u_h), \quad \forall v \in K_h^g, \tag{2.8}$$

where

$$K_h^g = \{v \in V_h \text{ such that } v = \pi_h g \text{ on } \partial\Omega \text{ and } v \le r_h\psi \text{ on } \Omega\}, \tag{2.9}$$

π_h is an interpolation operator on $\partial\Omega$ and r_h is the usual finite element restriction operator on Ω.

Theorem 1 (see [17]) *Under conditions* (2.3) *and* (2.4), *there exists a constant C independent of h such that*

$$\|\zeta - \zeta_h\|_{L^\infty(\Omega)} \le Ch^2|\log h|^2. \tag{2.10}$$

2.1 A Lipschitz continuous dependence with respect to both the boundary condition and the source term

This subsection is devoted to the establishment of a Lipschitz continuous dependence property of the solution with respect to the data whose proof is based on a monotonicity property of the solution of (2.6) with respect to the source term and the boundary condition by which we first set out and demonstrate our result.

Proposition 2 *Let* $(f;g)$; $(\tilde{f},\tilde{g})$ *be a pair of data and* $\zeta = \sigma(f,g)$; $\tilde{\zeta} = \sigma(\tilde{f},\tilde{g})$ *the corresponding solutions to* (2.6). *If* $f \le \tilde{f}$ *in* Ω *and* $g \le \tilde{g}$ *on* $\partial\Omega$ *then* $\zeta \le \tilde{\zeta}$ *in* $\overline{\Omega}$.

Proof According to [16], $\zeta = \max\{\underline{\zeta}\}$ where $\{\underline{\zeta}\}$ is the set of all the subsolutions of ζ. Hence $\forall \underline{\zeta} \in \{\underline{\zeta}\}$, $\underline{\zeta}$ satisfies

$$a(\underline{\zeta}, v) + c(\underline{\zeta}, v) \le (f, v), \quad \forall v \ge 0,$$

with

$$\underline{\zeta} \leq g \quad \text{on } \partial\Omega.$$

Then the two inequalities $f \leq \tilde{f}$ in Ω and $g \leq \tilde{g}$ on $\partial\Omega$ imply

$$a(\underline{\zeta}, v) + c(\underline{\zeta}, v) \leq (f, v) \leq (\tilde{f}, v), \quad \forall v \geq 0,$$

with

$$\underline{\zeta} \leq g \leq \tilde{g} \quad \text{on } \partial\Omega.$$

So, $\underline{\zeta}$ is a subsolution of $\tilde{\zeta} = \sigma(\tilde{f}, \tilde{g})$, that is, $\zeta \leq \tilde{\zeta}$. □

Proposition 3 *Under the conditions of Proposition* 2, *we have*

$$\|\zeta - \tilde{\zeta}\|_{L^\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta}\right)\|f - \tilde{f}\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}\right\}. \tag{2.11}$$

Proof First, set

$$\Phi = \max\left\{\left(\frac{1}{\beta}\right)\|f - \tilde{f}\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}\right\}. \tag{2.12}$$

Then

$$\begin{aligned}
\tilde{f} &\leq f + \|f - \tilde{f}\|_{L^\infty(\Omega)} \\
&\leq f + (1)\|f - \tilde{f}\|_{L^\infty(\Omega)} \\
&\leq f + \left(\frac{c}{\beta}\right)\|f - \tilde{f}\|_{L^\infty(\Omega)} \\
&\leq f + c\max\left\{\left(\frac{1}{\beta}\right)\|f - \tilde{f}\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}\right\} \\
&\leq f + c\Phi.
\end{aligned}$$

So,

$$\tilde{f} \leq f + c\Phi \quad \text{in } \Omega. \tag{2.13}$$

On the other hand, we have

$$\tilde{\zeta} = \tilde{g} \leq g + \Phi \quad \text{on } \partial\Omega. \tag{2.14}$$

Thus, making use of (2.13), (2.14), and Proposition 2, we obtain

$$\tilde{\zeta} \leq \sigma(f + c\Phi, g + \Phi) \quad \text{in } \overline{\Omega}. \tag{2.15}$$

Since $\zeta + \Phi$ is a solution of the following VI:

$$a\big(\zeta + \Phi, (v + \Phi) - (\zeta + \Phi)\big) + c\big(\zeta + \Phi, (v + \Phi) - (\zeta + \Phi)\big) \geq \big(f + c\Phi, (v + \Phi) - (\zeta + \Phi)\big)$$

with

$$v + \Phi, \zeta + \Phi \in K^{(g+\Phi)},$$

we have

$$\zeta + \Phi = \sigma(f + c\Phi, g + \Phi). \tag{2.16}$$

Equations (2.15) and (2.16) imply

$$\tilde{\zeta} \leq \zeta + \Phi \quad \text{in } \overline{\Omega},$$

thus

$$\tilde{\zeta} - \zeta \leq \Phi \quad \text{in } \overline{\Omega}. \tag{2.17}$$

Similarly, interchanging the roles of the couples $(f;g)$; $(\tilde{f},\tilde{g})$, we obtain

$$\zeta - \tilde{\zeta} \leq \Phi \quad \text{in } \overline{\Omega}, \tag{2.18}$$

which completes the proof. □

2.2 A Lipschitz discrete dependence with respect to both the boundary condition and the source term

Assuming that the discrete maximum principle (d.m.p.) is satisfied, *i.e.* the matrix resulting from the finite element discretization is an M-matrix (see [18, 19]), we prove the Lipschitz discrete dependence with respect to both the boundary condition and the source term by a similar study to that undertaken previously for the Lipschitz continuous dependence property.

Proposition 4 *Let* (f,g); $(\tilde{f},\tilde{g})$ *be a pair of data and* $\zeta_h = \sigma_h(f,g)$; $\tilde{\zeta}_h = \sigma_h(\tilde{f},\tilde{g})$ *the corresponding solutions to* (2.8). *If* $f \geq \tilde{f}$ *in* Ω *and* $g \geq \tilde{g}$ *on* $\partial\Omega$ *then* $\zeta_h \geq \tilde{\zeta}_h$ *in* $\overline{\Omega}$.

Proof The proof is similar to that of the continuous case. □

The proposition below establishes a Lipschitz discrete dependence of the solution with respect to the data.

Proposition 5 *Provided that the d.m.p. is verified, then under the conditions of Proposition* 4, *we have*

$$\|\zeta_h - \tilde{\zeta}_h\|_{L^\infty(\Omega)} \leq \max\left\{\left(\frac{1}{\beta}\right)\|f - \tilde{f}\|_{L^\infty(\Omega)}, \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}\right\}. \tag{2.19}$$

Proof The proof is similar to that of the continuous case. Indeed, as the basis functions ϕ_s of the space V_h are positive, it suffices to use the discrete maximum principle. □

3 Schwarz alternating methods for VI with nonlinear source terms

We consider the following variational inequality with nonlinear source term (1.1): Find $u \in K^g$, a solution of

$$a(u, v-u) + c(u, v-u) \geq \big(f(u), v-u\big), \quad \forall v \in K^g, \tag{3.1}$$

where

$$a(u,v) = \int_\Omega (\nabla u \nabla v)\,dx, \tag{3.2}$$

$f(\cdot)$ is a Lipschitz continuous nondecreasing nonlinear source term on $\mathbb{R}$,

$$\big|f(x) - f(y)\big| \leq k|x-y|, \quad \forall x, y \in \mathbb{R}, \tag{3.3}$$

with k satisfying

$$k < \beta, \tag{3.4}$$

where β is defined in (1.2).

Theorem 6 (see [20]) *Problem* (3.1) *has a unique solution.*

We decompose Ω into two overlapping smooth subdomains Ω_1 and Ω_2 such that

$$\Omega = \Omega_1 \cup \Omega_2. \tag{3.5}$$

We denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_i = \partial\Omega_i \cap \Omega_j$. We assume that the intersection of $\overline{\Gamma}_i$ and $\overline{\Gamma}_j$, $i \neq j$, is empty and we associate with problem (3.1) the following system: Find $(u_1, u_2) \in K_1^g \times K_2^g$, a solution of

$$\begin{cases} a_i(u_i, v-u_i) + c(u_i, v-u_i) \geq (f(u_i), v-u_i), & \forall v \in K_i^g, \\ u_i/_{\Gamma_i} = u_j/_{\Gamma_i}, & i,j = 1,2, i \neq j, \end{cases} \tag{3.6}$$

such that

$$K_i^g = \big\{v \in H^1(\Omega_i) \text{ such that } v = g \text{ on } \partial\Omega \cap \partial\Omega_i \text{ and } v \leq \psi \text{ on } \Omega_i\big\}, \tag{3.7}$$

$$a_i(u,v) = \int_{\Omega_i} (\nabla u.\nabla v)\,dx, \tag{3.8}$$

and

$$u_i = u/\Omega_i, \quad i = 1, 2. \tag{3.9}$$

Let

$$b_i(u,v) = a_i(u,v) + c(u,v).$$

3.1 The continuous Schwarz sequences

Let u_2^0 be an initialization in Γ_1 defined by

$$u_2^0 = \psi/\Omega_2. \tag{3.10}$$

We, respectively, define the alternating Schwarz sequences (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in K_1^g$ solves

$$\begin{cases} b_1(u_1^{n+1}, v - u_1^{n+1}) \geq (f(u_1^{n+1}), v - u_1^{n+1}), & \forall v \in K_1^g, \\ u_1^{n+1}/_{\Gamma_1} = u_2^n/_{\Gamma 1}, \end{cases} \tag{3.11}$$

and (u_2^{n+1}) on Ω_2 such that $u_2^{n+1} \in K_2^g$ solves

$$\begin{cases} b_2(u_2^{n+1}, v - u_2^{n+1}) \geq (f(u_2^{n+1}), v - u_2^{n+1}), & \forall v \in K_2^g, \\ u_2^{n+1}/_{\Gamma_2} = u_1^{n+1}/_{\Gamma 2}. \end{cases} \tag{3.12}$$

Theorem 7 (see [6]) *The two sequences* (3.11) *and* (3.12) *converge uniformly to the solution of* (3.6).

3.2 Nonmatching grids discretization

For $i = 1, 2$, let τ^{h_i} be a standard regular and quasi-uniform finite element triangulation in Ω_i; h_i being its mesh size. The two meshes being mutually independent on $\Omega_1 \cap \Omega_2$ in the sense that a triangle belonging to one triangulation does not necessarily belong to the other one. We consider the following discrete spaces:

$$V_{h_i} = \left\{ v \in C(\overline{\Omega}_i) \cap H^1(\Omega_i) \text{ such that } v/_T \in \mathcal{P}_1, \forall T \in \tau^{h_i} \right\}, \tag{3.13}$$

the convex sets

$$K_{h_i}^g = \{ v \in V_{h_i} \text{ such that } v = \pi_{h_i} g \text{ on } \partial\Omega \cap \partial\Omega_i \text{ and } v \leq r_{h_i}\psi \}, \tag{3.14}$$

where r_{h_i} denotes the restriction operator on the triangulation τ^{h_i}. Let also π_{h_i} denote the interpolation operator on Γ_i and ϕ_s^i, $s = 1, 2, \ldots, m(h_i)$, be the basis functions of V_{h_i}.

The discrete maximum principle (see [18, 19]) We assume that the respective matrices resulting from the discretization of problems (3.11) and (3.12) are M-matrices. Note that, as the two meshes h_1 and h_2 are independent over the overlapping subdomains, it is impossible to formulate a global approximate problem which would be the direct discrete counterpart of problem (3.1).

3.3 The discrete Schwarz sequences

Now, we define the discrete counterparts of the continuous Schwarz sequences defined in (3.11) and (3.12). Indeed, let $u_{h_2}^0$ be the discrete analog of u_2^0 defined in (3.10) that is, $u_{h_2}^0 = \pi_{h_2}(u_2^0) = \pi_{h_2}(\psi/\Omega_2)$. We, respectively, define $u_{h_1}^{n+1} \in K_{h_1}^g$ such that

$$\begin{cases} b_1(u_{h_1}^{n+1}, v - u_{h_1}^{n+1}) \geq (f(u_{h_1}^{n+1}), v - u_{h_1}^{n+1}), & \forall v \in K_{h_1}^g, \\ u_{h_1}^{n+1}/_{\Gamma_1} = \pi_{h_1}(u_{h_2}^n/_{\Gamma 1}), \end{cases} \tag{3.15}$$

and $u_{h_2}^{n+1} \in K_{h_2}^g$ such that

$$\begin{cases} b_2(u_{h_2}^{n+1}, v - u_{h_2}^{n+1}) \geq (f(u_{h_2}^{n+1}), v - u_{h_2}^{n+1}), & \forall v \in K_{h_2}^g, \\ u_{h_2}^{n+1}/_{\Gamma_2} = \pi_{h_2}(u_{h_1}^{n+1}/_{\Gamma 2}). \end{cases} \tag{3.16}$$

4 Maximum norm error

This section is devoted to the proof of the main result of the present paper. To that end, we begin by introducing two discrete auxiliary problems.

4.1 Two auxiliary problems

We define $w_{h_1} \in K_{h_1}^g$ such that w_{h_1} solves

$$\begin{cases} b_1(w_{h_1}, v - w_{h_1}) \geq (f(u_1), v - w_{h_1}), & \forall v \in K_{h_1}^g, \\ w_{h_1}/_{\Gamma_1} = \pi_{h_1}(u_2/_{\Gamma 1}), \end{cases} \tag{4.1}$$

and $w_{h_2} \in K_{h_2}^g$ such that w_{h_2} solves

$$\begin{cases} b_2(w_{h_2}, v - w_{h_2}) \geq (f(u_2), v - w_{h_2}), & \forall v \in K_{h_2}^g, \\ w_{h_2}/_{\Gamma_2} = \pi_{h_2}(u_1/_{\Gamma 2}). \end{cases} \tag{4.2}$$

It is then clear that w_{h_1} and w_{h_2} are the finite element approximations of u_1 and u_2 defined in (3.6) thus, making use of (2.10), we get

$$\|u_i - w_{h_i}\|_{L^\infty(\Omega_i)} \leq Ch^2 |\log h|^2, \tag{4.3}$$

where C is a constant independent of h.

Notation 8 From now on, we shall adopt the following notations:

$$|\cdot|_1 = \|\cdot\|_{L^\infty(\Gamma_1)}; \qquad |\cdot|_2 = \|\cdot\|_{L^\infty(\Gamma_2)},$$
$$\|\cdot\|_1 = \|\cdot\|_{L^\infty(\Omega_1)}; \qquad \|\cdot\|_2 = \|\cdot\|_{L^\infty(\Omega_2)},$$
$$\pi_{h_1} = \pi_{h_2} = \pi_h.$$

4.2 The main result

Theorem 9 *Let $h = \max(h_1, h_2)$ and let $\rho = \frac{k}{\beta} < 1$ then there exists a constant C independent of both h and n such that*

$$\left\| u_i - u_{h_i}^{n+1} \right\|_i \leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2, \quad i = 1, 2, n \geq 0. \tag{4.4}$$

Proof The proof of (4.4) will be carried out by induction, where the cases $\rho \in (0, \frac{1}{2}]$ and $\rho \in (\frac{1}{2}, 1)$ will be studied separately. Also, within each case, we will also discuss the two following situations:

$$\text{(A):} \quad \left\| u_2 - u_{h_2}^0 \right\|_2 \leq Ch^2 |\log h|^2 \tag{4.5}$$

and

$$\text{(B):}\quad Ch^2|\log h|^2 < \|u_2 - u_{h_2}^0\|_2, \tag{4.6}$$

where $u_{h_2}^0 = \pi_{h_2}(u_2^0) = \pi_{h_2}(\psi/\Omega_2)$. The basic idea of the proof is to define for each subdomain two approximations α_{h_i} and $\tilde{\alpha}_{h_i}$ in the L^∞-norm of u_i (a discrete subsolution and a discrete supersolution of $u_{h_i}^n$, $n \geq 1$), such that

$$\|\alpha_{h_i} - u_i\|_i \leq \frac{1}{(1-\rho)} Ch^2|\log h|^2$$

and

$$\|\tilde{\alpha}_{h_i} - u_i\|_i \leq \frac{1}{(1-\rho)} Ch^2|\log h|^2.$$

Part 1: The first part of the proof deals with $0 < \rho \leq \frac{1}{2}$. So

$$\frac{\rho}{1-\rho} \leq 1. \tag{4.7}$$

For $n = 1$, in domain 1, the discrete analog w_{h_1} of u_1 defined in (4.1) considered as the upper bound of the set of discrete subsolutions [16], satisfies

$$\begin{aligned} &b_1(w_{h_1}, \varphi_s^1) \leq (f(u_1), \varphi_s^1), \quad \forall s \in \{1, \ldots, m(h_1)\}, \\ &w_{h_1} = \pi_{h_1} u_2 \quad \text{on } \Gamma_1. \end{aligned}$$

Since the nonlinear functional is Lipschitz and according to (4.3), we get

$$f(u_1) - f(w_{h_1}) \leq kCh^2|\log h|^2.$$

Then

$$\begin{aligned} &b_1(w_{h_1}, \varphi_s^1) \leq (f(u_1), \varphi_s^1) \leq (f(w_{h_1}) + kCh^2|\log h|^2, \varphi_s^1), \\ &w_{h_1} = \pi_{h_1} u_2 \quad \text{on } \Gamma_1. \end{aligned}$$

Let

$$W_{h_1} = \sigma_{h_1}(f(w_{h_1}) + kCh^2|\log h|^2, \pi_{h_1} u_2); \tag{4.8}$$

therefore, w_{h_1} is a subsolution of W_{h_1},

$$w_{h_1} \leq W_{h_1} \quad \text{in } \Omega_1. \tag{4.9}$$

By applying (2.19), we get

$$\begin{aligned} \|W_{h_1} - u_{h_1}^1\|_1 &\leq \max\left\{\left(\frac{1}{\beta}\right)\|f(w_{h_1}) + kCh^2|\log h|^2 - f(u_{h_1}^1)\|_1; |u_2 - u_{h_2}^0|_1\right\} \\ &\leq \max\left\{\left(\frac{1}{\beta}\right)\|f(w_{h_1}) - f(u_{h_1}^1)\|_1 + \left(\frac{k}{\beta}\right)Ch^2|\log h|^2; \|u_2 - u_{h_2}^0\|_2\right\}. \end{aligned}$$

So

$$\left\| W_{h_1} - u_{h_1}^1 \right\|_1 \le \max\left\{ \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2; \left\| u_2 - u_{h_2}^0 \right\|_2 \right\}. \tag{4.10}$$

On the other hand, (4.9) generates two possibilities, that is,

$$(\mathrm{A}_1): \quad \left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \left\| W_{h_1} - u_{h_1}^1 \right\|_1$$

or

$$(\mathrm{A}_2): \quad \left\| W_{h_1} - u_{h_1}^1 \right\|_1 \le \left\| w_{h_1} - u_{h_1}^1 \right\|_1.$$

Case (A_1) in conjunction with (4.10) implies that

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \max\left\{ \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2; \left\| u_2 - u_{h_2}^0 \right\|_2 \right\},$$

which lets us distinguish the following two cases:

$$\begin{aligned} 1: \quad & \max\left\{ \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2; \left\| u_2 - u_{h_2}^0 \right\|_2 \right\} \\ & \quad = \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2 \end{aligned} \tag{4.11}$$

and

$$2: \quad \max\left\{ \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2; \left\| u_2 - u_{h_2}^0 \right\|_2 \right\} = \left\| u_2 - u_{h_2}^0 \right\|_2. \tag{4.12}$$

Equation (4.11) implies that

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2$$

and

$$\left\| u_2 - u_{h_2}^0 \right\|_2 \le \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2.$$

Then

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \frac{\rho}{1-\rho} C h^2 |\log h|^2$$

and

$$\begin{aligned} \left\| u_2 - u_{h_2}^0 \right\|_2 &\le \frac{\rho^2}{1-\rho} C h^2 |\log h|^2 + \rho C h^2 |\log h|^2 \\ &\le \frac{\rho}{1-\rho} C h^2 |\log h|^2 \le C h^2 |\log h|^2, \end{aligned}$$

which coincides with (4.5) and contradicts (4.6). So, (4.11) is only possible in situation (A).
Equation (4.12) implies that

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \left\| u_2 - u_{h_2}^0 \right\|_2 \tag{4.13}$$

and

$$\rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2 \le \left\| u_2 - u_{h_2}^0 \right\|_2.$$

So, by multiplying (4.13) by ρ and adding $\rho C h^2|\log h|^2$, we get

$$\rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho C h^2 |\log h|^2 \le \rho \left\| u_2 - u_{h_2}^0 \right\|_2 + \rho C h^2 |\log h|^2.$$

Then $\rho\|w_{h_1} - u_{h_1}^1\|_1 + \rho C h^2|\log h|^2$ is bounded by both $\|u_2 - u_{h_2}^0\|_2$ and $\rho\|u_2 - u_{h_2}^0\|_2 + \rho C h^2|\log h|$, so

$$\text{(a):} \quad \left\| u_2 - u_{h_2}^0 \right\|_2 \le \rho \left\| u_2 - u_{h_2}^0 \right\|_2 + \rho C h^2 |\log h|$$

or

$$\text{(b):} \quad \rho \left\| u_2 - u_{h_2}^0 \right\|_2 + \rho C h^2 |\log h| \le \left\| u_2 - u_{h_2}^0 \right\|_2.$$

That is,

$$\left\| u_2 - u_{h_2}^0 \right\|_2 \le \frac{\rho}{1-\rho} C h^2 |\log h|^2$$

or

$$\frac{\rho}{1-\rho} C h^2 |\log h|^2 \le \left\| u_2 - u_{h_2}^0 \right\|_2.$$

Thus

$$\left\| u_2 - u_{h_2}^0 \right\|_2 \le \frac{\rho}{1-\rho} C h^2 |\log h|^2 \le C h^2 |\log h|^2$$

or

$$\frac{\rho}{1-\rho} C h^2 |\log h|^2 \le \left\| u_2 - u_{h_2}^0 \right\|_2 \le C h^2 |\log h|^2.$$

So, the two cases (a) and (b) are true because they both coincide with (4.5). Therefore, there is either a contradiction and thus (4.12) is impossible or (4.12) is possible only if

$$\left\| u_2 - u_{h_2}^0 \right\|_2 = \frac{\rho}{1-\rho} C h^2 |\log h|^2.$$

Then (4.12) in situation (A) implies

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \left\| u_2 - u_{h_2}^0 \right\|_2 = \frac{\rho}{1-\rho} C h^2 |\log h|^2,$$

while in situation (B) only (b) is true and leads to

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \left\| u_2 - u_{h_2}^0 \right\|_2 \quad \text{and} \quad \frac{\rho}{1-\rho} C h^2 |\log h|^2 \le \left\| u_2 - u_{h_2}^0 \right\|_2.$$

Then

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \frac{\rho}{1-\rho} Ch^2 |\log h|^2 \le \left\| u_2 - u_{h_2}^0 \right\|_2$$

or

$$\frac{\rho}{1-\rho} Ch^2 |\log h|^2 \le \left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \left\| u_2 - u_{h_2}^0 \right\|_2.$$

We remark that both possibilities are true. There is either a contradiction and (4.12) is impossible or (4.12) is possible only if

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 = \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2.$$

So, in the two situations (A) and (B) and in the two cases (4.11) and (4.12) of situation (A_1), we get

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2, \tag{4.14}$$

which implies

$$w_{h_1} - \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2 \le u_{h_1}^1 \le w_{h_1} + \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2.$$

Let us denote

$$\alpha_{h_1} = w_{h_1} - \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2 \tag{4.15}$$

and

$$\tilde{\alpha}_{h_1} = w_{h_1} + \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2. \tag{4.16}$$

Then

$$\alpha_{h_1} \le u_{h_1}^1 \le \tilde{\alpha}_{h_1} \tag{4.17}$$

with

$$\begin{aligned} \|\alpha_{h_1} - u_1\|_1 &= \left\| w_{h_1} - \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2 - u_1 \right\|_1 \\ &\le \|w_{h_1} - u_1\|_1 + \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2 \\ &\le Ch^2 |\log h|^2 + \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2 \end{aligned}$$

by virtue of (4.3). So

$$\|\alpha_{h_1} - u_1\|_1 \le \frac{1}{(1-\rho)} Ch^2 |\log h|^2. \tag{4.18}$$

By using the same reasoning we see that (4.16) implies

$$\|\tilde{\alpha}_{h_1} - u_1\|_1 \leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2. \tag{4.19}$$

On the other hand, (4.17) implies

$$\alpha_{h_1} - u_1 \leq u^1_{h_1} - u_1 \leq \tilde{\alpha}_{h_1} - u_1 \tag{4.20}$$

so according to (4.18) and (4.19) we get

$$-\frac{1}{(1-\rho)} Ch^2 |\log h|^2 \leq u^1_{h_1} - u_1 \leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2 \tag{4.21}$$

thus

$$\left\| u_1 - u^1_{h_1} \right\|_1 \leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2. \tag{4.22}$$

Case (A_2) in conjunction with (4.10) implies that $\| W_{h_1} - u^1_{h_1}\|_1$ is bounded by the values $\|w_{h_1} - u^1_{h_1}\|_1$ and $\max\{\rho\|w_{h_1} - u^1_{h_1}\|_1 + \rho Ch^2|\log h|^2; \|u_2 - u^0_{h_2}\|_2\}$ which generates the two situations

$$\text{(c):} \quad \left\| w_{h_1} - u^1_{h_1} \right\|_1 \leq \max\left\{ \rho \left\| w_{h_1} - u^1_{h_1} \right\|_1 + \rho Ch^2 |\log h|^2; \left\| u_2 - u^0_{h_2} \right\|_2 \right\}$$

or

$$\text{(d):} \quad \max\left\{ \rho \left\| w_{h_1} - u^1_{h_1} \right\|_1 + \rho Ch^2 |\log h|^2; \left\| u_2 - u^0_{h_2} \right\|_2 \right\} \leq \left\| w_{h_1} - u^1_{h_1} \right\|_1. \tag{4.23}$$

It is clear that case (c) coincides with situation (A_1). Let us study case (d); as in case (A_1), $\max\{\rho\|w_{h_1} - u^1_{h_1}\|_1 + \rho Ch^2|\log h|^2; \|u_2 - u^0_{h_2}\|_2\}$ lets us distinguish the two cases (4.11) and (4.12). Equation (4.11) in conjunction with (d) implies

$$\left\| u_2 - u^0_{h_2} \right\|_2 \leq \rho \left\| w_{h_1} - u^1_{h_1} \right\|_1 + \rho Ch^2 |\log h|^2 \leq \left\| w_{h_1} - u^1_{h_1} \right\|_1$$

and (4.12) in conjunction with (d) implies

$$\rho \left\| w_{h_1} - u^1_{h_1} \right\|_1 + \rho Ch^2 |\log h|^2 \leq \left\| u_2 - u^0_{h_2} \right\|_2 \leq \left\| w_{h_1} - u^1_{h_1} \right\|_1.$$

Then it is clear that in the two cases (4.11) and (4.12), we obtain

$$\frac{\rho}{(1-\rho)} Ch^2 |\log h|^2 \leq \left\| w_{h_1} - u^1_{h_1} \right\|_1 \tag{4.24}$$

with

$$\left\| u_2 - u^0_{h_2} \right\|_2 \leq \left\| w_{h_1} - u^1_{h_1} \right\|_1. \tag{4.25}$$

Thus, $\|w_{h_1} - u^1_{h_1}\|_1$ is bounded below by both $\frac{\rho}{(1-\rho)} Ch^2|\log h|^2$ and $\|u_2 - u^0_{h_2}\|_2$ so we distinguish the two following possibilities:

$$\text{(e):}\quad \left\|u_2 - u^0_{h_2}\right\|_2 \le \frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \le Ch^2|\log h|^2$$

or

$$\text{(f):}\quad \frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \le \left\|u_2 - u^0_{h_2}\right\|_2 \le Ch^2|\log h|^2.$$

So, the two cases (e) and (f) are true because they both coincide with (4.5). Therefore, there is either a contradiction and thus cases (4.11) and (4.12) are impossible or the two cases (4.11) and (4.12) are possible in situation (A) only if

$$\left\|u_2 - u^0_{h_2}\right\|_2 = \frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \le \left\|w_{h_1} - u^1_{h_1}\right\|_1,$$

while in situation (B) only the case (f) is true and leads to

$$\frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \le \left\|u_2 - u^0_{h_2}\right\|_2 \le \left\|w_{h_1} - u^1_{h_1}\right\|_1.$$

In summary, in situation (A_2) and in the two cases (4.11) and (4.12) of situations (A) and (B), we get

$$\frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \le \left\|w_{h_1} - u^1_{h_1}\right\|_1. \tag{4.26}$$

Let us decompose the subdomain $\Omega_1 = \Omega_{1,1} \cup \Omega^c_{1,1}$ such that

$$\frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \le \left|w_{h_1} - u^1_{h_1}\right| \quad \text{on } \Omega_{1,1} \tag{4.27}$$

and

$$\left|w_{h_1} - u^1_{h_1}\right| < \frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \quad \text{on } \Omega^c_{1,1}. \tag{4.28}$$

We begin with $\Omega_{1,1}$. If $w_{h_1} - u^1_{h_1} \ge 0$ on $\Omega_{1,1}$ then (4.27) implies $u^1_{h_1} \le \alpha_{h_1}$; thus,

$$u^1_{h_1} - u_1 \le \alpha_{h_1} - u_1 \le \frac{1}{(1-\rho)} Ch^2|\log h|^2 \tag{4.29}$$

by virtue of (4.18). On the other hand, (4.18) leads also to

$$-\frac{1}{(1-\rho)} Ch^2|\log h|^2 \le \alpha_{h_1} - u_1.$$

So, $\alpha_{h_1} - u_1$ is bounded below by both $u^1_{h_1} - u_1$ and $-\frac{1}{(1-\rho)} Ch^2|\log h|^2$, which lets us distinguish the two following possibilities:

$$u^1_{h_1} - u_1 \le -\frac{1}{(1-\rho)} Ch^2|\log h|^2$$

or

$$-\frac{1}{(1-\rho)}Ch^2|\log h|^2 \le u^1_{h_1} - u_1.$$

Then

$$u^1_{h_1} - u_1 \le -\frac{1}{(1-\rho)}Ch^2|\log h|^2 \le w_{h_1} - u_1$$

or

$$-\frac{1}{(1-\rho)}Ch^2|\log h|^2 \le u^1_{h_1} - u_1 \le w_{h_1} - u_1.$$

So, both possibilities are true because they coincide with (4.3). So, there is either a contradiction and (4.27) is impossible or (4.27) is possible and we must have

$$\left\| u^1_{h_1} - u_1 \right\|_{L^\infty(\Omega_{1,1})} = \frac{1}{(1-\rho)}Ch^2|\log h|^2. \tag{4.30}$$

The case $w_{h_1} - u^1_{h_1} < 0$ on $\Omega_{1,1}$ is studied in a similar manner and leads to the same result (4.30). Equation (4.28) is studied in the same way as that for case (A_1) and leads to

$$\left\| u^1_{h_1} - u_1 \right\|_{L^\infty(\Omega^c_{1,1})} \le \frac{1}{(1-\rho)}Ch^2|\log h|^2. \tag{4.31}$$

Equations (4.30) and (4.31) imply

$$\left\| u^1_{h_1} - u_1 \right\|_1 \le \frac{1}{(1-\rho)}Ch^2|\log h|^2. \tag{4.32}$$

Finally, in the two cases (A_1) and (A_2) and in the two situations (A) and (B), we get

$$\left\| u_1 - u^1_{h_1} \right\|_1 \le \frac{1}{(1-\rho)}Ch^2|\log h|^2. \tag{4.33}$$

For $n = 1$ in domain 2, the discrete analog w_{h_2} of u_2, defined in (4.2) and considered as the upper bound of the set of discrete subsolutions [16], satisfies

$$\begin{aligned} &b_2\bigl(w_{h_2}, \varphi^2_s\bigr) \le \bigl(f(u_2), \varphi^2_s\bigr), \quad \forall s \in \bigl\{1, \ldots, m(h_2)\bigr\}, \\ &w_{h_2} = \pi_{h_2} u_1 \quad \text{on } \Gamma_2. \end{aligned}$$

The nonlinear functional is Lipschitz and according to (4.3)

$$f(u_2) - f(w_{h_2}) \le kCh^2|\log h|^2.$$

Then

$$\begin{aligned} &b_2\bigl(w_{h_2}, \varphi^2_s\bigr) \le \bigl(f(u_2), \varphi^2_s\bigr) \le \bigl(f(w_{h_2}) + kCh^2|\log h|^2, \varphi^2_s\bigr), \\ &w_{h_2} = \pi_{h_2} u_1 \quad \text{on } \Gamma_2. \end{aligned}$$

Let

$$W_{h_2} = \sigma_{h_2}\big(f(w_{h_2}) + kCh^2|\log h|^2, \pi_{h_2}u_1\big); \tag{4.34}$$

therefore, w_{h_2} is a subsolution of W_{h_2}, so

$$w_{h_2} \leq W_{h_2} \quad \text{in } \Omega_2. \tag{4.35}$$

By applying (2.19), we get

$$\begin{aligned}\|W_{h_2} - u^1_{h_2}\|_2 &\leq \max\left\{\left(\frac{1}{\beta}\right)\big\|f(w_{h_2}) + kCh^2|\log h| - f\big(u^1_{h_2}\big)\big\|_2; \big|u_1 - u^1_{h_1}\big|_2\right\} \\ &\leq \max\left\{\left(\frac{1}{\beta}\right)\big\|f(w_{h_2}) - f\big(u^1_{h_2}\big)\big\|_2 + \left(\frac{k}{\beta}\right)Ch^2|\log h|^2; \big\|u_1 - u^1_{h_1}\big\|_1\right\}.\end{aligned}$$

So

$$\|W_{h_2} - u^1_{h_2}\|_2 \leq \max\big\{\rho\|w_{h_2} - u^1_{h_2}\|_2 + \rho Ch^2|\log h|^2; \|u_1 - u^1_{h_1}\|_1\big\}. \tag{4.36}$$

On the other hand, (4.35) generates two possibilities, that is,

$$(\mathrm{B}_1): \quad \|w_{h_2} - u^1_{h_2}\|_2 \leq \|W_{h_2} - u^1_{h_2}\|_2$$

or

$$(\mathrm{B}_2): \quad \|W_{h_2} - u^1_{h_2}\|_2 \leq \|w_{h_2} - u^1_{h_2}\|_2.$$

Case (B_1) in conjunction with (4.36) implies that

$$\|w_{h_2} - u^1_{h_2}\|_2 \leq \max\big\{\rho\|w_{h_2} - u^1_{h_2}\|_2 + \rho Ch^2|\log h|^2; \|u_1 - u^1_{h_1}\|_1\big\},$$

which lets us distinguish the following two cases:

$$\begin{aligned}1: \quad &\max\big\{\rho\|w_{h_2} - u^1_{h_2}\|_2 + \rho Ch^2|\log h|^2; \|u_1 - u^1_{h_1}\|_1\big\} \\ &\quad = \rho\|w_{h_2} - u^1_{h_2}\|_2 + \rho Ch^2|\log h|^2\end{aligned} \tag{4.37}$$

and

$$2: \quad \max\big\{\rho\|w_{h_2} - u^1_{h_2}\|_2 + \rho Ch^2|\log h|^2; \|u_1 - u^1_{h_1}\|_1\big\} = \|u_1 - u^1_{h_1}\|_1. \tag{4.38}$$

Equation (4.37) implies that

$$\|w_{h_2} - u^1_{h_2}\|_2 \leq \rho\|w_{h_2} - u^1_{h_2}\|_2 + \rho Ch^2|\log h|^2$$

and

$$\|u_1 - u^1_{h_1}\|_1 \leq \rho\|w_{h_2} - u^1_{h_2}\|_2 + \rho Ch^2|\log h|^2.$$

Then

$$\left\|w_{h_2} - u_{h_2}^1\right\|_2 \le \frac{\rho}{1-\rho} Ch^2 |\log h|^2$$

and

$$\begin{aligned}\left\|u_1 - u_{h_1}^1\right\|_1 &\le \frac{\rho^2}{1-\rho} Ch^2 |\log h|^2 + \rho Ch^2 |\log h|^2 \\ &\le \frac{\rho}{1-\rho} Ch^2 |\log h|^2 < \frac{1}{1-\rho} Ch^2 |\log h|^2,\end{aligned}$$

which coincides with (4.33). Equation (4.38) implies that

$$\left\|w_{h_2} - u_{h_2}^1\right\|_2 \le \left\|u_1 - u_{h_1}^1\right\|_1 \tag{4.39}$$

and

$$\rho\left\|w_{h_2} - u_{h_2}^1\right\|_2 + \rho Ch^2 |\log h|^2 \le \left\|u_1 - u_{h_1}^1\right\|_1.$$

So, by multiplying (4.39) by ρ and adding $\rho Ch^2 |\log h|^2$ we get

$$\rho\left\|w_{h_2} - u_{h_2}^1\right\|_2 + \rho Ch^2 |\log h|^2 \le \rho\left\|u_1 - u_{h_1}^1\right\|_1 + \rho Ch^2 |\log h|^2;$$

then $\rho\|w_{h_2} - u_{h_2}^1\|_2 + \rho Ch^2 |\log h|^2$ is bounded above by $\|u_1 - u_{h_1}^1\|_1$ and $\rho\|u_1 - u_{h_1}^1\|_1 + \rho Ch^2 |\log h|$. So, either

$$\text{(a):}\quad \left\|u_1 - u_{h_1}^1\right\|_1 \le \rho\left\|u_1 - u_{h_1}^1\right\|_1 + \rho Ch^2 |\log h|$$

or

$$\text{(b):}\quad \rho\left\|u_1 - u_{h_1}^1\right\|_1 + \rho Ch^2 |\log h| \le \left\|u_1 - u_{h_1}^1\right\|_1,$$

that is,

$$\left\|u_1 - u_{h_1}^1\right\|_1 \le \frac{\rho}{1-\rho} Ch^2 |\log h|^2 < \frac{1}{1-\rho} Ch^2 |\log h|^2$$

or

$$\frac{\rho}{1-\rho} Ch^2 |\log h|^2 \le \left\|u_1 - u_{h_1}^1\right\|_1 \le \frac{1}{1-\rho} Ch^2 |\log h|^2.$$

So, the two cases (a) and (b) are true because they both coincide with (4.33). Therefore, there is either a contradiction and thus (4.38) is impossible or (4.38) is possible only if

$$\left\|u_1 - u_{h_1}^1\right\|_1 = \frac{\rho}{1-\rho} Ch^2 |\log h|^2$$

thus

$$\left\|w_{h_2} - u_{h_2}^1\right\|_2 \le \left\|u_1 - u_{h_1}^1\right\|_1 = \frac{\rho}{1-\rho} Ch^2 |\log h|^2.$$

In summary, in the two cases (4.37) and (4.38) of situation (B_1), we get

$$\left\| w_{h_2} - u_{h_2}^1 \right\|_2 \le \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2$$

so

$$w_{h_2} - \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2 \le u_{h_2}^1 \le w_{h_2} + \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2.$$

Let us denote

$$\alpha_{h_2} = w_{h_2} - \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2 \tag{4.40}$$

and

$$\tilde{\alpha}_{h_2} = w_{h_2} + \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2; \tag{4.41}$$

then

$$\alpha_{h_2} \le u_{h_2}^1 \le \tilde{\alpha}_{h_2}. \tag{4.42}$$

By using a same reasoning as adopted in subdomain Ω_1 for (4.15) and (4.16), we get

$$\|\alpha_{h_2} - u_2\|_2 \le \frac{1}{(1-\rho)} Ch^2 |\log h|^2 \tag{4.43}$$

and

$$\|\tilde{\alpha}_{h_2} - u_2\|_2 \le \frac{1}{(1-\rho)} Ch^2 |\log h|^2. \tag{4.44}$$

Equation (4.42) implies

$$\alpha_{h_2} - u_2 \le u_{h_2}^1 - u_2 \le \tilde{\alpha}_{h_2} - u_2$$

and according to (4.43) and (4.44), we obtain

$$-\frac{1}{(1-\rho)} Ch^2 |\log h|^2 \le u_{h_2}^1 - u_2 \le \frac{1}{(1-\rho)} Ch^2 |\log h|^2, \tag{4.45}$$

that is,

$$\left\| u_2 - u_{h_2}^1 \right\|_2 \le \frac{1}{(1-\rho)} Ch^2 |\log h|^2.$$

Case (B_2) in conjunction with (4.36) implies that $\| W_{h_2} - u_{h_2}^1 \|_2$ is bounded by the values $\|w_{h_2} - u_{h_2}^1\|_2$ and $\max\{\rho\|w_{h_2} - u_{h_2}^1\|_1 + \rho Ch^2|\log h|^2; \|u_1 - u_{h_1}^1\|_1\}$, which generates two situations,

$$\text{(c):} \quad \left\| w_{h_2} - u_{h_2}^1 \right\|_2 \le \max\left\{ \rho \left\| w_{h_2} - u_{h_2}^1 \right\|_2 + \rho Ch^2 |\log h|^2; \left\| u_1 - u_{h_1}^1 \right\|_1 \right\}$$

or

$$\text{(d):}\quad \max\{\rho\|w_{h_2}-u_{h_2}^1\|_2+\rho Ch^2|\log h|^2;\|u_1-u_{h_1}^1\|_1\}\le\|w_{h_2}-u_{h_2}^1\|_2.$$

It is clear that case (c) coincides with case (B_1). Let us study case (d); as in case (B_1) $\max\{\rho\|w_{h_2}-u_{h_2}^1\|_2+\rho Ch^2|\log h|^2;\|u_1-u_{h_1}^1\|_1\}$ lets us distinguish the two cases (4.37) and (4.38). Equation (4.37) in conjunction with (d) implies

$$\|u_1-u_{h_1}^1\|_1\le\rho\|w_{h_2}-u_{h_2}^1\|_2+\rho Ch^2|\log h|^2\le\|w_{h_2}-u_{h_2}^1\|_2$$

and (4.38) in conjunction with (d) implies

$$\rho\|w_{h_2}-u_{h_2}^1\|_2+\rho Ch^2|\log h|^2\le\|u_1-u_{h_1}^1\|_1\le\|w_{h_2}-u_{h_2}^1\|_2.$$

Then the two cases (4.37) and (4.38) imply

$$\frac{\rho}{(1-\rho)}Ch^2|\log h|^2\le\|w_{h_2}-u_{h_2}^1\|_2$$

and

$$\|u_1-u_{h_1}^1\|_1\le\|w_{h_2}-u_{h_2}^1\|_2.$$

$\|w_{h_2}-u_{h_2}^1\|_2$ is bounded below by $\frac{\rho}{(1-\rho)}Ch^2|\log h|^2$ and $\|u_1-u_{h_1}^1\|_1$ so we distinguish the two following possibilities:

$$\text{(e):}\quad \|u_1-u_{h_1}^1\|_1\le\frac{\rho}{(1-\rho)}Ch^2|\log h|^2<\frac{1}{(1-\rho)}Ch^2|\log h|^2$$

or

$$\text{(f):}\quad \frac{\rho}{(1-\rho)}Ch^2|\log h|^2\le\|u_1-u_{h_1}^1\|_1\le\frac{1}{(1-\rho)}Ch^2|\log h|^2.$$

So, the two cases (e) and (f) are true because they both coincide with (4.33). Therefore, there is either a contradiction and thus cases (4.37) and (4.38) are impossible or the two cases (4.37) and (4.38) are possible only if

$$\|u_1-u_{h_1}^1\|_1=\frac{\rho}{(1-\rho)}Ch^2|\log h|^2\le\|w_{h_2}-u_{h_2}^1\|_2.$$

So, in the two cases (4.37) and (4.38) of situation (B_2), we get

$$\frac{\rho}{(1-\rho)}Ch^2|\log h|^2\le\|w_{h_2}-u_{h_2}^1\|_2.$$

The remainder of the proof related to situation (B_2) rests on the same arguments used in subdomain Ω_1 for situation (A_2) that is, on a decomposition of $\Omega_2=\Omega_{2,1}\cup\Omega_{2,1}^c$ and showing that

$$\|u_2-u_{h_2}^1\|_{L^\infty(\Omega_{2,1})}\le\frac{1}{(1-\rho)}Ch^2|\log h|^2$$

and

$$\left\| u_2 - u_{h_2}^1 \right\|_{L^\infty(\Omega_{2,1}^c)} \leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2.$$

Finally, in the two situations (B_1) and (B_2) we get

$$\left\| u_2 - u_{h_2}^1 \right\|_2 \leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2. \tag{4.46}$$

Now, let us assume that

$$\begin{aligned} \left\| u_1 - u_{h_1}^n \right\|_1 &\leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2, \\ \left\| u_2 - u_{h_2}^n \right\|_2 &\leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2, \end{aligned} \tag{4.47}$$

and prove that

$$\begin{aligned} \left\| u_1 - u_{h_1}^{n+1} \right\|_1 &\leq \frac{1}{(1-\rho)} Ch^2 |\log h|, \\ \left\| u_2 - u_{h_2}^{n+1} \right\|_2 &\leq \frac{1}{(1-\rho)} Ch^2 |\log h|. \end{aligned} \tag{4.48}$$

By using the definition of W_{h_1} in (4.8) and by applying (2.19), we get

$$\begin{aligned} \left\| W_{h_1} - u_{h_1}^{n+1} \right\|_1 &\leq \max\left\{ \left(\frac{1}{\beta}\right) \left\| f(w_{h_1}) + kCh^2 |\log h| - f\left(u_{h_1}^{n+1}\right) \right\|_1 ; \left| u_2 - u_{h_2}^n \right|_1 \right\} \\ &\leq \max\left\{ \left(\frac{1}{\beta}\right) \left\| f(w_{h_1}) - f\left(u_{h_1}^{n+1}\right) \right\|_1 + \left(\frac{k}{\beta}\right) Ch^2 |\log h|^2 ; \left\| u_2 - u_{h_2}^n \right\|_2 \right\} \end{aligned}$$

so

$$\left\| W_{h_1} - u_{h_1}^{n+1} \right\|_1 \leq \max\left\{ \rho \left\| w_{h_1} - u_{h_1}^{n+1} \right\|_1 + \rho Ch^2 |\log h|^2 ; \left\| u_2 - u_{h_2}^n \right\|_2 \right\}. \tag{4.49}$$

On the other hand, (4.9) generates two possibilities, that is

$$(C_1): \quad \left\| w_{h_1} - u_{h_1}^{n+1} \right\|_1 \leq \left\| W_{h_1} - u_{h_1}^{n+1} \right\|_1$$

or

$$(C_2): \quad \left\| W_{h_1} - u_{h_1}^{n+1} \right\|_1 \leq \left\| w_{h_1} - u_{h_1}^{n+1} \right\|_1.$$

Case (C_1) in conjunction with (4.49) implies that

$$\left\| w_{h_1} - u_{h_1}^{n+1} \right\|_1 \leq \max\left\{ \rho \left\| w_{h_1} - u_{h_1}^{n+1} \right\|_1 + \rho Ch^2 |\log h|^2 ; \left\| u_2 - u_{h_2}^n \right\|_2 \right\},$$

which lets us distinguish the following two cases:

$$\begin{aligned} 1: \quad & \max\left\{ \rho \left\| w_{h_1} - u_{h_1}^{n+1} \right\|_1 + \rho Ch^2 |\log h|^2 ; \left\| u_2 - u_{h_2}^n \right\|_2 \right\} \\ & = \rho \left\| w_{h_1} - u_{h_1}^{n+1} \right\|_1 + \rho Ch^2 |\log h|^2 \end{aligned} \tag{4.50}$$

and

$$2: \quad \max\{\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2; \|u_2 - u_{h_2}^n\|_2\} = \|u_2 - u_{h_2}^n\|_2. \tag{4.51}$$

Equation (4.50) implies that

$$\|w_{h_1} - u_{h_1}^{n+1}\|_1 \leq \rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2$$

and

$$\|u_2 - u_{h_2}^n\|_2 \leq \rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2.$$

Then

$$\|w_{h_1} - u_{h_1}^{n+1}\|_1 \leq \frac{\rho}{1-\rho} Ch^2|\log h|^2$$

and

$$\|u_2 - u_{h_2}^n\|_2 \leq \frac{\rho}{1-\rho} Ch^2|\log h|^2 < \frac{1}{1-\rho} Ch^2|\log h|^2,$$

which coincides with (4.47). Equation (4.51) implies that

$$\|w_{h_1} - u_{h_1}^{n+1}\|_1 \leq \|u_2 - u_{h_2}^n\|_2 \tag{4.52}$$

and

$$\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2 \leq \|u_2 - u_{h_2}^n\|_2.$$

So, by multiplying (4.52) by ρ and adding $\rho Ch^2|\log h|^2$ we get

$$\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2 \leq \rho\|u_2 - u_{h_2}^n\|_2 + \rho Ch^2|\log h|^2;$$

then $\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2$ is bounded by both $\|u_2 - u_{h_2}^n\|_2$ and $\rho\|u_2 - u_{h_2}^n\|_2 + \rho Ch^2|\log h|$. So

$$\text{(a):} \quad \|u_2 - u_{h_2}^n\|_2 \leq \rho\|u_2 - u_{h_2}^n\|_2 + \rho Ch^2|\log h|$$

or

$$\text{(b):} \quad \rho\|u_2 - u_{h_2}^n\|_2 + \rho Ch^2|\log h| \leq \|u_2 - u_{h_2}^n\|_2.$$

Thus

$$\|u_2 - u_{h_2}^n\|_2 \leq \frac{\rho}{1-\rho} Ch^2|\log h|^2 < \frac{1}{1-\rho} Ch^2|\log h|^2$$

or

$$\frac{\rho}{1-\rho} Ch^2 |\log h|^2 \leq \|u_2 - u_{h_2}^n\|_2 \leq \frac{1}{1-\rho} Ch^2 |\log h|^2.$$

So, the two cases (a) and (b) are true because they both coincide with (4.47). Therefore, there is either a contradiction and thus (4.51) is impossible or (4.51) is possible only if

$$\|u_2 - u_{h_2}^n\|_2 = \frac{\rho}{1-\rho} Ch^2 |\log h|^2.$$

Then (4.51) implies

$$\|w_{h_1} - u_{h_1}^{n+1}\|_1 \leq \|u_2 - u_{h_2}^n\|_2 = \frac{\rho}{1-\rho} Ch^2 |\log h|^2.$$

Thus, in situation (C_1) and in the two cases (4.50) and (4.51), we get

$$\|w_{h_1} - u_{h_1}^{n+1}\|_1 \leq \frac{\rho}{(1-\rho)} Ch^2 |\log h|^2$$

so

$$\alpha_{h_1} \leq u_{h_1}^{n+1} \leq \tilde{\alpha}_{h_1} \tag{4.53}$$

and

$$\alpha_{h_1} - u_1 \leq u_{h_1}^{n+1} - u_1 \leq \tilde{\alpha}_{h_1} - u_1.$$

So, according to (4.18) and (4.19), we get

$$-\frac{1}{(1-\rho)} Ch^2 |\log h|^2 \leq u_{h_1}^{n+1} - u_1 \leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2.$$

Thus

$$\|u_1 - u_{h_1}^{n+1}\|_1 \leq \frac{1}{(1-\rho)} Ch^2 |\log h|^2.$$

Case (C_2) in conjunction with (4.52) implies that $\|W_{h_1} - u_{h_1}^{n+1}\|_1$ is bounded by the values $\|w_{h_1} - u_{h_1}^{n+1}\|_1$ and $\max\{\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2; \|u_2 - u_{h_2}^n\|_2\}$, which generates two situations,

$$\text{(c):}\quad \|w_{h_1} - u_{h_1}^{n+1}\|_1 \leq \max\{\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2 |\log h|^2; \|u_2 - u_{h_2}^n\|_2\}$$

or

$$\text{(d):}\quad \max\{\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2 |\log h|^2; \|u_2 - u_{h_2}^n\|_2\} \leq \|w_{h_1} - u_{h_1}^{n+1}\|_1.$$

It is clear that case (c) coincides with case (C_1). Let us study case (d); as in case (C_1), $\max\{\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2; \|u_2 - u_{h_2}^n\|_2\}$ lets us distinguish the two cases (4.50) and (4.51). Equation (4.50) in conjunction with (d) implies

$$\|u_2 - u_{h_2}^n\|_2 \leq \rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2 \leq \|w_{h_1} - u_{h_1}^{n+1}\|_1$$

and (4.51) in conjunction with (d) implies

$$\rho\|w_{h_1} - u_{h_1}^{n+1}\|_1 + \rho Ch^2|\log h|^2 \leq \|u_2 - u_{h_2}^n\|_2 \leq \|w_{h_1} - u_{h_1}^{n+1}\|_1.$$

Then in the two cases (4.50) and (4.51), we get

$$\frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \leq \|w_{h_1} - u_{h_1}^{n+1}\|_1$$

and

$$\|u_2 - u_{h_2}^n\|_2 \leq \|w_{h_1} - u_{h_1}^{n+1}\|_1.$$

Hence, $\|w_{h_1} - u_{h_1}^{n+1}\|_1$ is bounded below by both $\frac{\rho}{(1-\rho)} Ch^2|\log h|^2$ and $\|u_2 - u_{h_2}^n\|_2$ so we distinguish the two following possibilities:

$$\text{(e):}\quad \|u_2 - u_{h_2}^n\|_2 \leq \frac{\rho}{(1-\rho)} Ch^2|\log h|^2 < \frac{1}{(1-\rho)} Ch^2|\log h|^2$$

or

$$\text{(f):}\quad \frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \leq \|u_2 - u_{h_2}^n\|_2 \leq \frac{1}{(1-\rho)} Ch^2|\log h|^2.$$

So, the two cases (e) and (f) are true because they both coincide with (4.47). Therefore, there is either a contradiction and the two cases (4.50) and (4.51) are impossible or the two cases (4.50) and (4.51) are possible only if

$$\|u_2 - u_{h_2}^n\|_2 = \frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \leq \|w_{h_1} - u_{h_1}^{n+1}\|_1;$$

thus, in the two cases (4.50) and (4.51) of situation (C_2), we get

$$\frac{\rho}{(1-\rho)} Ch^2|\log h|^2 \leq \|w_{h_1} - u_{h_1}^{n+1}\|_1.$$

The remainder of the proof related to situation (C_2) rests on the same arguments used in subdomain Ω_1 for situation (A_2) at iteration $n = 1$, that is, on a decomposition of $\Omega_1 = \Omega_{1,1} \cup \Omega_{1,1}^c$ and on showing that

$$\|u_1 - u_{h_1}^{n+1}\|_{L^\infty(\Omega_{1,1})} \leq \frac{1}{(1-\rho)} Ch^2|\log h|^2$$

and

$$\|u_1 - u_{h_1}^{n+1}\|_{L^\infty(\Omega_{1,1}^c)} \leq \frac{1}{(1-\rho)} Ch^2|\log h|^2.$$

Finally, in the two situations (C_1) and (C_2) we get the desired result,

$$\left\| u_1 - u_{h_1}^{n+1} \right\|_1 \le \frac{1}{(1-\rho)} Ch^2 |\log h|^2. \tag{4.54}$$

Estimate (4.48) in domain 2 can be proved similarly using estimate (4.54).

Part 2: This second part of the proof is devoted to $\frac{1}{2} < \rho < 1$. So

$$\frac{\rho}{1-\rho} > 1. \tag{4.55}$$

For $n = 1$, in domain 1, like as in part 1, (4.9) generates two different situations (A_1) and (A_2), which we study separately. According to (4.10), situation (A_1) in conjunction with (4.11) implies

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho Ch^2 |\log h|^2$$

and

$$\left\| u_2 - u_{h_2}^0 \right\|_2 \le \rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho Ch^2 |\log h|^2.$$

Then

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \frac{\rho}{1-\rho} Ch^2 |\log h|^2$$

and

$$\left\| u_2 - u_{h_2}^0 \right\|_2 \le \frac{\rho}{1-\rho} Ch^2 |\log h|^2.$$

So, we can write for (4.5)

$$\left\| u_2 - u_{h_2}^0 \right\|_2 \le Ch^2 |\log h|^2 < \frac{\rho}{1-\rho} Ch^2 |\log h|^2$$

and for (4.6)

$$Ch^2 |\log h|^2 < \left\| u_2 - u_{h_2}^0 \right\|_2 \le \frac{\rho}{1-\rho} Ch^2 |\log h|^2.$$

Equation (4.12) implies that

$$\left\| w_{h_1} - u_{h_1}^1 \right\|_1 \le \left\| u_2 - u_{h_2}^0 \right\|_2 \tag{4.56}$$

and

$$\rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho Ch^2 |\log h|^2 \le \left\| u_2 - u_{h_2}^0 \right\|_2.$$

So, by multiplying (4.56) by ρ and adding $\rho Ch^2 |\log h|^2$ we get

$$\rho \left\| w_{h_1} - u_{h_1}^1 \right\|_1 + \rho Ch^2 |\log h|^2 \le \rho \left\| u_2 - u_{h_2}^0 \right\|_2 + \rho Ch^2 |\log h|^2.$$

Then $\rho\|w_{h_1} - u^1_{h_1}\|_1 + \rho Ch^2|\log h|^2$ is bounded by $\|u_2 - u^0_{h_2}\|_2$ and $\rho\|u_2 - u^0_{h_2}\|_2 + \rho Ch^2|\log h|$, so

$$\text{(a):}\quad \|u_2 - u^0_{h_2}\|_2 \le \rho\|u_2 - u^0_{h_2}\|_2 + \rho Ch^2|\log h|$$

or

$$\text{(b):}\quad \rho\|u_2 - u^0_{h_2}\|_2 + \rho Ch^2|\log h| \le \|u_2 - u^0_{h_2}\|_2,$$

that is,

$$\|u_2 - u^0_{h_2}\|_2 \le Ch^2|\log h|^2 < \frac{\rho}{1-\rho}Ch^2|\log h|^2 \tag{4.57}$$

or

$$\frac{\rho}{1-\rho}Ch^2|\log h|^2 \le \|u_2 - u^0_{h_2}\|_2 \le Ch^2|\log h|^2.$$

It is clear that only case (a) is possible because it coincides with (4.5). Equations (4.56) and (4.57) imply

$$\|w_{h_1} - u^1_{h_1}\|_1 \le \|u_2 - u^0_{h_2}\|_2 \le \frac{\rho}{1-\rho}Ch^2|\log h|^2,$$

while in (4.6) the two cases (a) and (b) are true with

$$Ch^2|\log h|^2 < \|u_2 - u^0_{h_2}\|_2 \le \frac{\rho}{1-\rho}Ch^2|\log h|^2 \tag{4.58}$$

or

$$Ch^2|\log h|^2 < \frac{\rho}{1-\rho}Ch^2|\log h|^2 \le \|u_2 - u^0_{h_2}\|_2,$$

which leads to the unique possibility

$$\|u_2 - u^0_{h_2}\|_2 = \frac{\rho}{1-\rho}Ch^2|\log h|^2.$$

In brief, in the two cases (4.11) and (4.12) of situation (A_1) and in the two situations (A) and (B), we get

$$\|w_{h_1} - u^1_{h_1}\|_1 \le \frac{\rho}{(1-\rho)}Ch^2|\log h|^2.$$

The rest of the proof is similar to the part 1, situation (A_1), and leads to the result (4.33). According to (4.10), situation (A_2) like in part 1 focuses on the study of the case (d) and in the two cases (4.11) and (4.12), we get

$$\frac{\rho}{(1-\rho)}Ch^2|\log h|^2 \le \|w_{h_1} - u^1_{h_1}\|_1$$

with

$$\left\| u_2 - u_{h_2}^0 \right\|_2 \le \left\| w_{h_1} - u_{h_1}^1 \right\|_1.$$

The rest of the proof related to situation (A_2) is similar to part 1, situation (A_2), and leads to the result (4.33). That is, in the two situations (A) and (B) with $\frac{1}{2} < \rho < 1$, we get (4.33).

The remainder of the proof related to $\frac{1}{2} < \rho < 1$ is by induction and similar to part 1, by which we obtain the desired result (4.4). □

5 Conclusion

In this paper an optimal convergence order for finite element Schwarz alternating method for a class of VI with nonlinear source terms on two subdomains with nonmatching grids is obtained. The approach rests on a discrete Lipschitz dependence with respect to the both boundary condition and the source term. This approach offers practical perspectives in that it enables us to control the error, on each subdomain between the discrete Schwarz algorithm and the true solution.

Competing interests
The author declares that there are no competing interests.

Author's contributions
This work is the continuation of previous contributions of the author.

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