Huang et al. *Journal of Inequalities and Applications* (2016) 2016:167
DOI 10.1186/s13660-016-1109-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

An extension of the estimation for solutions of certain Laplace equations

Bin Huang[1], Jiaofeng Wang[2] and Maria Hampshire Zylbersztejn[3*]

*Correspondence: m.h.zylbersztejn@gmail.com
[3]Departamento de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, Rio de Janeiro, RJ 21945-970, Brazil
Full list of author information is available at the end of the article

Abstract
In this paper, by using a new type of Carleman formula with respect to a certain Laplace operator, we estimate the growth property for solutions of certain Laplace equations defined in a smooth cone.

Keywords: growth property; certain Laplace equation; cone

1 Introduction and results

Let $\mathbf{R}^n$ ($n \geq 2$) be the n-dimensional Euclidean space. A point in $\mathbf{R}^n$ is denoted by $V = (X, y)$, where $X = (x_1, x_2, \ldots, x_{n-1})$. The boundary and the closure of a set E in $\mathbf{R}^n$ are denoted by ∂E and $\overline{E}$, respectively.

We introduce a system of spherical coordinates (l, Λ), $\Lambda = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\mathbf{R}^n$ which are related to cartesian coordinates $(x_1, x_2, \ldots, x_{n-1}, y)$ by $y = l\cos\theta_1$.

The unit sphere in $\mathbf{R}^n$ is denoted by $\mathbf{S}^{n-1}$. For simplicity, a point $(1, \Lambda)$ on $\mathbf{S}^{n-1}$ and the set $\{\Lambda; (1, \Lambda) \in \Gamma\}$ for a set Γ, $\Gamma \subset \mathbf{S}^{n-1}$ are often identified with Λ and Γ, respectively. For two sets $\Xi \subset \mathbf{R}_+$ and $\Gamma \subset \mathbf{S}^{n-1}$, the set

$$\bigl\{(l, \Lambda) \in \mathbf{R}^n; l \in \Xi, (1, \Lambda) \in \Gamma\bigr\}$$

in $\mathbf{R}^n$ is simply denoted by $\Xi \times \Gamma$.

We denote the set $\mathbf{R}_+ \times \Gamma$ in $\mathbf{R}^n$ with the domain Γ on $\mathbf{S}^{n-1}$ by $T_n(\Gamma)$. We call it a cone. The sets $I \times \Gamma$ and $I \times \partial\Gamma$ with an interval on $\mathbf{R}$ are denoted by $T_n(\Gamma; I)$ and $\mathcal{S}_n(\Gamma; I)$, respectively. By $\mathcal{S}_n(\Gamma; l)$ we denote $T_n(\Gamma) \cap S_l$. We denote $\mathcal{S}_n(\Gamma; (0, +\infty))$ by $\mathcal{S}_n(\Gamma)$.

Let $\mathbb{G}_\Gamma(V, W)$ $(P, Q \in T_n(\Gamma))$ be the Green function in $T_n(\Gamma)$. Then the ordinary Poisson formula in $T_n(\Gamma)$ is defined by

$$c_n \mathbb{PI}_\Gamma(V, W) = \frac{\partial \mathbb{G}_\Gamma(V, W)}{\partial n_W},$$

where $\partial/\partial n_W$ denotes the differentiation at Q along the inward normal into $T_n(\Gamma)$. Here, $c_2 = 2$ and $c_n = (n-2)w_n$ when $n \geq 3$, where w_n is the surface area of $\mathbf{S}^{n-1}$.

Let Δ_n^* be the spherical version of the Laplace operator and Γ be a domain on $\mathbf{S}^{n-1}$ with smooth boundary $\partial\Gamma$. Consider the Dirichlet problem (see [1])

$$\bigl(\Delta_n^* + \tau\bigr)\psi = 0 \quad \text{on } \Gamma, \tag{1.1}$$

$$\psi = 0 \quad \text{on } \partial\Gamma. \tag{1.2}$$

We denote the least positive eigenvalue of (1.1) and (1.2) by τ and the normalized positive eigenfunction corresponding to τ by $\psi(\Lambda)$. In the sequel, for the sake of brevity, we shall write χ instead of $\aleph^+ - \aleph^-$, where

$$2\aleph^{\pm} = -n + 2 \pm \sqrt{(n-2)^2 + 4\tau}.$$

We use the standard notations $h^+ = \max\{h, 0\}$ and $h^- = -\min\{h, 0\}$. All constants appearing in the expressions in the following sections will be always written M, because we do not need to specify them. Throughout this paper, we will always assume that $\eta(t)$ and $\rho(t)$ are nondecreasing real-valued functions on an interval $[1, +\infty)$ and $\rho(t) > \aleph^+$ for any $t \in [1, +\infty)$.

Recently, Li and Vetro (see [2], Theorem 1) obtained the lower bounds for functions harmonic in a smooth cone. Similar results for solutions of p-Laplace equations under Neumann boundary condition, we refer the reader to the papers by Guo and Gao (see [3]) and Rao and Pu (see [4]).

Theorem A *Let K be a constant, $h(V)$ $(V = (R, \Lambda))$ be harmonic on $T_n(\Gamma)$ and continuous on $\overline{T_n(\Gamma)}$. If*

$$h(V) \le KR^{\rho(R)}, \qquad V = (R, \Lambda) \in T_n\bigl(\Gamma; (1, \infty)\bigr)$$

and

$$h(V) \ge -K, \quad R \le 1, \qquad V = (R, \Lambda) \in \overline{T_n(\Gamma)},$$

then

$$h(V) \ge -KM\bigl(1 + \rho(R)R^{\rho(R)}\bigr)\psi^{1-n}(\Lambda),$$

where $V \in T_n(\Gamma)$ and M is a constant independent of K, R, $\psi(\Lambda)$, and the function $h(V)$.

In this paper, we shall extend Theorem A to solutions of a certain Laplace equation (see [5] for the definition of this Laplace equation).

Theorem 1 *Let $h(V)$ $(V = (R, \Lambda))$ be solutions of certain Laplace equation defined on $T_n(\Gamma)$ and continuous on $\overline{T_n(\Gamma)}$. If*

$$h(V) \le \eta(R)R^{\rho(R)}, \qquad V = (R, \Lambda) \in T_n\bigl(\Gamma; (1, \infty)\bigr), \tag{1.3}$$

and

$$h(V) \ge -\eta(R), \quad R \le 1, \qquad V = (R, \Lambda) \in \overline{T_n(\Gamma)}, \tag{1.4}$$

then

$$h(V) \ge -M\eta(R)\bigl(1 + \rho(cR)R^{\rho(cR)}\bigr)\psi^{1-n}(\Lambda),$$

where $V \in T_n(\Gamma)$, c is a real number satisfying $c \geq 1$ and M is a constant independent of R, $\psi(\Lambda)$, the functions $\eta(R)$ and $h(V)$.

Remark In the case $c \equiv 1$ and $\eta(R) \equiv K$, where K is a constant, Theorem 1 reduces to Theorem A.

2 Lemmas

In order to prove our result, we first introduce a new type of Carleman formula for functions harmonic in a cone (see [6]). For the Carleman formula for harmonic functions and its application, we refer the reader to the paper by Yang and Ren (see [7], Lemma 1). Recently, it has been extended to Schrödinger subharmonic functions in a cone (see [8], Lemma 1). For applications, we also refer the reader to the paper by Wang *et al.* (see [8], Theorem 2).

Lemma 1 *Let h be harmonic on a domain containing $T_n(\Gamma;(1,R))$, where $R > 1$. Then*

$$\chi \int_{S_n(\Gamma;R)} h\psi R^{\aleph^- -1}\, dS_R + \int_{S_n(\Gamma;(1,R))} h\bigl(t^{\aleph^-} - t^{\aleph^+} R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W + d_1 + d_2 R^{-\chi} = 0,$$

where dS_R denotes the $(n-1)$-dimensional volume elements induced by the Euclidean metric on S_R, $\partial/\partial n$ denotes differentiation along the interior normal,

$$d_1 = \int_{S_n(\Gamma;1)} \aleph^- h\psi - \psi(\partial h/\partial n)\, dS_1$$

and

$$d_2 = \int_{S_n(\Gamma;1)} \psi(\partial h/\partial n) - \aleph^+ h\psi\, dS_1.$$

Lemma 2 (See [9], Lemma 4) *We have*

$$\mathcal{PI}_\Gamma(V,W) \leq M r^{\aleph^-} t^{\aleph^+ -1}\psi(\Lambda)\frac{\partial\psi(\Phi)}{\partial n_\Phi}$$

for any $V = (r,\Lambda) \in T_n(\Gamma)$ and any $W = (t,\Phi) \in S_n(\Gamma)$ satisfying $0 < \frac{t}{r} \leq \frac{4}{5}$.

$$\mathcal{PI}_\Gamma(V,W) \leq M\frac{\psi(\Lambda)}{t^{n-1}}\frac{\partial\psi(\Phi)}{\partial n_\Phi} + M\frac{r\psi(\Lambda)}{|P-Q|^n}\frac{\partial\psi(\Phi)}{\partial n_\Phi}$$

for any $V = (r,\Lambda) \in T_n(\Gamma)$ and any $W = (t,\Phi) \in S_n(\Gamma;(\frac{4}{5}r, \frac{5}{4}r))$.

Let $G_{\Gamma,R}(V,W)$ be the Green function of $T_n(\Gamma,(0,R))$. Then

$$\frac{\partial G_{\Gamma,R}(V,W)}{\partial R} \leq M r^{\aleph^+} R^{\aleph^- -1}\psi(\Lambda)\psi(\Phi),$$

where $V = (r,\Lambda) \in T_n(\Gamma)$ and $Q = (R,\Phi) \in S_n(\Gamma;R)$.

3 Proof of Theorem 1

We first apply Lemma 1 to $h = h^+ - h^-$ and obtain

$$\begin{aligned}
&\chi \int_{\mathcal{S}_n(\Gamma;R)} h^+ R^{\aleph^- -1}\psi\, dS_R + \int_{\mathcal{S}_n(\Gamma;(1,R))} h^+\bigl(t^{\aleph^-} - t^{\aleph^+}R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W + d_1 + d_2R^{-\chi} \\
&\quad = \chi \int_{\mathcal{S}_n(\Gamma;R)} h^- R^{\aleph^- -1}\psi\, dS_R + \int_{\mathcal{S}_n(\Gamma;(1,R))} h^-\bigl(t^{\aleph^-} - t^{\aleph^+}R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W, \qquad (3.1)
\end{aligned}$$

It is easy to see that

$$\chi \int_{\mathcal{S}_n(\Gamma;R)} h^+ R^{\aleph^- -1}\psi\, dS_R \le M\eta(R)R^{\rho(cR)-\aleph^+} \qquad (3.2)$$

and

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^+\bigl(t^{\aleph^-} - t^{\aleph^+}R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W \le M\eta(R)R^{\rho(cR)-\aleph^+} \qquad (3.3)$$

from (1.3).

We remark that

$$d_1 + d_2R^{-\chi} \le M\eta(R)R^{\rho(cR)-\aleph^+}. \qquad (3.4)$$

We have

$$\chi \int_{\mathcal{S}_n(\Gamma;R)} h^- R^{\aleph^- -1}\psi\, dS_R \le M\eta(R)R^{\rho(cR)-\aleph^+} \qquad (3.5)$$

and

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^-\bigl(t^{\aleph^-} - t^{\aleph^+}R^{-\chi}\bigr)\partial\psi/\partial n\, d\sigma_W \le M\eta(R)R^{\rho(cR)-\aleph^+} \qquad (3.6)$$

from (3.1), (3.2), (3.3), and (3.4).

It follows from (3.6) that

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^- t^{\aleph^-}\frac{\partial\psi}{\partial n}\, d\sigma_W \le M\eta(R)\frac{(\rho(cR)+1)^{\chi}}{(\rho(cR)+1)^{\chi} - (\rho(cR))^{\chi}}\left(\frac{\rho(cR)+1}{\rho(cR)}R\right)^{\rho(\frac{\rho(cR)+1}{\rho(cR)}R)-\aleph^+},$$

which shows that

$$\int_{\mathcal{S}_n(\Gamma;(1,R))} h^- t^{\aleph^-}\partial\psi/\partial n\, d\sigma_W \le M\eta(R)\rho(cR)R^{\rho(cR)-\aleph^+}. \qquad (3.7)$$

By the Riesz decomposition theorem (see [10]), we have

$$\begin{aligned}
-h(V) &= \int_{\mathcal{S}_n(\Gamma;(0,R))} \mathcal{PI}_\Gamma(V,W) - h(W)\, d\sigma_W \\
&\quad + \int_{\mathcal{S}_n(\Gamma;R)} \frac{\partial \mathbb{G}_{\Gamma,R}(V,W)}{\partial R} - h(W)\, dS_R, \qquad (3.8)
\end{aligned}$$

where $V = (l,\Lambda) \in T_n(\Gamma;(0,R))$.

We next distinguish three cases.

Case 1. $V = (l, \Lambda) \in T_n(\Gamma; (5/4, \infty))$ and $R = 5l/4$.

Since $-h(V) \leq h^-(V)$, we have

$$-h(V) = \sum_{i=1}^{4} U_i(V) \tag{3.9}$$

from (3.8), where

$$U_1(V) = \int_{\mathcal{S}_n(\Gamma;(0,1])} \mathcal{PI}_\Gamma(V, W) - h(W)\, d\sigma_W,$$

$$U_2(V) = \int_{\mathcal{S}_n(\Gamma;(1,4l/5])} \mathcal{PI}_\Gamma(V, W) - h(W)\, d\sigma_W,$$

$$U_3(V) = \int_{\mathcal{S}_n(\Gamma;(4l/5,R))} \mathcal{PI}_\Gamma(V, W) - h(W)\, d\sigma_W,$$

and

$$U_4(V) = \int_{\mathcal{S}_n(\Gamma;R)} \mathcal{PI}_\Gamma(V, W) - h(W)\, d\sigma_W.$$

We have the following estimates:

$$U_1(V) \leq M\eta(R)\psi(\Lambda) \tag{3.10}$$

and

$$U_2(V) \leq M\eta(R)\rho(cR)R^{\rho(cR)}\psi(\Lambda) \tag{3.11}$$

from Lemma 2 and (3.7).

We consider the inequality

$$U_3(V) \leq U_{31}(V) + U_{32}(V), \tag{3.12}$$

where

$$U_{31}(V) = M \int_{\mathcal{S}_n(\Gamma;(4l/5,R))} \frac{-h(W)\psi(\Lambda)}{t^{n-1}} \frac{\partial\phi(\Phi)}{\partial n_\Phi}\, d\sigma_W$$

and

$$U_{32}(V) = Mr\psi(\Lambda) \int_{\mathcal{S}_n(\Gamma;(4l/5,R))} \frac{-h(W)l\psi(\Lambda)}{|V-W|^n} \frac{\partial\phi(\Phi)}{\partial n_\Phi}\, d\sigma_W.$$

We first have

$$U_{31}(V) \leq M\eta(R)\rho(cR)R^{\rho(cR)}\psi(\Lambda) \tag{3.13}$$

from (3.7).

We shall estimate $U_{32}(V)$. Take a sufficiently small positive number d such that

$$\mathcal{S}_n\big(\Gamma;(4l/5,R)\big)\subset B(P,l/2)$$

for any $V=(l,\Lambda)\in\Pi(d)$, where

$$\Pi(d)=\Big\{V=(l,\Lambda)\in T_n(\Gamma);\inf_{(1,z)\in\partial\Gamma}\big|(1,\Lambda)-(1,z)\big|<d,0<r<\infty\Big\},$$

and divide $T_n(\Gamma)$ into two sets $\Pi(d)$ and $T_n(\Gamma)-\Pi(d)$.

If $V=(l,\Lambda)\in T_n(\Gamma)-\Pi(d)$, then there exists a positive d' such that $|V-W|\geq d'l$ for any $Q\in\mathcal{S}_n(\Gamma)$, and hence

$$U_{32}(V)\leq M\eta(R)\rho(cR)R^{\rho(cR)}\psi(\Lambda),\tag{3.14}$$

which is similar to the estimate of $U_{31}(V)$.

We shall consider the case $V=(l,\Lambda)\in\Pi(d)$. Now put

$$H_i(V)=\big\{Q\in\mathcal{S}_n\big(\Gamma;(4l/5,R)\big);2^{i-1}\delta(V)\leq|V-W|<2^i\delta(V)\big\},$$

where

$$\delta(V)=\inf_{Q\in\partial T_n(\Gamma)}|V-W|.$$

Since

$$\mathcal{S}_n(\Gamma)\cap\big\{Q\in\mathbf{R}^n:|V-W|<\delta(V)\big\}=\varnothing,$$

we have

$$U_{32}(V)=M\sum_{i=1}^{i(V)}\int_{H_i(V)}\frac{-h(W)r\psi(\Lambda)}{|V-W|^n}\frac{\partial\psi(\Phi)}{\partial n_\Phi}\,d\sigma_W,$$

where $i(V)$ is a positive integer satisfying $2^{i(V)-1}\delta(V)\leq\frac{r}{2}<2^{i(V)}\delta(V)$.

Since

$$r\psi(\Lambda)\leq M\delta(V),$$

where $V=(l,\Lambda)\in T_n(\Gamma)$, similar to the estimate of $U_{31}(V)$ we obtain

$$\int_{H_i(V)}\frac{-h(W)r\psi(\Lambda)}{|V-W|^n}\frac{\partial\psi(\Phi)}{\partial n_\Phi}\,d\sigma_W\leq M\eta(R)\rho(cR)R^{\rho(cR)}\psi^{1-n}(\Lambda)$$

for $i=0,1,2,\dots,i(V)$.

So

$$U_{32}(V)\leq M\eta(R)\rho(cR)R^{\rho(cR)}\psi^{1-n}(\Lambda).\tag{3.15}$$

From (3.12), (3.13), (3.14), and (3.15) we see that

$$U_3(V) \leq M\eta(R)\rho(cR)R^{\rho(cR)}\psi^{1-n}(\Lambda). \tag{3.16}$$

On the other hand, we have from Lemma 2 and (3.5)

$$U_4(V) \leq M\eta(R)R^{\rho(cR)}\psi(\Lambda). \tag{3.17}$$

We thus obtain from (3.10), (3.11), (3.16), and (3.17)

$$-h(V) \leq M\eta(R)\bigl(1+\rho(cR)R^{\rho(cR)}\bigr)\psi^{1-n}(\Lambda). \tag{3.18}$$

Case 2. $V=(l,\Lambda)\in T_n(\Gamma;(4/5,5/4])$ and $R=5l/4$.
It follows from (3.8) that

$$-h(V) = U_1(V)+U_5(V)+U_4(V),$$

where $U_1(V)$ and $U_4(V)$ are defined in Case 1 and

$$U_5(V) = \int_{\mathcal{S}_n(\Gamma;(1,R))} \mathcal{PI}_\Gamma(V,W) - h(W)\,d\sigma_W.$$

Similar to the estimate of $U_3(V)$ in Case 1 we have

$$U_5(V) \leq M\eta(R)\rho(cR)R^{\rho(cR)}\psi^{1-n}(\Lambda),$$

which together with (3.10) and (3.17) gives (3.18).
Case 3. $V=(l,\Lambda)\in T_n(\Gamma;(0,4/5])$.
It is evident from (1.4) that we have

$$-h \leq \eta(R),$$

which also gives (3.18).
We finally have

$$h(V) \geq -\eta(R)M\bigl(1+\rho(cR)R^{\rho(cR)}\bigr)\psi^{1-n}(\Lambda)$$

from (3.18), which is the conclusion of Theorem 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MZ completed the main study. BH carried out the results of this article. JW verified the calculation. All the authors read and approved the final manuscript.

Author details
[1]Mathematical and Statistical Sciences, Quzhou University, Quzhou, 324000, China. [2]Mathematical and Statistical Sciences, Quzhou Vocational and Technical College, Quzhou, 324000, China. [3]Departamento de Matemática, Universidade Federal do Rio de Janeiro, Caixa Postal 68530, Rio de Janeiro, RJ 21945-970, Brazil.

Acknowledgements
The authors are grateful to the two anonymous referees for their valuable comments, which led to a much improved version of the manuscript.

Received: 30 April 2016 Accepted: 21 June 2016 Published online: 29 June 2016

References

1. Carleman, T: Über die Approximation analytischer Funktionen durch lineare Aggregate von vorgegebenen Potenzen. Ark. Mat. Astron. Fys. **17**(66), 1-30 (1923)
2. Li, Z, Vetro, M: Levin's type boundary behaviors for functions harmonic and admitting certain lower bounds. Bound. Value Probl. **2015**, 159 (2015)
3. Guo, B, Gao, W: Non-extinction of solutions to a fast diffusive *p*-Laplace equation with Neumann boundary conditions. J. Math. Anal. Appl. **422**(2), 1527-1531 (2015)
4. Rao, R, Pu, Z: Stability analysis for impulsive stochastic fuzzy *p*-Laplace dynamic equations under Neumann or Dirichlet boundary condition. Bound. Value Probl. **2013**, 133 (2013)
5. Yan, Z, Ychussie, B: Normal families and asymptotic behaviors for solutions of certain Laplace equations. Adv. Differ. Equ. **2015**, 226 (2015)
6. Qiao, L, Zhang, S: Asymptotic analysis of a coupled nonlinear parabolic system. Front. Math. China **3**(1), 87-99 (2008)
7. Yang, D, Ren, Y: Dirichlet problem on the upper half space. Proc. Indian Acad. Sci. Math. Sci. **124**(2), 175-178 (2014)
8. Wang, J, Pu, J, Zama, A: Solutions of the Dirichlet-Schrödinger problems with continuous data admitting arbitrary growth property in the boundary. Adv. Differ. Equ. **2016**, 33 (2016)
9. Azarin, VS: Generalization of a theorem of Hayman's on a subharmonic function in an *n*-dimensional cone. Mat. Sb. (N.S.) **108**(66), 248-264 (1965) (in Russian)
10. Axler, S, Bourdon, P, Wade, R: Harmonic Function Theory. Graduate Texts in Mathematics, vol. 137. Springer, New York (1992)