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The monotonicity and convexity of a function involving psi function with applications

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Abstract
In this paper, we prove that the function

$$x \mapsto \exp\left(\psi\left(x+\frac{1}{2}\right)-\frac{1}{24}\frac{1}{x^2+7/40}\right)-x$$

is decreasing from $(-1/2,\infty)$ onto $(0,1/2)$ and convex on $(-1/2,\infty)$. As a consequence of the main theorem, various type of bounds for the psi function are presented, which essentially generalize or improve some known results.

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1 Introduction

For $x > 0$, the classical Euler gamma function Γ and psi (digamma) function ψ are defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.1}$$

respectively. Furthermore, the derivatives $\psi', \psi'', \psi''', \ldots$ are known as polygamma functions.

As an important role played in many branches, such as mathematical physics, probability, statistics, and engineering, the gamma and polygamma functions have attracted the attention of many scholars. Recently, many authors showed numerous interesting inequalities for the digamma (psi) function ψ and the Euler constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right) = 0.577215664\cdots,$$

where $\sum_{k=1}^{n}\frac{1}{k} := H_n$ is called the harmonic number. In particular, there has many approximation formulas for psi function and harmonic number, which can be found in [1–18], and closely related references therein.

We would like to mention DeTemple and Wang's paper [19] for half-integer approximation formulas

$$\gamma + \ln\left(n + \frac{1}{2}\right) + \frac{1}{24(n+1)^2} < H_n < \gamma + \ln\left(n + \frac{1}{2}\right) + \frac{1}{24n^2} \tag{1.2}$$

and

$$\gamma + \ln\left(n + \frac{1}{2}\right) + \frac{1}{24n^2} + \frac{7}{960}\frac{1}{(n+1)^4} < H_n < \gamma + \ln\left(n + \frac{1}{2}\right) + \frac{1}{24n^2} + \frac{7}{960}\frac{1}{n^4}. \tag{1.3}$$

It was also proved in [13, 20], [21], Lemma 1.7, that

$$\ln\left(x + \frac{1}{2}\right) < \psi(x+1) \le \ln\left(x + e^{-\gamma}\right) \tag{1.4}$$

for $x > 0$, where $\frac{1}{2}$ and $e^{-\gamma}$ are the best possible constants, and $\gamma = 0.577215664\cdots$ is the Euler-Mascheroni constant. Thanks to formula (1.4) and the relation $H_n = \gamma + \psi(n+1)$, we have

$$\gamma + \ln\left(n + \frac{1}{2}\right) < H_n \le \gamma + \ln\left(n + e^{1-\gamma} - 1\right) \tag{1.5}$$

for any $n \in \mathbb{N}$. In 2011, Batir [22] further proved that

$$\frac{1}{2}\ln\left(x^2 + x + e^{-2\gamma}\right) \le \psi(x+1) < \frac{1}{2}\ln\left(x^2 + x + \frac{1}{3}\right) \tag{1.6}$$

for all $x > 0$, where $e^{-2\gamma}$ and $1/3$ are the best possible. As a direct consequence, he showed that, for $n \in \mathbb{N}$,

$$\gamma + \frac{1}{2}\ln\left(n^2 + n + e^{2-2\gamma} - 2\right) \le H_n < \gamma + \frac{1}{2}\ln\left(n^2 + n + \frac{1}{3}\right). \tag{1.7}$$

Batir [23, 24] provided another interesting bound for the psi function:

$$a - \ln\left(e^{1/x} - 1\right) < \psi(x) < b - \ln\left(e^{1/x} - 1\right), \tag{1.8}$$

where $x > 0$, $a \le \ln 2$, and $b \ge 0$. Consequently, the double inequality

$$\ln\frac{\pi^2}{6} - \ln\left(e^{1/(n+1)} - 1\right) < H_n < \gamma - \ln\left(e^{1/(n+1)} - 1\right) \tag{1.9}$$

for all $n \in \mathbb{N}$ was attained in Corollary 2.2 in [23]. Later, this inequality was sharpened to

$$1 + \ln(\sqrt{e} - 1) - \ln\left(e^{1/(n+1)} - 1\right) < H_n < \gamma - \ln\left(e^{1/(n+1)} - 1\right) \tag{1.10}$$

for all $n \in \mathbb{N}$ by Alzer [25].

For reader's convenience, here we name this class of bounds for the psi function and harmonic numbers the *Batir-type bounds* and call the corresponding inequality a *Batir-type inequality.*

Also, inequalities (1.10) are equivalent to

$$a \le e^{H_{n+1}} - e^{H_n} < b \tag{1.11}$$

with the best constants $a = e(\sqrt{e} - 1) \approx 1.7634$ and $b = e^{\gamma} \approx 1.7810$ (see also [26]). For a more general result, see [3], Theorem 1.3. Similarly, in the context, we call inequalities (1.11) the *Alzer-type ones*.

Batir [22] further proved some new Batir-type inequalities for the psi functions and harmonic number, in particular,

$$\frac{1}{2}\ln\frac{2x+b}{e^{2/(x+1)}-1} \le \psi(x+1) \le \frac{1}{2}\ln\frac{2x+b}{e^{2/(x+1)}-1} \quad \text{for } x \ge 0$$

with the best constants $a = 2$ and $b = e^{-2\gamma}(e^2 - 1)$. This implies that, for $n \in \mathbb{N}$, we have

$$\frac{1}{2}\ln\frac{2n+b}{e^{2/(n+1)}-1} \le H_n \le \frac{1}{2}\ln\frac{2n+b}{e^{2/(n+1)}-1},$$

where $a = 2$ and $b = e^{2-2\gamma}(e-1) - 2 \approx 2.0024$ are the best possible. Obviously, it is equivalent to the double inequalities

$$e^{2\gamma}(2n+2) < e^{2H_{n+1}} - e^{2H_n} \le e^{2\gamma}(2n + 2.0024\cdots).$$

Clearly, it is an Alzer-type inequality.

On the other hand, Villarino [27], Theorem 1.7, proved that the sequence

$$d_n = \frac{1}{\psi(n+1) - \ln(n+1/2)} - \frac{1}{24}\left(n + \frac{1}{2}\right)^2$$

is increasing for $n \in \mathbb{N}$, and meanwhile DeTemple and Wang [19] by an approximation argument for the harmonic number showed, for $n \in \mathbb{N}$, the following inequality:

$$\frac{1}{24(n+1/2)^2 + 21/5} < H_n - \ln\left(n + \frac{1}{2}\right) - \gamma < \frac{1}{24(n+1/2)^2 + 1/(1 - \ln 3 + \ln 2 - \gamma) - 54}$$

with the best constants $\frac{21}{5}$ and $1/(1 - \ln 3 + \ln 2 + \psi(1)) - 54 \approx 3.7393$. Yang *et al.* [28], Theorem 2, further showed that the function

$$x \mapsto F_a(x) = 24\left(x^2 + a\right)\left[\psi(x + 1/2) - \ln x\right] - 1$$

is strictly completely monotonic on $(0, \infty)$ if and only if $a \ge 7/40$.

Motivated by all these recent papers, the aim of this paper is to investigate the monotonicity and convexity of the function related to the psi function and present some new general, Batir-type, and Alzer-type inequalities for the psi function and harmonic number.

2 Preliminaries

In this section, let us recall a few of involving lemmas and some basic facts.

Lemma 1 ([29], pp.258-260) *Let $x > 0$ and $n \in \mathbb{N}$. Then*

$$\psi^{(n)}(x+1) - \psi^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}. \tag{2.1}$$

Lemma 2 ([29], pp.258-260) *As $x \to \infty$, we have*

$$\psi(x) \sim \ln x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6}, \tag{2.2}$$

$$\psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}, \tag{2.3}$$

$$\psi''(x) \sim -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + \frac{1}{6x^6} - \frac{1}{6x^8}. \tag{2.4}$$

Lemma 3 ([30]) *Let f be a function on an interval I such that $\lim_{x\to\infty} f(x) = 0$. If $f(x+1) - f(x) > 0$ for all $x \in (a,\infty)$, then $f(x) < 0$. Conversely, if $f(x+1) - f(x) < 0$ for all $x \in (a,\infty)$, then $f(x) > 0$.*

Lemma 4 ([31], Lemma 7) *For $n \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ with $n > m$, let $P_n(t)$ be a polynomial with n degrees defined by*

$$P_n(t) = \sum_{i=m+1}^{n} a_i t^i - \sum_{i=0}^{m} a_i t^i, \tag{2.5}$$

where $a_n, a_m > 0$ and $a_i \geq 0$ for $0 \leq i \leq n-1$ with $i \neq m$. Then there exists an unique number $t_{m+1} \in (0,\infty)$ satisfying $P_n(t) = 0$ such that $P_n(t) < 0$ for $t \in (0, t_{m+1})$ and $P_n(t) > 0$ for $t \in (t_{m+1}, \infty)$.

Lemma 5 *Let u be the function on $(-\infty,\infty)$ defined by*

$$u(x) = \frac{5}{120x^2 + 21}. \tag{2.6}$$

Then, for $x \neq -1/2$, we have

$$p(x) = -\frac{1}{(x+1/2)^2} - u'(x+1) + u'(x) < 0. \tag{2.7}$$

Proof Differentiation leads to

$$u'(x) = -\frac{400}{3}\frac{x}{(40x^2+7)^2}. \tag{2.8}$$

Factoring gives

$$\begin{aligned} p(x) &= -\frac{1}{(x+1/2)^2} + \frac{400}{3}\frac{x+1}{(40(x+1)^2+7)^2} - \frac{400}{3}\frac{x}{(40x^2+7)^2} \\ &= -\frac{1}{(x+1/2)^2} - \frac{400}{3}\frac{4{,}800x^4 + 9{,}600x^3 + 6{,}960x^2 + 2{,}160x - 49}{(40(x+1)^2+7)^2(40x^2+7)^2}. \end{aligned}$$

Replacing x by $(t-1/2)$ and factoring, we get

$$p(x) = -\frac{7{,}680{,}000t^8 - 384{,}000t^6 + 2{,}851{,}200t^4 - 531{,}760t^2 + 250{,}563}{3t^2(40(t-1/2)^2+7)^2(40(t+1/2)^2+7)^2},$$

where $t = x + 1/2$.

Note that the numerator of this fraction can be written as

$$\begin{aligned} p_1(t) &= 7{,}680{,}000t^8 - 384{,}000t^6 + 2{,}851{,}200t^4 - 531{,}760t^2 + 250{,}563 \\ &= 4{,}800t^4\left(40t^2-1\right)^2 + \frac{7{,}692{,}800}{3}t^4 + \frac{1}{3}\left(920t^2-867\right)^2 > 0, \end{aligned} \tag{2.9}$$

and the desired result easily follows. □

Lemma 6 *Let u and p be defined by* (2.6) *and* (2.7). *Suppose that q and r are defined on* $(-1/2,\infty)$ *by*

$$\begin{aligned} q(x) &= \frac{2}{(x+1/2)^3} - u''(x+1) + u''(x), \\ r(x) &= -\frac{1}{(x+1/2)^2} - u'(x+1) - u'(x). \end{aligned}$$

Then, for $x > -1/2$, we have

$$S(x) := -\frac{2}{(x+1/2)^2} + r(x+1) + \frac{q(x+1)}{p(x+1)} - r(x) - \frac{q(x)}{p(x)} < 0. \tag{2.10}$$

Proof An immediate computation yields

$$u''(x) = \frac{400}{3}\frac{120x^2-7}{(40x^2+7)^3}. \tag{2.11}$$

Then we get

$$q(x) = \frac{2}{(x+1/2)^3} - \frac{400}{3}\frac{120(x+1)^2-7}{(40(x+1)^2+7)^3} + \frac{400}{3}\frac{120x^2-7}{(40x^2+7)^3}, \tag{2.12}$$

$$r(x) = -\frac{1}{(x+1/2)^2} + \frac{400}{3}\frac{x+1}{(40(x+1)^2+7)^2} + \frac{400}{3}\frac{x}{(40x^2+7)^2}. \tag{2.13}$$

Substituting $p(x)$, $q(x)$, $r(x)$ into $S(x)$ and factoring it give

$$S(x) = -\frac{16\times 10^{11}}{3}\frac{S_2(x)}{S_1(x)},$$

where

$$\begin{aligned} S_1(x) &= (2x+1)^2(2x+3)^2\left(40x^2+7\right)^2\left(40x^2+160x+167\right)^2 \\ &\quad \times \left(7{,}680{,}000x^8 + 92{,}160{,}000x^7 + 483{,}456{,}000x^6 + 1{,}448{,}064{,}000x^5\right. \\ &\quad + 2{,}711{,}491{,}200x^4 + 3{,}257{,}107{,}200x^3 + 2{,}458{,}239{,}440x^2 \end{aligned}$$

$$
\begin{aligned}
&+ 1{,}069{,}159{,}920x + 205{,}944{,}303\big) \\
&\times \big(7{,}680{,}000x^8 + 30{,}720{,}000x^7 + 53{,}376{,}000x^6 + 52{,}608{,}000x^5 \\
&+ 35{,}011{,}200x^4 + 18{,}182{,}400x^3 + 6{,}745{,}040x^2 \\
&+ 1{,}301{,}840x + 319{,}823\big),
\end{aligned} \tag{2.14}
$$

$$
\begin{aligned}
S_2(x) = {}& 19{,}544{,}408{,}064x^{20} + 390{,}888{,}161{,}280x^{19} + \frac{18{,}277{,}115{,}756{,}544}{5}x^{18} \\
&+ \frac{106{,}181{,}831{,}688{,}192}{5}x^{17} + \frac{2{,}147{,}345{,}768{,}669{,}184}{25}x^{16} \\
&+ \frac{6{,}422{,}868{,}775{,}010{,}304}{25}x^{15} + \frac{368{,}287{,}175{,}087{,}671{,}296}{625}x^{14} \\
&+ \frac{662{,}967{,}302{,}010{,}630{,}144}{625}x^{13} + \frac{4{,}756{,}138{,}453{,}310{,}742{,}528}{3{,}125}x^{12} \\
&+ \frac{5{,}496{,}272{,}101{,}145{,}296{,}896}{3{,}125}x^{11} + \frac{25{,}764{,}390{,}625{,}415{,}987{,}616}{15{,}625}x^{10} \\
&+ \frac{3{,}939{,}903{,}200{,}496{,}190{,}272}{3{,}125}x^{9} + \frac{12{,}333{,}289{,}847{,}706{,}921{,}772}{15{,}625}x^{8} \\
&+ \frac{6{,}339{,}553{,}647{,}515{,}390{,}816}{15{,}625}x^{7} + \frac{26{,}760{,}964{,}338{,}980{,}254{,}467}{156{,}250}x^{6} \\
&+ \frac{4{,}616{,}788{,}558{,}072{,}176{,}841}{78{,}125}x^{5} + \frac{513{,}733{,}037{,}814{,}725{,}250{,}509}{31{,}250{,}000}x^{4} \\
&+ \frac{27{,}954{,}545{,}230{,}825{,}852{,}509}{7{,}812{,}500}x^{3} + \frac{44{,}860{,}757{,}315{,}321{,}422{,}071}{78{,}125{,}000}x^{2} \\
&+ \frac{2{,}404{,}936{,}823{,}928{,}444{,}981}{39{,}062{,}500}x + \frac{389{,}355{,}305{,}888{,}516{,}211{,}027}{100{,}000{,}000{,}000}.
\end{aligned}
$$

We further prove that $S_1(x), S_2(x) > 0$ for $x > -1/2$. In fact, replacing x by $(t-1/2)$ in (2.14) and arranging, we obtain

$$
\begin{aligned}
S_1(x) = {}& 16t^2(t+1)^2\big(40t^2 - 40t + 17\big)^2\big(120t + 40t^2 + 97\big)^2 \\
&\times \big(7{,}680{,}000t^8 + 61{,}440{,}000t^7 + 214{,}656{,}000t^6 + 427{,}776{,}000t^5 \\
&+ 534{,}691{,}200t^4 + 433{,}804{,}800t^3 + 225{,}855{,}440t^2 + 69{,}477{,}280t \\
&+ 9{,}866{,}003\big) \times p_1(t),
\end{aligned}
$$

where $p_1(t)$ is defined by (2.9), and $t = x + 1/2 > 0$. Clearly, $S_1(x) > 0$ for $x > -1/2$.

Similarly, we have

$$S_2(x) = S_3(t) + t^2 S_4(t),$$

where

$$
\begin{aligned}
S_3(t) = {}& 19{,}544{,}408{,}064t^{20} + 195{,}444{,}080{,}640t^{19} + \frac{4{,}351{,}725{,}010{,}944}{5}t^{18} \\
&+ \frac{11{,}314{,}743{,}607{,}296}{5}t^{17} + \frac{94{,}415{,}373{,}656{,}064}{25}t^{16} + \frac{104{,}269{,}760{,}888{,}832}{25}t^{15}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1{,}887{,}440{,}973{,}367{,}296}{625}t^{14}+\frac{888{,}059{,}638{,}659{,}072}{625}t^{13}\\
&+\frac{1{,}921{,}294{,}271{,}712{,}768}{3{,}125}t^{12}+\frac{1{,}795{,}217{,}217{,}788{,}928}{3{,}125}t^{11}\\
&+\frac{8{,}617{,}726{,}188{,}296{,}736}{15{,}625}t^{10}+\frac{928{,}558{,}757{,}330{,}976}{3{,}125}t^{9}\\
&+\frac{1{,}217{,}964{,}858{,}530{,}932}{15{,}625}t^{8}+\frac{353{,}703{,}953{,}859{,}088}{15{,}625}t^{7}\\
&+\frac{3{,}731{,}661{,}240{,}517{,}347}{156{,}250}t^{6}+\frac{151{,}574{,}123{,}955{,}363{,}957}{156{,}250{,}000}t\\
&+\frac{20{,}166{,}045{,}810{,}484{,}915{,}467}{100{,}000{,}000{,}000},
\end{aligned}
$$

$$
\begin{aligned}
S_4(t) ={}& \frac{1{,}496{,}705{,}911{,}425{,}721}{156{,}250}t^{3}-\frac{78{,}452{,}580{,}264{,}405{,}241}{31{,}250{,}000}t^{2}\\
&-\frac{25{,}060{,}239{,}724662{,}741}{15{,}625{,}000}t+\frac{11{,}608{,}440{,}209{,}633{,}547}{9{,}765{,}625}.
\end{aligned}
$$

It is clear that $S_3(t) > 0$ for $t = x + 1/2 > 0$. To prove that $S_2(x) > 0$ for $x > -1/2$, it suffices to prove that $S_4(t) > 0$ for $t > 0$. In fact, it is easy to check that

$$
\begin{aligned}
S_4'(t) ={}& \frac{4{,}490{,}117{,}734{,}277{,}163}{156{,}250}t^{2}-\frac{78{,}452{,}580{,}264{,}405{,}241}{15{,}625{,}000}t\\
&-\frac{25{,}060{,}239{,}724{,}662{,}741}{15{,}625{,}000}
\end{aligned}
$$

has a positive zero point $t_0 \in (1/3, 1/2)$ such that $S_4'(t) < 0$ for $t \in (0, t_0)$ and $S_4'(t) < 0$ for $t \in (t_0, \infty)$. Since $S_4(0), S_4(\infty) > 0$ and

$$
\begin{aligned}
S_4(t_0) ={}& \frac{1{,}496{,}705{,}911{,}425{,}721}{156{,}250}t_0^{3}+\frac{11{,}608{,}440{,}209{,}633{,}547}{9{,}765{,}625}\\
&-\left(\frac{78{,}452{,}580{,}264{,}405{,}241}{31{,}250{,}000}t_0^{2}+\frac{25{,}060{,}239{,}724{,}662{,}741}{15{,}625{,}000}t_0\right)\\
>{}& \frac{1{,}496{,}705{,}911{,}425{,}721}{156{,}250}\left(\frac{1}{3}\right)^{3}+\frac{11{,}608{,}440{,}209{,}633{,}547}{9{,}765{,}625}\\
&-\left(\frac{78{,}452{,}580{,}264{,}405{,}241}{31{,}250{,}000}\left(\frac{1}{2}\right)^{2}+\frac{25{,}060{,}239{,}724{,}662{,}741}{15{,}625{,}000}\frac{1}{2}\right)\\
={}& \frac{1{,}922{,}580{,}540{,}937{,}065{,}541}{16{,}875{,}000{,}000} > 0,
\end{aligned}
$$

so we get $S_4(t) > 0$ for $t > 0$, which proves that $S_2(x) > 0$ for $x > -1/2$ and completes the proof. □

3 Monotonicity and convexity

In this section, we state and prove Theorems 1-3 on the monotonicity and convexity of three important functions $f_1(x)$, $f_2(x)$, and $f_3(x)$ concerning the psi function, respectively.

Theorem 1 *The function*

$$x \mapsto f_1(x) = \exp\left(\psi\left(x+\frac{1}{2}\right) - \frac{1}{24}\frac{1}{x^2+7/40}\right) - x$$

is decreasing from $(-1/2,\infty)$ *onto* $(0,1/2)$ *and convex on* $(-1/2,\infty)$.

Proof With (2.6) in hand, $f_1(x)$ can be written as

$$f_1(x) = e^{\psi(x+1/2)-u(x)} - x.$$

Differentiation of this formula yields

$$f_1'(x) = \left(\psi'(x+1/2) - u'(x)\right)e^{\psi(x+1/2)-u(x)} - 1, \tag{3.1}$$

$$\begin{aligned} f_1''(x) &= \frac{\psi''(x+1/2) - u''(x) + (\psi'(x+1/2) - u'(x))^2}{e^{-\psi(x+1/2)+u(x)}} \\ &:= \frac{g(x)}{e^{-\psi(x+1/2)+u(x)}}. \end{aligned} \tag{3.2}$$

By (2.1) this yields

$$\begin{aligned} &g(x+1) - g(x) \\ &\quad = \left[\psi(x+3/2,2) - u''(x+1)\right] + \left[\psi(x+3/2,1) - u'(x+1)\right]^2 \\ &\qquad - \left\{\psi(x+1/2,2) - u''(x) + \left[\psi(x+1/2,1) - u'(x)\right]^2\right\} \\ &\quad = p(x)\left[2\psi(x+1/2,1) + r(x) + \frac{q(x)}{p(x)}\right] \\ &\quad := p(x)h(x), \end{aligned} \tag{3.3}$$

where $p(x)$, $q(x)$, $r(x)$ are defined by (2.7), (2.12), (2.13), respectively.

Similarly, we get

$$h(x+1) - h(x) = -\frac{2}{(x+1/2)^2} + r(x+1) + \frac{q(x+1)}{p(x+1)} - r(x) - \frac{q(x)}{p(x)} = S(x).$$

By Lemma 6 we have $h(x+1) - h(x) < 0$. It follows from $\lim_{x\to\infty} h(x) = 0$ and Lemma 3 that $h(x) > \lim_{x\to\infty} h(x) = 0$. Thanks to inequality (3.3), $p(x) < 0$, and Lemma 5, it follows that

$$g(x+1) - g(x) = p(x)h(x) < 0,$$

which implies by Lemma 3 that $g(x) > \lim_{x\to\infty} g(x) = 0$. Thus, in combination with (3.2), this leads to $f_1''(x) > 0$, that is, f_1 is convex on $(-1/2,\infty)$, and f_1' is increasing on $(-1/2,\infty)$. Utilizing the asymptotic formulas (2.2)-(2.3), this yields

$$\lim_{x\to\infty} f_1'(x) = \lim_{x\to\infty} f_1(x) = 0.$$

Therefore, we get that $f_1'(x) < \lim_{x\to\infty} f_1'(x) = 0$, which implies that $f_1(x)$ is decreasing on $(-1/2, \infty)$. Moreover, we conclude that

$$0 = \lim_{x\to\infty} f_1(x) < f_1(x) < \lim_{x\to -1/2^+} f_1(x) = \frac{1}{2},$$

which completes the proof. □

Theorem 2 *The function*

$$x \mapsto f_2(x) = \exp\psi\left(x + \frac{1}{2}\right) - x\exp\left(\frac{1}{24}\frac{1}{x^2 + 7/40}\right)$$

is decreasing from $(0, \infty)$ *onto* $(0, e^{-\gamma}/4)$ *and convex on* $(1/2, \infty)$.

Proof We have

$$\begin{aligned} f_2(x) &= \exp\left(\frac{1}{24}\frac{1}{x^2 + 7/40}\right)\left\{\exp\left[\psi\left(x + \frac{1}{2}\right) - \frac{1}{24}\frac{1}{x^2 + 7/40}\right] - x\right\} \\ &= \exp\left(\frac{1}{24}\frac{1}{x^2 + 7/40}\right) f_1(x) := f_0(x) f_1(x). \end{aligned}$$

Noting that

$$\begin{aligned} f_0'(x) &= -\frac{1}{12}\frac{x}{(x^2 + 7/40)^2}\exp\left(\frac{1}{24}\frac{1}{x^2 + 7/40}\right) \le 0 \quad \text{for } x \ge 0, \\ f_0''(x) &= \frac{1}{57{,}600}\frac{14{,}400x^4 + 2{,}080x^2 - 147}{(x^2 + 7/40)^4}\exp\left(\frac{1}{24}\frac{1}{x^2 + 7/40}\right) > 0 \quad \text{for } x \ge \frac{1}{2}, \end{aligned}$$

we have

$$\begin{aligned} f_2'(x) &= f_0'(x) f_1(x) + f_0(x) f_1'(x) < 0 \quad \text{for } x \ge 0, \\ f_2''(x) &= f_0''(x) f_1(x) + 2 f_0'(x) f_1'(x) + f_0(x) f_1''(x) > 0 \quad \text{for } x \ge \frac{1}{2}. \end{aligned}$$

A simple calculation leads to

$$\lim_{x\to\infty} f_2(x) = \lim_{x\to\infty}\exp\left(\frac{1}{24}\frac{1}{x^2 + 7/40}\right)\lim_{x\to\infty} f_1(x) = 0 \quad \text{and} \quad \lim_{x\to 0+} f_2(x) = \frac{1}{4}e^{-\gamma},$$

which completes the proof. □

Theorem 3 *Let* $a \ge 0$. *Then the function*

$$x \mapsto f_3(x) = \psi\left(x + \frac{1}{2}\right) - \ln x - \frac{1}{24}\frac{1}{x^2 + a}$$

is decreasing and convex on $(0, \infty)$ *if and only if* $a \ge 7/40$.

Proof The necessity is obvious; it follows from the inequality $\lim_{x\to\infty} x^5 f_3'(x) \le 0$. Indeed, using the asymptotic formulas (2.2), we have

$$\lim_{x\to\infty} x^5 f_3'(x) = \frac{7}{240} - \frac{1}{6}a \le 0,$$

which yields $a \ge 7/40$.

We now are a position to prove the sufficiency. By differentiation we have

$$f_3'(x) = \psi'\left(x+\frac{1}{2}\right) - \frac{1}{x} + \frac{1}{12}\frac{x}{(x^2+a)^2},$$
$$f_3''(x) = \psi''\left(x+\frac{1}{2}\right) + \frac{1}{x^2} + \frac{1}{12}\frac{1}{(x^2+a)^2} - \frac{1}{3}\frac{x^2}{(x^2+a)^3}.$$

Using (2.1), we have

$$\begin{aligned} f_3''(x+1) - f_3''(x) &= \psi''\left(x+\frac{3}{2}\right) + \frac{1}{(x+1)^2} + \frac{1}{12}\frac{1}{((x+1)^2+a)^2} - \frac{1}{3}\frac{x^2}{((x+1)^2+a)^3} \\ &\quad - \psi''\left(x+\frac{1}{2}\right) - \frac{1}{x^2} - \frac{1}{12}\frac{1}{(x^2+a)^2} + \frac{1}{3}\frac{x^2}{(x^2+a)^3} \\ &= \frac{2}{(x+1/2)^3} + \frac{1}{(x+1)^2} + \frac{1}{12}\frac{1}{((x+1)^2+a)^2} \\ &\quad - \frac{1}{3}\frac{x^2}{((x+1)^2+a)^3} - \frac{1}{x^2} - \frac{1}{12}\frac{1}{(x^2+a)^2} + \frac{1}{3}\frac{x^2}{(x^2+a)^3} \\ &= -\frac{P(x)}{12x^2(2x+1)^3(x^2+a)^3(x+1)^2(x^2+2x+a+1)^3}, \end{aligned}$$

where

$$\begin{aligned} P(x) &= 12(40a-7)x^{12} + 72(40a-7)x^{11} + (1{,}536a^2 + 7{,}576a - 1{,}290)x^{10} \\ &\quad + (7{,}680a^2 + 11{,}480a - 1{,}830)x^9 + (2{,}016a^3 + 16{,}788a^2 + 11{,}038a - 1{,}563)x^8 \\ &\quad + (8{,}064a^3 + 21{,}072a^2 + 6{,}952a - 816)x^7 \\ &\quad + (1{,}440a^4 + 14{,}112a^3 + 16{,}710a^2 + 2{,}840a - 252)x^6 \\ &\quad + (4{,}320a^4 + 14{,}112a^3 + 8{,}634a^2 + 724a - 42)x^5 \\ &\quad + (576a^5 + 5{,}652a^4 + 8{,}862a^3 + 2{,}853a^2 + 115a - 3)x^4 \\ &\quad + (1{,}152a^5 + 4{,}104a^4 + 3{,}612a^3 + 540a^2 + 14a)x^3 \\ &\quad + (96a^6 + 936a^5 + 1{,}764a^4 + 966a^3 + 39a^2 + a)x^2 \\ &\quad + (96a^6 + 360a^5 + 432a^4 + 168a^3)x + (12a^6 + 36a^5 + 36a^4 + 12a^3). \end{aligned}$$

It is easy to check that the coefficients of the polynomial $P(x)$ are nonnegative for $a \ge 7/40$. Then we have $f_3''(x+1) - f_3''(x) < 0$ for $x > 0$. By Lemma 3 we get that $f_3''(x) > \lim_{x\to\infty} f_3''(x) = 0$, which implies that $f_3'(x)$ is increasing on $(0,\infty)$. Therefore, $f_3'(x) < \lim_{x\to\infty} f_3'(x) = 0$, which completes our proof. □

4 Inequalities for the psi function and harmonic number

Denote $F_i(x) = f_i(x + 1/2)$ $(i = 1, 2, 3)$ in Theorems 1 and 2. Then, since F_1 is decreasing, $F_1(0) = e^{-\gamma-5/51}$, and $\lim_{x\to\infty} F(x) = 1/2$, we get the following conclusion.

Corollary 1

(i) *For* $x > -1/2$, *we have*

$$\psi(x+1) > \ln\left(x+\frac{1}{2}\right) + \frac{1}{24}\frac{1}{x^2+x+17/40}.$$

(ii) *For* $x \geq 0$, *we have*

$$\ln\left(x+\frac{1}{2}\right) + \frac{1}{24}\frac{1}{x^2+x+17/40} < \psi(x+1) < \ln(x+\alpha_0) + \frac{1}{24}\frac{1}{x^2+x+17/40}, \tag{4.1}$$

where the constants $1/2$ *and* $\alpha_0 = e^{-\gamma-5/51} \approx 0.50903$ *are the best possible.*

Remark 1 Comparing (4.1) with (1.4), we find that, for $x > 0$,

$$\begin{aligned}\ln\left(x+\frac{1}{2}\right) &< \ln\left(x+\frac{1}{2}\right) + \frac{1}{24}\frac{1}{x^2+x+17/40} < \psi(x+1)\\ &< \ln(x+\alpha_0) + \frac{1}{24}\frac{1}{x^2+x+17/40} < \ln\left(x+e^{-\gamma}\right).\end{aligned}$$

In fact, it suffices to show that the last inequality is valid for $x > 0$. Let

$$D_1(x) = \ln(x+\alpha_0) + \frac{1}{24}\frac{1}{x^2+x+17/40} < \ln\left(x+e^{-\gamma}\right).$$

By differentiation we have

$$\begin{aligned}D_1'(x) &= \frac{1}{x+\alpha_0} - \frac{1}{24}\frac{2x+1}{(x^2+x+17/40)^2} - \frac{1}{x+\beta}\\ &= \frac{D_2(x)}{3(x+\alpha_0)(x+\beta)(40x^2+40x+17)^2},\end{aligned}$$

where

$$\begin{aligned}D_2(x) &= 4{,}800(e^{-\gamma}-\alpha_0)x^4 + 400(24e^{-\gamma}-24\alpha_0-1)x^3 - 40(232\alpha_0-212e^{-\gamma}+5)x^2\\ &\quad - 40(107\alpha_0-97e^{-\gamma}+10\alpha_0e^{-\gamma})x - (867\alpha_0-867e^{-\gamma}+200\alpha_0e^{-\gamma})\\ &:= a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.\end{aligned}$$

It is easy to check that $a_4, a_3 > 0$ and $a_2, a_1, a_0 < 0$. Then by Lemma 4 we see that there is $x_0 > 0$ such that $D_2(x) < 0$ for $x \in (0, x_0)$ and $D_2(x) > 0$ for $x \in (x_0, \infty)$. This indicates that D_1 is decreasing on $(0, x_0)$ and increasing on (x_0, ∞). Therefore, we conclude that, for $x > 0$,

$$D_1(x) < \max\left(D_1(0), D_1(\infty)\right) = 0.$$

Remark 2 Similarly, we get the following inequalities:

$$\frac{1}{2}\ln\bigl(x^2+x+e^{-2\gamma}\bigr) < \ln\left(x+\frac{1}{2}\right)+\frac{1}{24}\frac{1}{x^2+x+17/40} < \psi(x+1)$$
$$< \frac{1}{2}\ln\left(x^2+x+\frac{1}{3}\right) < \ln(x+\alpha_0)+\frac{1}{24}\frac{1}{x^2+x+17/40}$$

for $x > 0$. A direct computation shows that

$$\lim_{x\to\infty}\frac{\psi(x+1)-\frac{1}{2}\ln(x^2+x+\frac{1}{3})}{x^{-4}} = -\frac{1}{180}, \tag{4.2}$$
$$\lim_{x\to\infty}\frac{\psi(x+1)-\ln(x+\frac{1}{2})-\frac{1}{24}\frac{1}{x^2+x+17/40}}{x^{-6}} = \frac{2{,}071}{806{,}400}, \tag{4.3}$$

which implies that the approximation formula of the psi function given in (4.1) is superior to (1.6).

Since $F_1(1) = e^{286/291-\gamma}$, by the relation $\psi(n+1) = H_n - \gamma$ we deduce the following:

Corollary 2 *For $n \in \mathbb{N}$, we have*

$$\gamma+\frac{1}{24}\frac{1}{n^2+n+17/40}+\ln(n+1/2) < H_n < \gamma+\frac{1}{24}\frac{1}{n^2+n+17/40}+\ln(n+\alpha_1),$$

where $1/2$ *and* $\alpha_1 = e^{286/291-\gamma} \approx 0.50021$ *are the best possible.*

Since F_2 is decreasing on $(-1/2,\infty)$, $\lim_{x\to\infty} F_2(x) = 0$, and

$$F_2(0) = e^{-\gamma}-\frac{1}{2}e^{5/51} = \frac{1}{2}\beta_0, \qquad F_2(1) = e^{1-\gamma}-\frac{3}{2}e^{5/291} = \frac{1}{2}\beta_1,$$

we get the following:

Corollary 3 *For $x \geq 0$, we have*

$$\ln\left(x+\frac{1}{2}\right)+\frac{1}{24}\frac{1}{x^2+x+17/40} < \psi(x+1) < \ln\left(x+\frac{1}{2}\right)+\frac{1}{24}\frac{1}{x^2+x+17/40}$$
$$+\ln\left[1+\frac{\beta_0}{2x+1}\exp\left(-\frac{1}{24}\frac{1}{x^2+x+17/40}\right)\right],$$

where $\beta_0 = 2e^{-\gamma}-e^{5/51} \approx 0.019913$ *is the best constant.*

Corollary 4 *For $n \in \mathbb{N}$, we have*

$$\gamma+\ln\left(n+\frac{1}{2}\right)+\frac{1}{24}\frac{1}{n^2+n+17/40} < H_n < \gamma+\ln\left(n+\frac{1}{2}\right)+\frac{1}{24}\frac{1}{n^2+n+17/40}$$
$$+\ln\left[1+\frac{\beta_1}{2n+1}\exp\left(-\frac{1}{24}\frac{1}{n^2+n+17/40}\right)\right],$$

where $\beta_1 = 2e^{1-\gamma}-3e^{5/291} \approx 0.00041845$ *is the best constant.*

For $a = 7/40$, since F_3 is decreasing on $(-1/2, \infty)$, $\lim_{x\to\infty} F_3(x) = 0$, and

$$F_3(0) = \ln 2 - \frac{5}{51} - \gamma = \delta_0, \qquad F_3(1) = \frac{286}{291} - \ln\frac{3}{2} - \gamma = \delta_1,$$

we deduce the following:

Corollary 5 *For $x \geq 0$, we have*

$$\ln\left(x + \frac{1}{2}\right) + \frac{1}{24}\frac{1}{x^2 + x + 17/40} + \delta_0^* < \psi(x+1) < \ln\left(x + \frac{1}{2}\right) + \frac{1}{24}\frac{1}{x^2 + x + 17/40} + \delta_0,$$

where $\delta_0^* = 0$ and $\delta_0 = \ln 2 - 5/51 - \gamma \approx 0.017892$ are the best constants.

Corollary 6 *For $n \in \mathbb{N}$, we have*

$$\gamma + \ln\left(n + \frac{1}{2}\right) + \frac{1}{24}\frac{1}{n^2 + n + 17/40} + \delta_1^* < H_n < \gamma + \ln\left(n + \frac{1}{2}\right) + \frac{1}{24}\frac{1}{n^2 + n + 17/40} + \delta_1,$$

where $\delta_1^* = 0$ and $\delta_1 = 286/291 - \ln(3/2) - \gamma \approx 0.00013710$ are the best constants.

From Theorem 1 we can obtain the following Batir-type inequalities for the psi function and harmonic number.

Corollary 7 *For $x \geq 0$, we have*

$$\frac{1}{24(x^2 + x + 17/40)} - \ln\left(e^{Q(x)} - 1\right) + c_0 < \psi(x+1) < \frac{1}{24(x^2 + x + 17/40)} - \ln\left(e^{Q(x)} - 1\right) + c_0^*$$

with the best constants $c_0 = \ln(e^{5,347/4,947} - 1) - 5/51 - \gamma \approx -0.0088601$ and $c_0^* = 0$, where

$$Q(x) = \frac{4,800x^4 + 19,200x^3 + 28,480x^2 + 18560x + 5,347}{3(x+1)(40x^2 + 40x + 17)(40x^2 + 120x + 97)}.$$

Proof Let $G_1(x) = F_1(x+1) - F_1(x) = f_1(x + 3/2) - f_1(x + 1/2)$. Since f_1 is convex on $(-1/2, \infty)$, we have

$$G_1'(x) = f_1'(x + 3/2) - f_1'(x + 1/2) = f_1''(x + 1/2 + \theta) > 0,$$

which means that G_1 is increasing on $(0, \infty)$. Considering

$$\begin{aligned} G_1(x) &= \exp\left(\psi(x+2) - \frac{1}{24}\frac{1}{(x+3/2)^2 + 7/40}\right) \\ &\quad - \exp\left(\psi(x+1) - \frac{1}{24}\frac{1}{(x+1/2)^2 + 7/40}\right) - 1 \\ &= \exp\left(\psi(x+1) - \frac{1}{24}\frac{1}{(x+1/2)^2 + 7/40}\right)\left(e^{Q(x)} - 1\right) - 1, \end{aligned}$$

$$G_1(0) = \exp\left(\frac{286}{291} - \gamma\right) - e\left(-\frac{5}{51} - \gamma\right) - 1 \quad \text{and} \quad \lim_{x\to\infty} G_1(x) = 0,$$

we have

$$G_1(0) < \exp\left(\psi(x+1) - \frac{1}{24}\frac{1}{(x+1/2)^2+7/40}\right)\left(e^{Q(x)}-1\right) - 1 < 0,$$

which attains the desired inequality. □

The increasing property of G_1 and

$$G_1(1) = \exp\left(\frac{2{,}303}{1{,}542} - \gamma\right) - \exp\left(\frac{286}{291} - \gamma\right) - 1$$

yield a sharp bound of harmonic number.

Corollary 8 *For $n \in \mathbb{N}$, we have*

$$\gamma + \frac{1}{24(n^2+n+17/40)} - \ln\left(e^{Q(n)}-1\right) + c_1 < H_n < \gamma + \frac{1}{24(n^2+n+17/40)} - \ln\left(e^{Q(n)}-1\right) + c_1^*,$$

where $c_1 = 286/291 - \gamma + \ln(e^{76{,}387/149{,}574} - 1) \approx -0.00018438$ and $c_1^* = 0$ are the best constants.

Remark 3 Note that since $\psi(n+1) = H_n - \gamma$, $G_1(n)$ is written as

$$G_1(n) = \exp\left(H_{n+1} - \gamma - \frac{1}{24}\frac{1}{(n+3/2)^2+7/40}\right) - \exp\left(H_n - \gamma - \frac{1}{24}\frac{1}{(n+1/2)^2+7/40}\right) - 1.$$

Then by $G_1(1) \le G_1(n) < G_1(\infty)$ we have the following Alzer-type inequalities:

$$1.7807 \approx \exp\left(\frac{2{,}303}{1{,}542}\right) - \exp\left(\frac{286}{291}\right) < e^{H_{n+1}-u_{n+1}} - e^{H_n - u_n} < e^{\gamma} \approx 1.7811,$$

where

$$u_n = \frac{1}{24}\frac{1}{(n+1/2)^2+7/40}. \tag{4.4}$$

Remark 4 Similarly, it is easy to check that

$$G_2(x) = F_2(x+1) - F_2(x) = f_2(x+3/2) - f_2(x+1/2)$$

is increasing on $(0,\infty)$. Then, from

$$-0.00018779 \approx \frac{3}{2}e^{5/291} - \frac{5}{2}e^{5/771} + e^{1-\gamma}\left(e^{1/2}-1\right) = G_2(1) \le G_2(n) < G_2(\infty) = 0$$

we derive other Alzer-type inequalities:

$$\left(n+\frac{3}{2}\right)u_{n+1}-\left(n+\frac{1}{2}\right)u_n+d_1<e^{H_{n+1}}-e^{H_n}<\left(n+\frac{3}{2}\right)u_{n+1}-\left(n+\frac{1}{2}\right)u_n+d_0,$$

where $d_0=0$ and $d_1=3e^{5/291}/2-5e^{5/771}/2+e^{1-\gamma}(e^{1/2}-1)\approx -0.00018779$ are the best constants with u_n as in (4.4).

Using the increasing property of F_1', and noting that $F_1'(-1^+)=-1$, $F_1'(0)=(\pi^2/6+200/867)e^{-\gamma-5/51}-1$, and $F_1(\infty)=0$, we get

$$\left(\frac{1}{6}\pi^2+\frac{200}{867}\right)e^{-\gamma-5/51}-1<\left(\psi'(x+1)-u'(x)\right)e^{\psi(x+1)-u(x)}-1<0,$$

which implies the following:

Corollary 9

(i) *For* $x>-1$, *we have*

$$\psi'(x+1)<-\frac{1}{12}\frac{x+1/2}{(x^2+x+17/40)^2}+\exp\left(-\psi(x+1)+\frac{1}{24(x^2+x+17/40)}\right).$$

(ii) *For* $x\geq 0$, *we have the double inequalities*

$$\begin{aligned}&-\frac{1}{12}\frac{x+1/2}{(x^2+x+17/40)^2}+\lambda_1\exp\left(-\psi(x+1)+\frac{1}{24(x^2+x+17/40)}\right)\\&\quad<\psi'(x+1)<-\frac{1}{12}\frac{x+1/2}{(x^2+x+17/40)^2}\\&\qquad+\lambda_2\exp\left(-\psi(x+1)+\frac{1}{24(x^2+x+17/40)}\right),\end{aligned}$$

where $\lambda_1=(\pi^2/6+200/867)e^{-\gamma-5/51}\approx 0.95474$ *and* $\lambda_2=1$ *are the best constants.*

Remark 5 Elezovic *et al.* [32] proved the inequality

$$\psi'(x)<e^{-\psi(x)} \tag{4.5}$$

for $x>0$. It has been improved by Batir [22] as

$$\left(x+a^*\right)e^{-2\psi(x)}<\psi'(x+1)<\left(x+b^*\right)e^{-2\psi(x)}$$

for $x>0$ with the best constants $a^*=1/2$ and $b^*=\pi^2e^{-2\gamma}/6$. The last corollary gives another improvement of (4.5).

From the proof of Theorem 1 we see that $g(x)>0$ for $x>-1/2$, which can be written as the following corollary.

Corollary 10 *For* $x>0$, *we have*

$$\psi''(x)-u''(x-1/2)+\left(\psi'(x)-u'(x-1/2)\right)^2>0, \tag{4.6}$$

where $u(x)$ *is defined by* (2.6).

Remark 6 Batir [23] showed that, for $x > 0$,

$$\psi'(x)^2 + \psi''(x) > 0. \tag{4.7}$$

Therefore, inequality (4.6) can be written as

$$\psi''(x) + \psi'(x)^2 > \Delta(x),$$

where

$$\Delta(x) = -\frac{400}{3}\frac{2x-1}{(40x^2-40x+17)^2} \times \left(\psi'(x) - \frac{14{,}400x^4 - 28{,}800x^3 + 22{,}880x^2 - 8{,}480x + 1{,}073}{3(2x-1)(40x^2-40x+17)^2}\right).$$

Indeed, this result is optimal due to

$$\begin{aligned} &\psi''(x) - \frac{400}{3}\frac{120(x-1/2)^2-7}{(40(x-1/2)^2+7)^3} + \left(\psi'(x) + \frac{400}{3}\frac{(x-1/2)}{(40(x-1/2)^2+7)^2}\right)^2 \\ &\quad = \psi''(x) + \psi'(x)^2 - \frac{400}{3}\frac{1}{(40x^2-40x+17)^2}\Delta(x), \end{aligned}$$

where

$$\Delta(x) = -(2x-1)\psi'(x) + \frac{14{,}400x^4 - 28{,}800x^3 + 22{,}880x^2 - 8{,}480x + 1{,}073}{3(40x^2-40x+17)^2}.$$

A numeric computation shows that $\Delta(x) > 0$ for $0 < x < 2/5$ and $x > 3/2$, and so inequality (4.6) is better than (4.7).

From the inequalities $f_3'(x) < 0$ and $f_3''(x) > 0$ on $(0, \infty)$ for $a = 7/40$, which are given in the proof of Theorem 3, we have the following:

Corollary 11 *For* $x > 0$, *we have the following inequalities*:

$$\begin{aligned} &\psi'\left(x+\frac{1}{2}\right) < \frac{1}{3}\frac{4{,}800x^4 + 1{,}280x^2 + 147}{x(40x^2+7)^2}, \\ &\psi''\left(x+\frac{1}{2}\right) > -\frac{1}{3}\frac{192{,}000x^6 + 52{,}800x^4 + 20{,}440x^2 + 1{,}029}{x^2(40x^2+7)^3}, \\ &\psi'''\left(x+\frac{1}{2}\right) < 2\frac{2{,}560{,}000x^8 + 512{,}000x^6 + 694{,}400x^4 + 54{,}880x^2 + 2{,}401}{x^3(40x^2+7)^4}. \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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