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Robust portfolio selection under norm uncertainty

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Abstract
In this paper, we consider the robust portfolio selection problem which has a data uncertainty described by the (p, w)-norm in the objective function. We show that the robust formulation of this problem is equivalent to a linear optimization problem. Moreover, we present some numerical results concerning our robust portfolio selection problem.

MSC: 91G10; 90C05; 90C90

Keywords: portfolio selection; (p, w)-norm; uncertainty set; linear optimization problem

1 Introduction

Portfolio selection is the problem of allocating capital over a number of available assets in order to maximize the return on the investment while minimizing the risk. The first systematic approach to the problem of asset allocation under uncertainty is attributed to Markowitz [1]. An underlying assumption of Markowitz's model is that precise estimates of μ_i and σ_i have been obtained. Consequently, μ_i and σ_i are treated as known constants; however, asset returns are variable. It is reasonable to conclude that a model which treats returns as known constants will produce a portfolio whose realized return is different from the optimal portfolio return given by the objective function value. In particular, when the realized asset returns are less than the estimates used to optimize the model, the realized portfolio return will be less than the optimal portfolio return given by the objective. Therefore, it is worthwhile exploring alternative frameworks, such as robust optimization, for application to the portfolio selection problem.

Although the distributions of asset returns are uncertain, in the robust optimization framework, we may assert that μ or σ, or both, belong to an uncertainty set, the bounds of which we can define. Most robust portfolio models describe asset returns by ellipsoidal uncertainty sets, based on the methodology of Ben-Tal and Nemirovski [2–4] and El Ghaoui and Lebret [5], in which the user defined parameter Ω adjusts the guaranteed and achieved robustness of the portfolio. Previously, robustness has been evaluated based upon performance, particularly the worst case performance, then compared to the worst case performance of a non-robust model such as the expected value-variance model. In addition to the worst case performance, we suggest that it is also important to evaluate robustness based upon whether a model yields portfolios that achieve their guaranteed robustness in practice. In 2000, Lobo and Boyd [6] presented several different methods for modeling the uncertainty sets for the expected returns vector and covariance matrices, such as box or ellipsoidal sets. Each robust model was a semi-definite program solved via interior point methods. Their results focused on the performance of the solution method rather than on the robustness of the optimal portfolios. Goldfarb and Iyengar [7] defined asset returns by robust factor models in which the uncertainty was modeled by ellipsoidal sets. The robustness was evaluated based on performance, particularly in worst case scenarios, and compared to the expected value-variance portfolio model. Results showed the worst case performance of the robust model was approximately 200% better than the non-robust model; thus, they concluded that robust portfolios were more apt to withstand noisy data. For more about the robust portfolio selection problems with the ellipsoidal sets, we refer to [8–13].

Recently, Bertsimas and Sim [14] proposed a different approach for robust linear optimization with polyhedral (as opposed to ellipsoidal) uncertainty sets. An attractive aspect of their method is that the new robust formulation is also a linear optimization problem. They also extended their methods to discrete optimization problems in a tractable way. In 2004, Bertsimas *et al.* [15] characterized the robust counterpart of a linear programming problem with an uncertainty set. They also showed that the approach of [14] follows from their conclusion by considering a norm, called the D-norm, and its dual. Recently, Wang and Luo [16] considered the linear optimization problem which has a data uncertainty described by the (p, w)-norm. They showed that the (p, w)-norm includes the polyhedral norms L_1, L_∞, and the D-norm as special cases not only to make up for the disadvantages of the uncertain parameters of all possible values that will give the same weight, but also to consider the robust cost of the robust optimization model which is mentioned in [14, 15]. They also provided probabilistic guarantees on the feasibility of an optimal robust solution when the uncertain coefficients obey independent and identically distributed normal distributions.

Motivated and inspired by the work mentioned above, in this paper, we consider the robust portfolio selection problem with an uncertainty set described by the (p, w)-norm. We see that the robust formulation of this problem is a linear optimization problem. Moreover, we present some numerical results about our robust portfolio selection problem.

Here is the structure of this paper. In Section 2, we consider the robust portfolio selection problem which has data uncertainty described by (p, w)-norm in the objective function and we show that the robust formulation of this problem is equivalent to a linear optimization problem. In Section 3, we present some computational results on the performance of our robust portfolio selection problem. Section 4 concludes with a summary of this paper.

2 Robust portfolio selection

In this section, we discuss the formulation of the robust counterpart to the portfolio optimization problem. First of all, we consider the robust portfolio selection problem which has a data uncertainty described by the (p, w)-norm in the objective function. Then we show that the robust formulation of this problem is equivalent to a linear programming.

It is well known that the classical portfolio selection problem can be formulated as follows:

$$\begin{aligned} &\max \sum_{i=1}^{n} r_i x_i - \phi \sum_{i=1}^{n} \sigma_i^2 x_i^2 \\ &\text{s.t.} \ \sum_{i=1}^{n} x_i = 1, \\ &\quad x_i \geq 0, \end{aligned} \tag{2.1}$$

where r_i is the return of ith stock, x_i is the wealth invested in stock i, σ_i is the standard deviation of the return for the ith stock, and ϕ is a parameter that controls the tradeoff between risk and return.

Next, we assume $\tilde{r}_i$ is uncertain, which is a random variable that has an arbitrary independent symmetric distribution in the interval $[r_i - \sigma_i, r_i + \sigma_i]$ and r_i is the expected return for the ith stock. Then the robust counterpart of problem (2.1) is defined as follows:

$$\begin{aligned} &\max \sum_{i=1}^{n} \tilde{r}_i x_i - \phi \sum_{i=1}^{n} \sigma_i^2 x_i^2 \\ &\text{s.t.} \ \sum_{i=1}^{n} x_i = 1, \\ &\quad x_i \geq 0, \\ &\quad \tilde{r}_i \in [r_i - \sigma_i, r_i + \sigma_i]. \end{aligned} \tag{2.2}$$

To make (2.2) more tractable, we add an artificial variable z and rewrite the problem as follows:

$$\begin{aligned} &\max z \\ &\text{s.t.} \ z \leq \sum_{i=1}^{n} \tilde{r}_i x_i - \phi \sum_{i=1}^{n} \sigma_i^2 x_i^2, \\ &\quad \sum_{i=1}^{n} x_i = 1, \\ &\quad x_i \geq 0, \\ &\quad \tilde{r}_i \in [r_i - \sigma_i, r_i + \sigma_i]. \end{aligned} \tag{2.3}$$

We denote by J the set of coefficients r_i, $i \in J$, that are subject to parameter uncertainty; *i.e.*, $\tilde{r}_i$, $i \in J$ takes values according to a symmetric distribution with mean equal to the nominal value r_i in the interval $[r_i - \sigma_i, r_i + \sigma_i]$. For every i, we introduce a parameter p, which takes values in the interval $[0, |J|]$. The case is unlikely that all of the r_i, $i \in J$, will change, which is proposed by [17]. Our goal is to protect for the cases that up to $\lceil p \rceil$ of these coefficients are allowed to change and take the worst case values at the same time. Next, we introduce the following definition of the (p, w)-norm.

Definition 2.1 (see [16]) For a given nonzero vector $w \in R^n$ with $w_j > 0$, $j = 1, \dots, n$, we define the (p, w)-norm as

$$\|y\|_{p,w} = \max_{\{S \mid S \subseteq J, |S| \leq \lceil p \rceil\}} \left\{ \sum_{j \in S} w_j |y_j| \right\}$$

with $y \in R^n$.

Remark 2.1 (see [16])

(1) $\|y\|_{p,w}$ is indeed a norm.

(2) If

- $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$,
- $w_1 = w_2 = \cdots = w_{\lfloor p \rfloor} = 1$, $w_{\lceil p \rceil} = p - \lfloor p \rfloor$, $w_i \leq w_{\lceil p \rceil}$, $\lceil p \rceil < i \leq n$,

then the (p, w)-norm degenerates into D-norm studied by Bertsimas *et al.* [15], *i.e.*,

$$\|y\|_p = \max_{\{S \cup t \mid S \subseteq J, |S| \leq \lfloor p \rfloor, t \in J \setminus S\}} \left\{ \sum_{j \in S} |y_j| + \big(p - \lfloor p \rfloor\big) |y_t| \right\}.$$

(3) If $w = (1, \dots, 1)^T$ and $p = n$, then (p, w)-norm degenerates into L^1 and one has $\|y\|_{p,w} = \|y\|_{n,e} = \sum_{i=1}^n y_i$, $i = 1, \dots, n$.

(4) If $w = (1, \dots, 1)^T$ and $p = 1$, then (p, w)-norm degenerates into L^∞ and $\|y\|_{p,w} = \|y\|_{1,e} = \max |y_i|$, $i = 1, \dots, n$.

Next, we will solve instead the following problem of (2.3) with the (p, w)-norm:

$$\begin{aligned} &\max z \\ &\quad \text{s.t. } z \leq \sum_{i=1}^n r_i x_i - \beta(x, p), \\ &\quad \sum_{i=1}^n x_i = 1, \\ &\quad x_i \geq 0, \end{aligned} \tag{2.4}$$

where

$$\beta(x, p) = \max_{\{S \mid S \subseteq N, |S| = \lceil p \rceil\}} \left\{ \sum_{j \in S} \sigma_j w_j x_j \right\},$$

and in this setting, p is the protection level of the actual portfolio return.

We need the following proposition to reformulate (2.4) as a linear optimization problem.

Proposition 2.1 *Given a vector x^*, the protection function,*

$$\beta\big(x^*, p\big) = \max_{\{S \mid S \subseteq J, |S| = \lceil p \rceil\}} \left\{ \sum_{j \in S} \sigma_j w_j x_j^* \right\} \tag{2.5}$$

is equivalent to the following linear optimization problem:

$$\begin{aligned} \beta(x^*, p) = \max & \sum_{j\in J} \sigma_j w_j x_j^* q_j \\ \text{s.t.} & \sum_{j\in J} q_j \le \lceil p \rceil, \\ & 0 \le q_j \le 1, \quad \forall j \in J, \\ & w_j \ge 0, \quad \forall j \in J. \end{aligned} \tag{2.6}$$

Proof An optimal solution of problem (2.6) obviously consists of $\lceil p \rceil$ variables at 1, which is equivalent to a subset $\{S | S \subseteq J, |S| = \lceil p \rceil\}$. The objective function of problem (2.6) converts to $\sum_{j\in S} \sigma_j w_j x_j^*$, which is equivalent to problem (2.5). □

Next we will reformulate problem (2.4) as a linear optimization problem.

Theorem 2.1 *Problem* (2.4) *is equivalent to the following linear optimization problem*:

$$\begin{aligned} \max\, & z \\ \text{s.t. } & z \le \sum_{i=1}^n r_i x_i - \sum_{j\in J} t_j - q\lceil p \rceil, \\ & t_j + q \ge \sigma_j w_j x_j, \quad \forall j \in J, \\ & \sum_{i=1}^n x_i = 1, \\ & x_i \ge 0, \quad i = 1, \ldots, n, \\ & t_j \ge 0, \quad \forall j \in J, \\ & q \ge 0, \\ & w_j \ge 0, \quad \forall j \in J. \end{aligned} \tag{2.7}$$

Proof First, we consider the dual problem of (2.6):

$$\begin{aligned} \min & \sum_{j\in J} t_j + q\lceil p \rceil \\ \text{s.t. } & t_j + q \ge \sigma_j w_j x_j, \quad \forall j \in J, \\ & t_j \ge 0, \quad \forall j \in J, \\ & q \ge 0, \\ & w_j \ge 0, \quad \forall j \in J. \end{aligned} \tag{2.8}$$

Since problem (2.6) is feasible and bounded for all $p \in [0, |J|]$, by strong duality, we know that the dual problem (2.8) is also feasible and bounded and their objective values coincide. By the proposition, we see that $\beta(x^*, p)$ is equivalent to the objective function value of (2.8).

Substituting into problem (2.4), we see that problem (2.4) equals the linear programming problem (2.7). □

3 Computational results

In this section, the experimental results show that our approach can get better risk-adjusted returns than Bertsimas and Sim [14] with the same protection level, while the risk deviation is significantly smaller than Bertsimas and Sim [14] and our method can capture the balance between risks and benefits that is similar to the mean-variance model, and also it is more simple to get its linear structure.

Assume that x^* is the optimal solution of problem (2.4). In this paper, we also consider 150 stocks, and let

$$r_i^* = \lambda^* + i\delta, \qquad \lambda^* = 1.15, \qquad \delta = \frac{0.05}{150}, \qquad \left[r_1^* \approx 1.15, r_{150}^* = 1.2\right],$$
$$\sigma_i = \frac{1}{3}\delta\sqrt{2in(n+1)} \approx 0.0236\sqrt{i}, \qquad [\sigma_1 = 0.0236, \sigma_{150} = 0.2896].$$

Optimization results. Assume that $x^*(p)$ is an optimal solution of problem (2.4) corresponding to the protection level p, the standard deviation is

$$S_t(p) = \sqrt{\sum_{i\in N}\left(\sigma_i^2\right)\left(w_i^2\right)\left(x_i^*(p)\right)^2}.$$

We will use Matlab 2010 and Cplex 12.6 to solve this problem.

Figure 1 illustrates the performance of the robust solution as a function of the protection level p. The stable returns we get is similar to the expected revenues in the case of anti-interference, it is the result which investors want to see. Furthermore, when $p \geq 40$, the risk-adjusted returns and expectations are all insensitive to the protection level. The

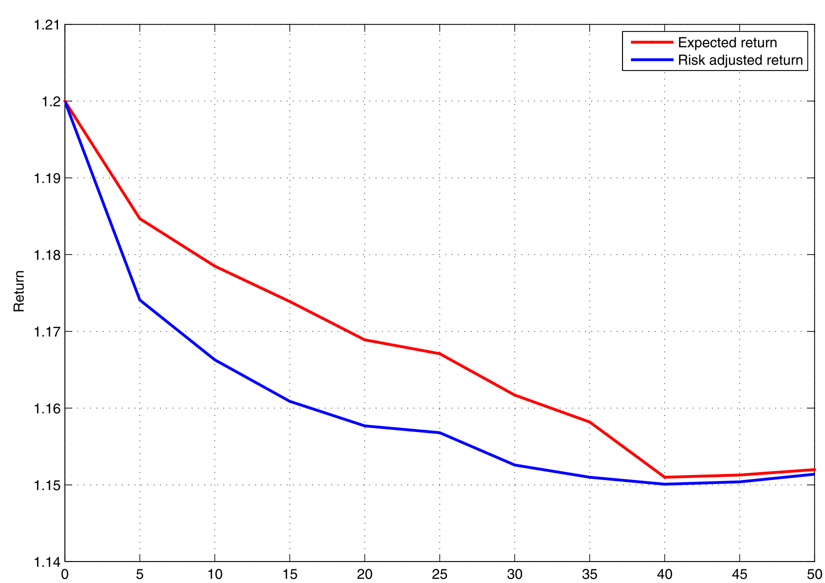

Figure 1 Expected return and risk adjusted return with (p, w)-norm uncertainty.

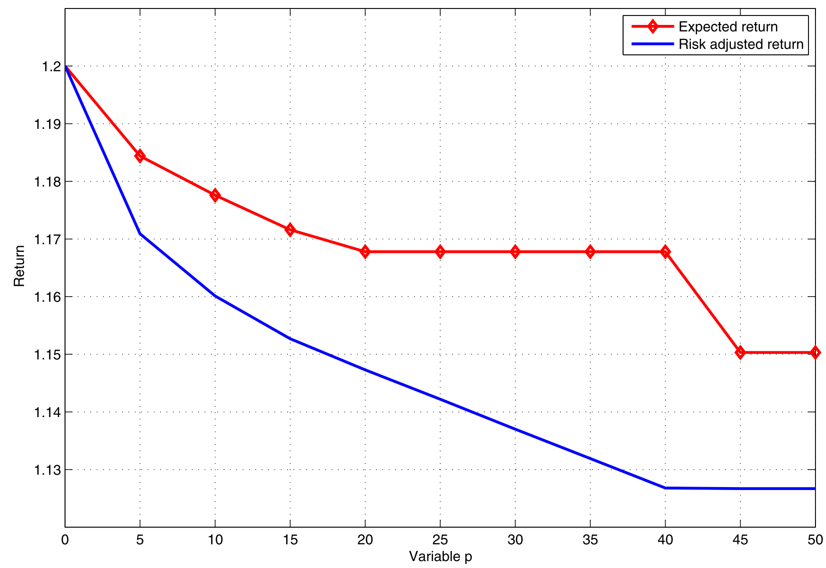

Figure 2 Expected return and risk adjusted return with Bertsimas and Sim approach.

Table 1 (p,w)-Norm uncertainty approach

p	p-exp. return	p-min. return	p-max. return	p-dev
0	1.2000	0.9115	1.4885	0.2885
5	1.1847	1.1108	1.2586	0.0186
10	1.1785	1.1298	1.2272	0.0123
15	1.1739	1.1349	1.2130	0.0094
20	1.1689	1.1409	1.1968	0.0063
25	1.1671	1.1444	1.1898	0.0047
30	1.1617	1.1435	1.1799	0.0036
35	1.1582	1.1449	1.1715	0.0025
40	1.1510	1.1501	1.1519	8.8294e–004
45	1.1513	1.1504	1.1523	9.2802e–004

situation of $p = 40$ is similar to the results of Soyster [17], but our approach reduces the computation complexity and can get a stable objective value under a small protective level. Figure 2 is the simulation result of Bertsimas and Sim [14]. Comparing with Figure 2, we know that Figure 1 does not show the situation of phase transitions, and its risk-adjusted returns and expected returns maintain a consistent trend. The more important aspect is that the objective value in Figure 1 is better than Figure 2.

As we can see from Figure 1, Figure 2, Table 1, Table 2, comparing the results of Figure 3 and Figure 4 we see that our approach can get better risk-adjusted returns than Bertsimas and Sim [14] with the same protection level, while the risk deviation is significantly smaller than Bertsimas and Sim [14]. For instance, when $p = 20$, the risk-adjusted return of our approach is 1.1577, and the risk deviation is 0.0063, while the result of Bertsimas and Sim [14] are 1.1473 and 0.0126. On the other hand, from Figure 5, we can see that the solutions of our method and the approach of Bertsimas and Sim [14], are more balanced than the ellipsoidal method, and the solutions of these methods have all diversities.

According to Table 1, Table 2, and Table 3, we obtain some related empirical results of the robust portfolio problem under the (p,w)-norm, D-norm, and ellipsoid uncertainty sets. The robust counterpart of (p,w)-norm has smaller deviation and the results come

Table 2 Bertsimas and Sim approach

p	D-exp. return	D-min. return	D-max. return	D-dev
0	1.2000	0.9115	1.4885	0.2885
5	1.1844	1.0896	1.2793	0.0254
10	1.1776	1.1075	1.2478	0.0192
15	1.1716	1.1147	1.2285	0.0151
20	1.1678	1.1165	1.2190	0.0126
25	1.1678	1.1114	1.2241	0.0126
30	1.1678	1.1063	1.2293	0.0126
35	1.1678	1.1012	1.2344	0.0126
40	1.1678	1.0960	1.2395	0.0126
45	1.1503	1.1267	1.1740	0.0236

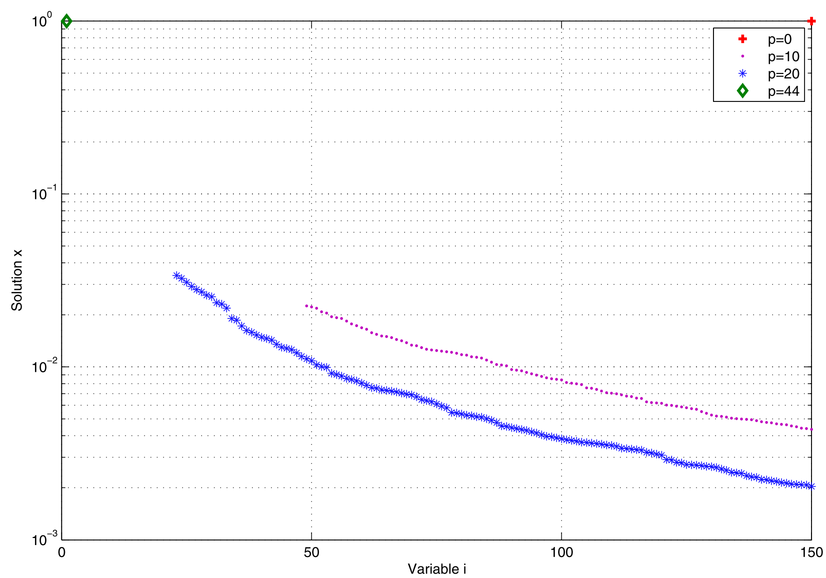

Figure 3 Solutions of (p, w)-norm uncertainty approach.

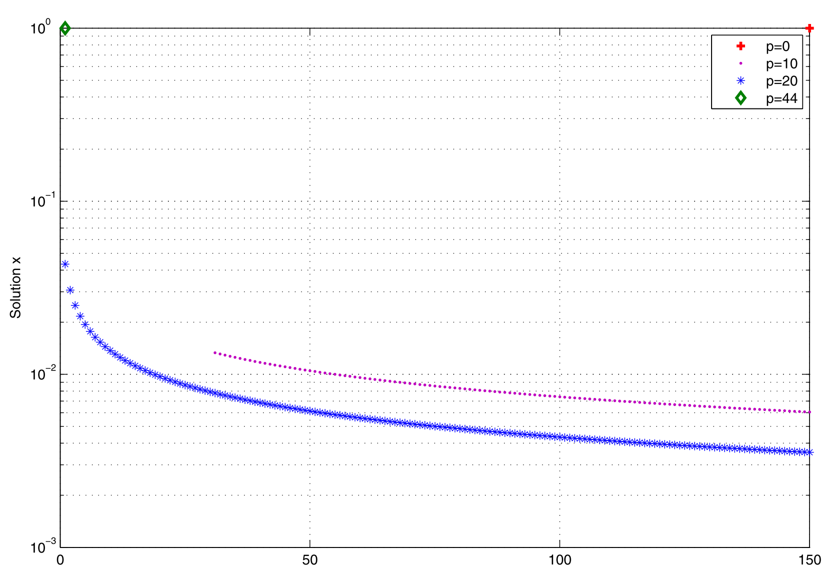

Figure 4 Solutions of Bertsimas and Sim approach.

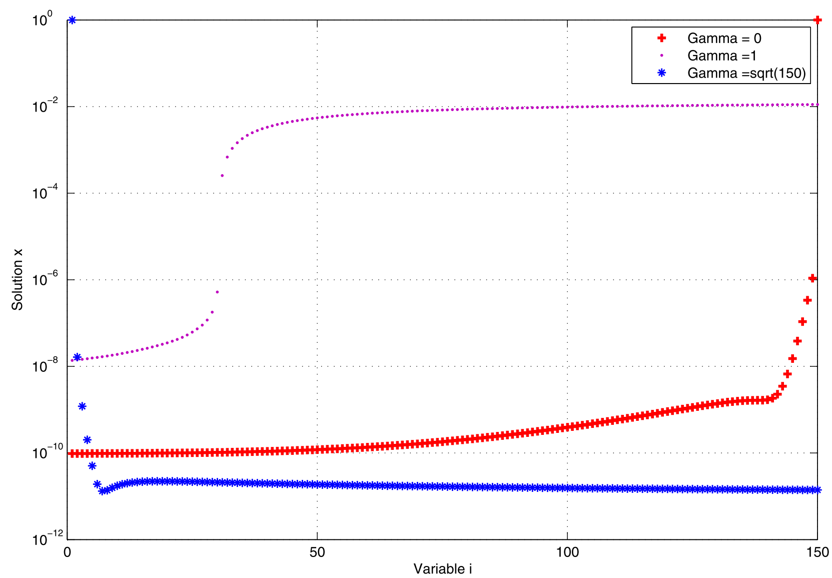

Figure 5 Solutions of ellipsoidal method.

Table 3 Ellipsoidal method

θ	*S*-exp. return	*S*-min. return	*S*-max. return	*S*-dev
0	1.2000	0.9104	1.4896	0.2896
1	1.1836	0.9492	1.1481	0.0235
$\sqrt{150}$	1.1503	1.1267	1.1740	0.0236

quicker as one gets a stable robust value. This example shows that our method capturing the balance between risks and benefits is similar to the mean-variance model, and it is more simple to get its linear structure.

4 Conclusions

In this paper, we consider the robust portfolio selection problem which has data uncertainty described by the (p, w)-norm not only to make up for the disadvantages of the uncertain parameters of all possible values that will give the same weight, but also to consider the robust cost of the robust optimization model which is mentioned in [14]. We see that the robust formulation of this problem is a linear optimization problem. Moreover, we present some numerical results concerning our robust portfolio selection problem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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