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Extremum of geometric functionals involving general L_p-projection bodies

Weidong Wang* and Jianye Wang

*Correspondence:
wdwxh722@163.com
Department of Mathematics, China Three Gorges University, Yichang, 443002, P.R. China

Abstract

Following the discovery of general L_p-projection bodies by Ludwig, Haberl and Schuster determined the extremum of the volume of the polars of this family of L_p-projection bodies. In this paper, the result of Haberl and Schuster is extended to all dual quermassintegrals, and a dual counterpart for the quermassintegrals of general L_p-projection bodies is also obtained. Moreover, the extremum of the L_q-dual affine surface areas of polars of general L_p-projection bodies are determined.

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1 Introduction

Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in the n-dimensional Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}_o^n$ and $\mathcal{K}_{os}^n$, respectively. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and denote by $V(K)$ the n-dimensional volume of the body K. For the standard unit ball B in $\mathbb{R}^n$, write $V(B) = \omega_n$.

For $K \in \mathcal{K}^n$, its support function $h_K = h(K, \cdot) : \mathbb{R}^n \longrightarrow (-\infty, +\infty)$ is defined by (see [1])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

The projection body of a convex body was introduced by Minkowski at the turn of the previous century. For $K \in \mathcal{K}^n$, the projection body ΠK of K is the origin-symmetric convex body whose support function is defined by (see [1])

$$h_{\Pi K}(u) = \frac{1}{2} \int_{S^{n-1}} |u \cdot v| \, dS(K, v)$$

for all $u \in S^{n-1}$. Here, $S(K, \cdot)$ denotes the surface area measure of the convex body K. Classical projection bodies are a very important notion in the Brunn-Minkowski theory. During the past four decades, a number of important results regarding classical projection bodies were obtained (see [1–12]).

The notion of an L_p-projection body was introduced by Lutwak, Yang, and Zhang [13]. For $K \in \mathcal{K}_o^n$ and $p \geq 1$, the L_p-projection body $\Pi_p K$ of K is the origin-symmetric convex body whose support function is given by

$$h^p_{\Pi_p K}(u) = \alpha_{n,p} \int_{S^{n-1}} |u \cdot v|^p \, dS_p(K, v) \tag{1.1}$$

for all $u \in S^{n-1}$. Here

$$\alpha_{n,p} = \frac{1}{n\omega_n c_{n-2,p}} \tag{1.2}$$

with $c_{n,p} = \omega_{n+p}/\omega_2\omega_n\omega_{p-1}$, and $S_p(K,\cdot)$ is the L_p-surface area measure of K that has the Radon-Nikodym derivative

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)} = h(K,\cdot)^{1-p}. \tag{1.3}$$

The unusual normalization of definition (1.1) is chosen so that for the unit ball B, we have $\Pi_p B = B$. In particular, for $p = 1$, $\Pi_1 K$ is just the classical projection body ΠK of K under the different normalization of definition (1.1).

L_p-projection bodies belong to the L_p-Brunn-Minkowski theory, which is an extension of the classical Brunn-Minkowski theory. Apart from [13], L_p-projection bodies have been investigated intensively in recent years (see [6, 14–21]).

Through the characterization of so-called L_p-Minkowski valuations, Ludwig [15] discovered (see also [22–29] for related results) an asymmetric L_p-projection body $\Pi_p^+ K$ of $K \in \mathcal{K}_o^n$, whose support function is defined by

$$h^p_{\Pi_p^+ K}(u) = 2\alpha_{n,p} \int_{S^{n-1}} (u \cdot v)_+^p \, dS_p(K, v), \tag{1.4}$$

where $(u \cdot v)_+ = \max\{u \cdot v, 0\}$. From (1.2) and (1.4) we see $\Pi_p^+ B = B$.

Moreover, Ludwig [15] introduced the function $\varphi_\tau : \mathbb{R} \longrightarrow [0, +\infty)$ given by

$$\varphi_\tau(t) = |t| + \tau t$$

for $\tau \in [-1, 1]$. For $K \in \mathcal{K}_o^n$, $p \geq 1$, let $\Pi_p^\tau K \in \mathcal{K}_o^n$ with support function

$$h^p_{\Pi_p^\tau K}(u) = \alpha_{n,p}(\tau) \int_{S^{n-1}} \varphi_\tau(u \cdot v)^p \, dS_p(K, v), \tag{1.5}$$

where

$$\alpha_{n,p}(\tau) = \frac{2\alpha_{n,p}}{(1+\tau)^p + (1-\tau)^p}.$$

The normalization is chosen such that $\Pi_p^\tau B = B$ for every $\tau \in [-1, 1]$. Here $\Pi_p^\tau K$ is called the general L_p-projection body of K. Obviously, if $\tau = 0$, then $\Pi_p^\tau K = \Pi_p^0 K = \Pi_p K$.

Following the discovery of Ludiwg, Haberl and Schuster [30] defined

$$\Pi_p^- K = \Pi_p^+(-K). \tag{1.6}$$

From (1.4), (1.5), and (1.6) they (see [30]) deduced that for $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, and all $u \in S^{n-1}$,

$$h^p_{\Pi_p^\tau K}(u) = f_1(\tau) h^p_{\Pi_p^+ K}(u) + f_2(\tau) h^p_{\Pi_p^- K}(u), \tag{1.7}$$

that is,

$$\Pi_p^\tau K = f_1(\tau) \cdot \Pi_p^+ K +_p f_2(\tau) \cdot \Pi_p^- K, \tag{1.8}$$

where '$+_p$' denotes the L_p-Minkowski addition of convex bodies, and

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.9}$$

If $\tau = \pm 1$, then $\Pi_p^\tau K = \Pi_p^\pm K$.

For general L_p-projection bodies, Haberl and Schuster [30] not only established a general version of the L_p-Petty projection inequality but also determined the following extremum of volume for their polars.

Theorem 1.A *If $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$V\big(\Pi_p^* K\big) \leq V\big(\Pi_p^{\tau,*} K\big) \leq V\big(\Pi_p^{\pm,*} K\big). \tag{1.10}$$

If K is not origin-symmetric and p is not an odd integer, then there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$. Here, $\Pi_p^{\tau,*} K$ denotes the polar of the general L_p-projection body $\Pi_p^\tau K$ of $K \in \mathcal{K}_o^n$.

Apart from [30], general L_p-projection bodies were studied by various authors; for example, Wang and Wan [31] investigated related Shephard-type problems, Wang and Feng [32] established Petty's affine projection inequality for them. General L_p-projection bodies are a central notion in a new and rapidly evolving asymmetric L_p-Brunn-Minkowski theory (see [14, 15, 30–47]).

In this paper, we first extend inequality (1.10) to dual quermassintegrals forms, that is, the extremums of dual quermassintegrals for the polars of general L_p-projection bodies are obtained.

Theorem 1.1 *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, and real $i \neq n$, then, for $i < n$ or $i > n+p$,*

$$\widetilde{W}_i\big(\Pi_p^* K\big) \leq \widetilde{W}_i\big(\Pi_p^{\tau,*} K\big) \leq \widetilde{W}_i\big(\Pi_p^{\pm,*} K\big), \tag{1.11}$$

and, for $n < i < n+p$,

$$\widetilde{W}_i\big(\Pi_p^* K\big) \geq \widetilde{W}_i\big(\Pi_p^{\tau,*} K\big) \geq \widetilde{W}_i\big(\Pi_p^{\pm,*} K\big). \tag{1.12}$$

In each case, if K is not origin-symmetric and p is not an odd integer, then there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm 1$. For $i = n+p$, (1.11) *and* (1.12) *become equalities. Here $\widetilde{W}_i(Q)$ (i is any real) denote the dual quermassintegrals of the star body Q.*

If $i = 0$, then since $\widetilde{W}_0(Q) = V(Q)$, Theorem 1.1 reduces to Theorem 1.A.

Next, we obtain the extremums of quermassintegrals of general L_p-projection bodies.

Theorem 1.2 *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $\tau \in [-1,1]$, and $i = 0,1,\ldots,n-1$, then*

$$W_i(\Pi_p K) \geq W_i(\Pi_p^\tau K) \geq W_i(\Pi_p^\pm K). \tag{1.13}$$

If K is not origin-symmetric and p is not an odd integer, then there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm1$. Here $W_i(Q)$ $(i = 0,1,\ldots,n-1)$ denote the quermassintegrals of $Q \in \mathcal{K}_o^n$.

Taking $i = 0$ in Theorem 1.2, we obtain the following:

Corollary 1.1 *If $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$V(\Pi_p K) \geq V(\Pi_p^\tau K) \geq V(\Pi_p^\pm K). \tag{1.14}$$

If K is not origin-symmetric and p is not an odd integer, then there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm1$.

Inequality (1.14) can be viewed as a dual version of inequality (1.10).

Finally, we determine the extremal values of the L_q-dual affine surface area (see Section 2) of the polars of general L_p-projection bodies.

Theorem 1.3 *If $K \in \mathcal{K}_o^n$, $p \geq 1$, $0 < q < n$, and $\tau \in [-1,1]$, then*

$$\widetilde{\Omega}_q(\Pi_p^* K) \leq \widetilde{\Omega}_q(\Pi_p^{\tau,*} K) \leq \widetilde{\Omega}_q(\Pi_p^{\pm,*} K). \tag{1.15}$$

If K is not origin-symmetric and p is not an odd integer, then there is equality in the left inequality if and only if $\tau = 0$ and equality in the right inequality if and only if $\tau = \pm1$. Here $\widetilde{\Omega}_q(Q)$ denotes the L_q-dual affine surface area of the star body Q.

This paper is organized as follows. In Section 2, we provide some preliminary results. Then, in Section 3, we recall some basic properties of general L_p-projection bodies. Section 4 contains the proofs of Theorems 1.1-1.3.

2 Basic notions

2.1 Radial functions and polar bodies

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \backslash \{0\} \longrightarrow [0,+\infty)$ is defined by (see [1])

$$\rho(K,x) = \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \backslash \{0\}.$$

If ρ_K is positive and continuous, then K is called a star body (about the origin). For the set of star bodies containing the origin in their interiors and the set of origin-symmetric star bodies in $\mathbb{R}^n$, we write $\mathcal{S}_o^n$ and $\mathcal{S}_{os}^n$, respectively. Two star bodies K and L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If E is a nonempty subset of $\mathbb{R}^n$, then the polar set E^* of E is defined by (see [1])

$$E^* = \{x : x \cdot y \le 1, y \in E\}, \quad x \in \mathbb{R}^n. \tag{2.1}$$

From (2.1) it follows that if $K \in \mathcal{K}_o^n$, then

$$h_{K^*} = 1/\rho_K \quad \text{and} \quad \rho_{K^*} = 1/h_K. \tag{2.2}$$

2.2 L_p-Minkowski and L_p-harmonic radial combinations

For $K, L \in \mathcal{K}_o^n$, $p \ge 1$, and $\lambda, \mu \ge 0$ (not both zero), the L_p-Minkowski combination (also called the Firey L_p-combination) $\lambda \cdot K +_p \mu \cdot L \in \mathcal{K}_o^n$ of K and L is defined by (see [48, 49])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K, \cdot)^p + \mu h(L, \cdot)^p, \tag{2.3}$$

where '$\cdot$' in $\lambda \cdot K$ denotes the L_p-Minkowski scalar multiplication.

For $K, L \in \mathcal{S}_o^n$, $p \ge 1$, and $\lambda, \mu \ge 0$ (not both zero), the L_p-harmonic radial combination $\lambda \circ K \,\tilde{+}_{-p}\, \mu \circ L \in \mathcal{S}_o^n$ of K and L is defined by (see [50])

$$\rho(\lambda \circ K \,\tilde{+}_{-p}\, \mu \circ L, \cdot)^{-p} = \lambda \rho(K, \cdot)^{-p} + \mu \rho(L, \cdot)^{-p}. \tag{2.4}$$

From (2.2), (2.3), and (2.4) we easily see that if $K, L \in \mathcal{K}_o^n$, $p \ge 1$, and $\lambda, \mu \ge 0$ (not both zero), then

$$(\lambda \cdot K +_p \mu \cdot L)^* = \lambda \circ K^* \,\tilde{+}_{-p}\, \mu \circ L^*. \tag{2.5}$$

2.3 L_p-mixed and dual mixed volumes

Lutwak [51] gave the definition of L_p-mixed volume associated with L_p-Minkowski combinations of convex bodies: For $K, L \in \mathcal{K}_o^n$, $\varepsilon > 0$, and $p \ge 1$, the L_p-mixed volume $V_p(K, L)$ of K and L is defined by

$$\frac{n}{p} V_p(K, L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Corresponding to each $K \in \mathcal{K}_o^n$, Lutwak [51] proved that, for each $L \in \mathcal{K}_o^n$,

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(v)\, dS_p(K, v). \tag{2.6}$$

From (2.6) and (1.3) it follows immediately that, for each $K \in \mathcal{K}_o^n$,

$$V_p(K, K) = V(K) = \frac{1}{n} \int_{S^{n-1}} h_K(v)\, dS(K, v). \tag{2.7}$$

The L_p-Minkowski inequality states the following (see [51]):

Theorem 2.A *If $K, L \in \mathcal{K}_o^n$, and $p \ge 1$, then*

$$V_p(K, L) \ge V(K)^{(n-p)/n} V(L)^{p/n} \tag{2.8}$$

with equality for $p > 1$ if and only if K and L are dilates and for $p = 1$ if and only if K and L are homothetic.

Haberl [35] (also see [52]) introduced the notion of L_p-dual mixed volume as follows. For $K, L \in \mathcal{S}_o^n$, $p > 0$, and $\varepsilon > 0$, the L_p-dual mixed volume $\widetilde{V}_p(K,L)$ of K and L is defined by

$$\widetilde{V}_p(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n-p}(u)\rho_L^p(u)\,dS(u), \tag{2.9}$$

where the integration is with respect to spherical Lebesgue measure on S^{n-1}.

2.4 L_p-dual affine surface areas

Based on the L_p-dual mixed volume, Wang, Yuan, and He [53] defined the notion of L_p-dual affine surface area. For $K \in \mathcal{S}_o^n$ and $0 < p < n$, the L_p-dual affine surface area $\widetilde{\Omega}_p(K)$ of K is defined by

$$n^{-\frac{p}{n}}\widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\bigl\{n\widetilde{V}_p\bigl(K,Q^*\bigr)V(Q)^{\frac{p}{n}} : Q \in \mathcal{K}_c^n\bigr\}, \tag{2.10}$$

where $\mathcal{K}_c^n$ denotes the set of convex bodies whose centroids lie at the origin in $\mathbb{R}^n$.

We extend definition (2.10) from $Q \in \mathcal{K}_c^n$ to $Q \in \mathcal{S}_{os}^n$ as follows: For $K \in \mathcal{S}_o^n$ and $0 < p < n$, the L_p-dual affine surface area $\widetilde{\Omega}_p(K)$ of K is defined by

$$n^{-\frac{p}{n}}\widetilde{\Omega}_p(K)^{\frac{n+p}{n}} = \sup\bigl\{n\widetilde{V}_p\bigl(K,Q^*\bigr)V(Q)^{\frac{p}{n}} : Q \in \mathcal{S}_{os}^n\bigr\}. \tag{2.11}$$

2.5 Quermassintegrals and dual quermassintegrals

For $K \in \mathcal{K}^n$, $i = 0, 1, \ldots, n-1$, the quermassintegrals $W_i(K)$ of K are given by (see [1, 49])

$$W_i(K) = \frac{1}{n}\int_{S^{n-1}} h(K,u)\,dS_i(K,u), \tag{2.12}$$

where $S_i(K,\cdot)$ ($i = 0, 1, \ldots, n-1$) denotes the ith surface area measure of K, and $S_0(K,\cdot) = S(K,\cdot)$. From (2.12) and (2.7) we easily see that $W_0(K) = V(K)$.

For the L_p-Minkowski combination, Lutwak [51] proved the following Brunn-Minkowski inequality for quermassintegrals.

Theorem 2.B *If $K, L \in \mathcal{K}_o^n$, $p \geq 1$, $i = 0, 1, \ldots, n-1$, and $\lambda, \mu \geq 0$ (not both zero), then*

$$W_i(\lambda \cdot K +_p \mu \cdot L)^{\frac{p}{n-i}} \geq \lambda W_i(K)^{\frac{p}{n-i}} + \mu W_i(L)^{\frac{p}{n-i}} \tag{2.13}$$

with equality for $p = 1$ if and only if K and L are homothetic and for $p > 1$ if and only if K and L are dilates.

For $K \in \mathcal{S}_o^n$ and any real i, the dual quermassintegrals $\widetilde{W}_i(K)$ of K are defined by (see [54])

$$\widetilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}\,dS(u). \tag{2.14}$$

Obviously, (2.14) implies

$$\widetilde{W}_0(K) = V(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n \, dS(u).$$

Associated with the L_p-harmonic radial combinations of star bodies, Wang and Leng [55] established the following Brunn-Minkowski inequality for dual quermassintegrals.

Theorem 2.C *If $K, L \in \mathcal{S}_o^n$, $p \geq 1$, $\lambda, \mu \geq 0$ (not both zero), and real $i \neq n$, then, for $i < n$ or $n < i < n + p$,*

$$\widetilde{W}_i(\lambda \circ K \tilde{+}_{-p} \mu \circ L)^{-\frac{p}{n-i}} \geq \lambda \widetilde{W}_i(K)^{-\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{-\frac{p}{n-i}} \tag{2.15}$$

and, for $i > n + p$,

$$\widetilde{W}_i(\lambda \circ K \tilde{+}_{-p} \mu \circ L)^{-\frac{p}{n-i}} \leq \lambda \widetilde{W}_i(K)^{-\frac{p}{n-i}} + \mu \widetilde{W}_i(L)^{-\frac{p}{n-i}}. \tag{2.16}$$

In each inequality, equality holds if and only if K and L are dilates. For $i = n + p$, (2.15) *and* (2.16) *become equalities.*

3 Some properties of general L_p-projection bodies

In this section, we recall some basic properties of general L_p-projection bodies.

Theorem 3.1 *If $K \in \mathcal{K}_o^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$\Pi_p^\tau(-K) = \Pi_p^{-\tau} K = -\Pi_p^\tau K. \tag{3.1}$$

Proof From (1.5) it follows that, for all $u \in S^{n-1}$,

$$\begin{aligned} h^p_{-\Pi_p^\tau K}(u) = h^p_{\Pi_p^\tau K}(-u) &= \alpha_{n,p}(\tau)\int_{S^{n-1}} \varphi_\tau(-u\cdot v)^p \, dS_p(K,v) \\ &= \alpha_{n,p}(\tau)\int_{S^{n-1}} \varphi_\tau\big(u\cdot(-v)\big)^p \, dS_p(-K,-v) = h^p_{\Pi_p^\tau(-K)}(u). \end{aligned}$$

This gives

$$\Pi_p^\tau(-K) = -\Pi_p^\tau K. \tag{3.2}$$

In addition, by (1.9) we have that

$$f_1(\tau) + f_2(\tau) = 1, \tag{3.3}$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau). \tag{3.4}$$

From (3.3) and (3.4), together with (1.6) and (1.8), we obtain

$$\begin{aligned} \Pi_p^{-\tau} K &= f_1(-\tau)\cdot \Pi_p^+ K +_p f_2(-\tau)\cdot \Pi_p^- K \\ &= f_2(\tau)\cdot \Pi_p^-(-K) +_p f_1(\tau)\cdot \Pi_p^+(-K) = \Pi_p^\tau(-K). \end{aligned} \tag{3.5}$$

Obviously, (3.2) and (3.5) yield (3.1). □

Theorem 3.2 *If $K \in \mathcal{K}_o^n$, $p \ge 1$, $\tau \in [-1,1]$, and $\tau \neq 0$, then*

$$\Pi_p^\tau K = \Pi_p^{-\tau} K \quad \Longleftrightarrow \quad \Pi_p^+ K = \Pi_p^- K.$$

Proof From (1.8) and (3.4) it follows that, for $K \in \mathcal{K}_o^n$, $p \ge 1$, and $\tau \in [-1,1]$,

$$\Pi_p^{-\tau} K = f_2(\tau) \cdot \Pi_p^+ K +_p f_1(\tau) \cdot \Pi_p^- K,$$

that is,

$$h^p_{\Pi_p^{-\tau} K}(u) = f_2(\tau) h^p_{\Pi_p^+ K}(u) + f_1(\tau) h^p_{\Pi_p^- K}(u) \tag{3.6}$$

for all $u \in S^{n-1}$. Therefore, by (3.3), (1.7), and (3.6), if $\Pi_p^+ K = \Pi_p^- K$, then

$$h^p_{\Pi_p^\tau K}(u) = h^p_{\Pi_p^{-\tau} K}(u)$$

for all $u \in S^{n-1}$. This gives $\Pi_p^\tau K = \Pi_p^{-\tau} K$.

Conversely, if $\Pi_p^\tau K = \Pi_p^{-\tau} K$, then (1.7) and (3.6) yield

$$\bigl[f_1(\tau) - f_2(\tau)\bigr] h^p_{\Pi_p^+ K}(u) = \bigl[f_1(\tau) - f_2(\tau)\bigr] h^p_{\Pi_p^- K}(u)$$

for all $u \in S^{n-1}$. Since $f_1(\tau) - f_2(\tau) \neq 0$ when $\tau \neq 0$, we get $\Pi_p^+ K = \Pi_p^- K$. □

Haberl and Schuster [30] proved the following fact.

Theorem 3.A *If $K \in \mathcal{K}_o^n$, $p \ge 1$, and p is not odd integer, then $\Pi_p^+ K = \Pi_p^- K$ if and only if $K \in \mathcal{K}_{os}^n$.*

According to Theorems 3.A and 3.2, we get the following:

Theorem 3.3 *If $K \in \mathcal{K}_o^n$, $p \ge 1$, and p is not odd integer, then, for $\tau \in [-1,1]$ and $\tau \neq 0$, $\Pi_p^\tau K = \Pi_p^{-\tau} K$ if and only if $K \in \mathcal{K}_{os}^n$.*

Theorem 3.4 *If $K \in \mathcal{K}_o^n$, $p \ge 1$, and $\tau \in [-1,1]$, then*

$$\Pi_p^\tau K +_p \Pi_p^{-\tau} K = \Pi_p^+ K +_p \Pi_p^- K. \tag{3.7}$$

Proof From (1.7) and (3.6), using (3.3), we have that, for any $u \in S^{n-1}$,

$$h\bigl(\Pi_p^\tau K, u\bigr)^p + h\bigl(\Pi_p^{-\tau} K, u\bigr)^p = h\bigl(\Pi_p^+ K, u\bigr)^p + h\bigl(\Pi_p^- K, u\bigr)^p,$$

that is,

$$h\bigl(\Pi_p^\tau K +_p \Pi_p^{-\tau} K, u\bigr)^p = h\bigl(\Pi_p^+ K +_p \Pi_p^- K, u\bigr)^p.$$

This is the desired relation. □

From Theorem 3.4 we deduce the following:

Corollary 3.1 *If $K \in \mathcal{K}_{\mathrm{o}}^n$, $p \geq 1$, and $\tau \in [-1,1]$, then*

$$\Pi_p K = \frac{1}{2} \cdot \Pi_p^{\tau} K +_p \frac{1}{2} \cdot \Pi_p^{-\tau} K. \tag{3.8}$$

Proof Taking $\tau = 0$ in (1.8) and combining with (1.9) yield

$$\Pi_p K = \frac{1}{2} \cdot \Pi_p^{+} K +_p \frac{1}{2} \cdot \Pi_p^{-} K. \tag{3.9}$$

From (3.9) and (3.7) we immediately get (3.8). □

Theorem 3.5 *If $K, L \in \mathcal{K}_{\mathrm{os}}^n$, $p \geq 1$ is not an even integer, and $\tau \in [-1,1]$, then*

$$\Pi_p^{\tau} K = \Pi_p^{\tau} L \quad \Longrightarrow \quad K = L.$$

The proof of Theorem 3.5 requires the following two lemmas.

Lemma 3.1 *If $K, L \in \mathcal{K}_{\mathrm{o}}^n$, and $p \geq 1$ is not an even integer, then $\Pi_p K = \Pi_p L$ if and only if $V_p(K, Q) = V_p(L, Q)$ for any $Q \in \mathcal{K}_{\mathrm{os}}^n$.*

Proof From (1.1) we know that, for any $u \in S^{n-1}$,

$$\begin{aligned} h_{\Pi_p(-K)}^p(u) &= \alpha_{n,p} \int_{S^{n-1}} |u \cdot v|^p \, dS_p(-K, v) \\ &= \alpha_{n,p} \int_{S^{n-1}} \bigl|u \cdot (-v)\bigr|^p \, dS_p(K, -v) = h_{\Pi_p K}^p(u), \end{aligned}$$

which implies $\Pi_p(-K) = \Pi_p K$. Thus, for any $u \in S^{n-1}$,

$$\begin{aligned} h_{\Pi_p K}^p(u) &= \frac{1}{2} h_{\Pi_p K}^p(u) + \frac{1}{2} h_{\Pi_p(-K)}^p(u) \\ &= \frac{1}{2} \alpha_{n,p} \int_{S^{n-1}} |u \cdot v|^p \bigl[dS_p(K, v) + dS_p(-K, v)\bigr]. \end{aligned}$$

Thus, if $\Pi_p K = \Pi_p L$, then, for any $u \in S^{n-1}$,

$$\int_{S^{n-1}} |u \cdot v|^p \bigl[dS_p(K, v) + dS_p(-K, v) - dS_p(L, v) - dS_p(-L, v)\bigr] = 0.$$

Letting $\mu(v) = S_p(K, v) + S_p(-K, v) - S_p(L, v) - S_p(-L, v)$, we have

$$\int_{S^{n-1}} |u \cdot v|^p \, d\mu(v) = 0. \tag{3.10}$$

Since $\mu(v)$ is an even Borel measure on S^{n-1} and $p \geq 1$ is not an even integer, it follows from (3.10) that $\mu(v) = 0$ (see, e.g., [30]), that is,

$$S_p(K, \cdot) + S_p(-K, \cdot) = S_p(L, \cdot) + S_p(-L, \cdot). \tag{3.11}$$

Since $Q \in \mathcal{K}^n_{\mathrm{os}}$, we have $h_Q(-v) = h_Q(v)$ for all $v \in S^{n-1}$. Therefore, by (2.6) we get

$$V_p(K,Q) = \frac{1}{n}\int_{S^{n-1}} h^p_Q(-v)\,dS_p(K,-v) = \frac{1}{n}\int_{S^{n-1}} h^p_Q(v)\,dS_p(-K,v).$$

This and (2.6) yield

$$V_p(K,Q) = \frac{1}{2n}\int_{S^{n-1}} h^p_Q(v)\big[dS_p(K,v) + dS_p(-K,v)\big]$$

for any $Q \in \mathcal{K}^n_{\mathrm{os}}$. By (3.11) we see that if $\Pi_p K = \Pi_p L$, then $V_p(K,Q) = V_p(L,Q)$ for any $Q \in \mathcal{K}^n_{\mathrm{os}}$.

Conversely, if $Q \in \mathcal{K}^n_{\mathrm{os}}$, let $Q = [-u,u]$ ($u \in S^{n-1}$). Then $h_Q(v) = |u \cdot v|$ for any $v \in S^{n-1}$. This, together with (2.6), yields

$$\begin{aligned} V_p(K,Q) &= \frac{1}{n}\int_{S^{n-1}} h^p_Q(v)\,dS_p(K,v) \\ &= \frac{1}{n}\int_{S^{n-1}} |u \cdot v|^p\,dS_p(K,v) = \frac{1}{n\alpha_{n,p}} h^p(\Pi_p K,u). \end{aligned}$$

Hence, if $V_p(K,Q) = V_p(L,Q)$ for any $Q \in \mathcal{K}^n_{\mathrm{os}}$, then $\Pi_p K = \Pi_p L$. □

Lemma 3.2 *If $K, L \in \mathcal{K}^n_{\mathrm{os}}$ and $p \geq 1$ is not an even integer, then*

$$\Pi_p K = \Pi_p L \quad \Longrightarrow \quad K = L.$$

Proof By Lemma 3.1, if $\Pi_p K = \Pi_p L$ and p is not an even integer, then, for any $Q \in \mathcal{K}^n_{\mathrm{os}}$,

$$V_p(K,Q) = V_p(L,Q). \tag{3.12}$$

Taking K for Q in (3.12) and using (2.7) and (2.8), we obtain $V(K) \geq V(L)$ with equality for $p > 1$ if and only if K and L are dilates (for $p = 1$, if and only if K and L are homothetic). Similarly, taking L for Q in (3.12) yields $V(K) \leq V(L)$, and equality holds for $p > 1$ if and only if K and L are dilates (for $p = 1$, if and only if K and L are homothetic). Therefore, $V(K) = V(L)$, and K and L are dilates when $p > 1$ (K and L are homothetic when $p = 1$). Since $K, L \in \mathcal{K}^n_{\mathrm{os}}$, we have that, for $p \geq 1$, $K = L$. □

Proof of Theorem 3.5 If $K \in \mathcal{K}^n_{\mathrm{os}}$, then by (3.5) and Corollary 3.1 we have that

$$\Pi_p K = \Pi^{\tau}_p K = \Pi^{-\tau}_p K.$$

Therefore, if $K, L \in \mathcal{K}^n_{\mathrm{os}}$, then, for $\tau \in [-1,1]$,

$$\Pi^{\tau}_p K = \Pi^{\tau}_p L \quad \Longleftrightarrow \quad \Pi_p K = \Pi_p L.$$

This, together with Lemma 3.2, completes the proof of Theorem 3.5. □

4 Proofs of the main results

In this section, we will complete the proofs of Theorems 1.1-1.3.

Proof of Theorem 1.1 From (1.8) and (2.5) we have

$$\Pi_p^{\tau,*}K = f_1(\tau)\circ\Pi_p^{+,*}K \tilde{+}_{-p} f_2(\tau)\circ\Pi_p^{-,*}K. \tag{4.1}$$

Hence, for $i < n$ or $n < i < n+p$, using (4.1) and (2.15), we have that

$$\widetilde{W}_i\bigl(\Pi_p^{\tau,*}K\bigr)^{-\frac{p}{n-i}} \geq f_1(\tau)\widetilde{W}_i\bigl(\Pi_p^{+,*}K\bigr)^{-\frac{p}{n-i}} + f_2(\tau)\widetilde{W}_i\bigl(\Pi_p^{-,*}K\bigr)^{-\frac{p}{n-i}}. \tag{4.2}$$

But (3.1) yields $\Pi_p^{-}K = \Pi_p^{+}(-K) = -\Pi_p^{+}K$, which implies $\widetilde{W}_i(\Pi_p^{+,*}K) = \widetilde{W}_i(\Pi_p^{-,*}K)$. Hence, by (4.2) and (3.3) we obtain

$$\widetilde{W}_i\bigl(\Pi_p^{\tau,*}K\bigr)^{-\frac{p}{n-i}} \geq \widetilde{W}_i\bigl(\Pi_p^{\pm,*}K\bigr)^{-\frac{p}{n-i}}. \tag{4.3}$$

Now, if $i < n$, then

$$\widetilde{W}_i\bigl(\Pi_p^{\tau,*}K\bigr) \leq \widetilde{W}_i\bigl(\Pi_p^{\pm,*}K\bigr). \tag{4.4}$$

Inequality (4.4) is just the right-hand side inequality of (1.11). If $n < i < n+p$, then by (4.3) we get

$$\widetilde{W}_i\bigl(\Pi_p^{\tau,*}K\bigr) \geq \widetilde{W}_i\bigl(\Pi_p^{\pm,*}K\bigr), \tag{4.5}$$

which gives the right-hand side inequality of (1.12).

For $i > n+p$, using (4.1) and (2.16), we arrive at

$$\widetilde{W}_i\bigl(\Pi_p^{\tau,*}K\bigr)^{-\frac{p}{n-i}} \leq \widetilde{W}_i\bigl(\Pi_p^{\pm,*}K\bigr)^{-\frac{p}{n-i}},$$

which yields (4.4).

According to the conditions of equality in (2.15) and (2.16), we have that equality holds in (4.4) and (4.5) if and only if $\Pi_p^{+,*}K$ and $\Pi_p^{-,*}K$ are dilates. From this, letting $\Pi_p^{+,*}K = c\Pi_p^{-,*}K$ $(c > 0)$ and using that $\widetilde{W}_i(\Pi_p^{+,*}K) = \widetilde{W}_i(\Pi_p^{-,*}K)$, it follows that $c = 1$, that is, $\Pi_p^{+,*}K = \Pi_p^{-,*}K$. This means that $\Pi_p^{+}K = \Pi_p^{-}K$. Hence, from Theorem 3.A we see that if K is not origin-symmetric and p is not an odd integer, then equality holds in the right-hand side inequalities of (1.11) and (1.12) if and only if $\tau = \pm 1$.

Now we prove the left-hand side inequalities of (1.11) and (1.12).

From (3.8) and (2.5) we have that

$$\Pi_p^{*}K = \frac{1}{2}\circ\Pi_p^{\tau,*}K \tilde{+}_{-p} \frac{1}{2}\circ\Pi_p^{-\tau,*}K. \tag{4.6}$$

Using (3.1) and respectively combining with inequalities (2.15) and (2.16), we obtain the left-hand side inequalities of (1.11) and (1.12).

Moreover, by the conditions of equality in (2.15) and (2.16) we see that equality holds in the left-hand side inequalities of (1.11) and (1.12) if and only if $\Pi_p^{\tau}K = \Pi_p^{-\tau}K$. This, together with Theorem 3.3, yields that if K is not origin-symmetric and p is not an odd integer, then equality holds in the left-hand side inequalities of (1.11) and (1.12) if and only if $\tau = 0$. □

Proof of Theorem 1.2 Using (1.8) and inequality (2.13), we have

$$W_i(\Pi_p^{\tau}K)^{\frac{p}{n-i}} \geq f_1(\tau)W_i(\Pi_p^{+}K)^{\frac{p}{n-i}} + f_2(\tau)W_i(\Pi_p^{-}K)^{\frac{p}{n-i}},$$

which, combined with (3.3), yields

$$W_i(\Pi_p^{\tau}K) \geq W_i(\Pi_p^{\pm}K).$$

This gives the right-hand side inequality of (1.13).

According to the condition of equality in (2.13), we see that equality holds in the right-hand side inequality of (1.13) for $p > 1$ if and only if $\Pi_p^{+}K$ and $\Pi_p^{-}K$ are dilates (for $p = 1$, if and only if $\Pi_p^{+}K$ and $\Pi_p^{-}K$ are homothetic), which yields $\Pi_p^{+}K = \Pi_p^{-}K$. Thus, from Theorem 3.A it follows that if K is not origin-symmetric and p is not an odd integer, then equality holds in the right-hand side inequality of (1.13) if and only if $\tau = \pm 1$.

Meanwhile, from (3.8) and inequality (2.13) we obtain

$$W_i(\Pi_p K)^{\frac{p}{n-i}} \geq \frac{1}{2}W_i(\Pi_p^{\tau}K)^{\frac{p}{n-i}} + \frac{1}{2}W_i(\Pi_p^{-\tau}K)^{\frac{p}{n-i}},$$

which, together with (3.1), yields

$$W_i(\Pi_p K) \geq W_i(\Pi_p^{\tau}K).$$

This is the left-hand side inequality of (1.13), where equality holds if and only if $\Pi_p^{\tau}K = \Pi_p^{-\tau}K$. This, together with Theorem 3.3, shows that if K is not origin-symmetric and p is not an odd integer, then equality holds in the left-hand side inequality of (1.13) if and only if $\tau = 0$. □

The proof of Theorem 1.3 requires the following two lemmas.

Lemma 4.1 *If $K, L \in \mathcal{S}_o^n$, $0 < q < n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), then, for any $Q \in \mathcal{S}_o^n$,*

$$\widetilde{V}_q(\lambda \circ K \tilde{+}_{-p} \mu \circ L, Q)^{-\frac{p}{n-q}} \geq \lambda \widetilde{V}_q(K, Q)^{-\frac{p}{n-q}} + \mu \widetilde{V}_q(L, Q)^{-\frac{p}{n-q}} \tag{4.7}$$

with equality if and only if K and L are dilates.

Proof Since $0 < q < n$ and $p \geq 1$, we have $-p/(n-q) < 0$. Hence, from (2.9), (2.4), and the Minkowski integral inequality (see [56]), we obtain that, for any $Q \in \mathcal{S}_o^n$,

$$\begin{aligned} \widetilde{V}_q(\lambda \circ K \tilde{+}_{-p} \mu \circ L, Q)^{-\frac{p}{n-q}} &= \left[\frac{1}{n}\int_{S^{n-1}} \rho_{\lambda\circ K \tilde{+}_{-p}\mu\circ L}^{n-q}(u)\rho_Q^q(u)\,dS(u)\right]^{-\frac{p}{n-q}} \\ &= \left[\frac{1}{n}\int_{S^{n-1}} \left(\rho_{\lambda\circ K \tilde{+}_{-p}\mu\circ L}^{-p}(u)\right)^{-\frac{n-q}{p}}\rho_Q^q(u)\,dS(u)\right]^{-\frac{p}{n-q}} \\ &= \left[\frac{1}{n}\int_{S^{n-1}} \left(\lambda\rho_K^{-p}(u) + \mu\rho_L^{-p}(u)\right)^{-\frac{n-q}{p}}\rho_Q^q(u)\,dS(u)\right]^{-\frac{p}{n-q}} \end{aligned}$$

$$\begin{aligned}
&\geq \left[\frac{1}{n}\int_{S^{n-1}} \lambda\rho_K^{n-q}(u)\rho_Q^q(u)\,dS(u)\right]^{-\frac{p}{n-q}} \\
&\quad + \left[\frac{1}{n}\int_{S^{n-1}} \mu\rho_L^{n-q}(u)\rho_Q^q(u)\,dS(u)\right]^{-\frac{p}{n-q}} \\
&= \lambda\widetilde{V}_q(K,Q)^{-\frac{p}{n-q}} + \mu\widetilde{V}_q(L,Q)^{-\frac{p}{n-q}}.
\end{aligned}$$

Thus, inequality (4.7) is proven.

According to the equality condition of the Minkowski integral inequality, equality holds in (4.7) if and only if there exists a constant $c > 0$ such that

$$\frac{\rho_K(u)^{n-q}\rho_Q^q(u)}{\rho_L(u)^{n-q}\rho_Q^q(u)} = c$$

for any $u \in S^{n-1}$, that is, K and L are dilates. □

Lemma 4.2 *If $K, L \in \mathcal{S}_o^n$, $0 < q < n$, $p \geq 1$, and $\lambda, \mu \geq 0$ (not both zero), then*

$$\widetilde{\Omega}_q(\lambda \circ K \mp_{-p} \mu \circ L)^{-\frac{p(n+q)}{n(n-q)}} \geq \lambda\widetilde{\Omega}_q(K)^{-\frac{p(n+q)}{n(n-q)}} + \mu\widetilde{\Omega}_q(L)^{-\frac{p(n+q)}{n(n-q)}} \tag{4.8}$$

with equality if and only if K and L are dilates.

Proof For a bounded function $\varphi > 0$, we have

$$(\sup\varphi)^{-1} = \inf\varphi^{-1}. \tag{4.9}$$

Thus, by (2.11), (4.7), and (4.9), noticing that $-\frac{p}{n-q} < 0$ when $0 < q < n$ and $p \geq 1$, we have that

$$\begin{aligned}
&\left[n^{-\frac{q}{n}}\widetilde{\Omega}_q(\lambda \circ K \mp_{-p} \mu \circ L)^{\frac{n+q}{n}}\right]^{-\frac{p}{n-q}} \\
&\quad = \left[\sup\left\{n\widetilde{V}_q\left(\lambda \circ K \mp_{-p} \mu \circ L, Q^*\right)V(Q)^{\frac{q}{n}} : Q \in \mathcal{S}_{os}^n\right\}\right]^{-\frac{p}{n-q}} \\
&\quad = \inf\left\{\left[n\widetilde{V}_q\left(\lambda \circ K \mp_{-p} \mu \circ L, Q^*\right)V(Q)^{\frac{q}{n}}\right]^{-\frac{p}{n-q}} : Q \in \mathcal{S}_{os}^n\right\} \\
&\quad = \inf\left\{\left[n\widetilde{V}_q\left(\lambda \circ K \mp_{-p} \mu \circ L, Q^*\right)\right]^{-\frac{p}{n-q}}\left[V(Q)^{\frac{q}{n}}\right]^{-\frac{p}{n-q}} : Q \in \mathcal{S}_{os}^n\right\} \\
&\quad \geq \inf\left\{\lambda\left[n\widetilde{V}_q\left(K, Q^*\right)V(Q)^{\frac{q}{n}}\right]^{-\frac{p}{n-q}} + \mu\left[n\widetilde{V}_q\left(L, Q^*\right)V(Q)^{\frac{q}{n}}\right]^{-\frac{p}{n-q}} : Q \in \mathcal{S}_{os}^n\right\} \\
&\quad \geq \lambda\inf\left\{\left[n\widetilde{V}_q\left(K, Q^*\right)V(Q)^{\frac{q}{n}}\right]^{-\frac{p}{n-q}} : Q \in \mathcal{S}_{os}^n\right\} \\
&\qquad + \mu\inf\left\{\left[n\widetilde{V}_q\left(L, Q^*\right)V(Q)^{\frac{q}{n}}\right]^{-\frac{p}{n-q}} : Q \in \mathcal{S}_{os}^n\right\} \\
&\quad = \lambda\left[\sup\left\{n\widetilde{V}_q\left(K, Q^*\right)V(Q)^{\frac{q}{n}} : Q \in \mathcal{S}_{os}^n\right\}\right]^{-\frac{p}{n-q}} \\
&\qquad + \mu\left[\sup\left\{n\widetilde{V}_q\left(L, Q^*\right)V(Q)^{\frac{q}{n}} : Q \in \mathcal{S}_{os}^n\right\}\right]^{-\frac{p}{n-q}} \\
&\quad = \lambda\left[n^{-\frac{q}{n}}\widetilde{\Omega}_q(K)^{\frac{n+q}{n}}\right]^{-\frac{p}{n-q}} + \mu\left[n^{-\frac{q}{n}}\widetilde{\Omega}_q(L)^{\frac{n+q}{n}}\right]^{-\frac{p}{n-q}}.
\end{aligned}$$

This gives (4.8).

According to the equality condition of inequality (4.7), equality holds in inequality (4.8) if and only if K and L are dilates. □

Proof of Theorem 1.3 From (4.1) and (4.8) we have that, for $0 < q < n$ and $p \geq 1$,

$$\begin{aligned} \widetilde{\Omega}_q\big(\Pi_p^{\tau,*}K\big)^{-\frac{p(n+q)}{n(n-q)}} &= \widetilde{\Omega}_q\big(f_1(\tau)\circ\Pi_p^{+,*}K \,\tilde{+}_{-p} f_2(\tau)\circ\Pi_p^{-,*}K\big)^{-\frac{p(n+q)}{n(n-q)}} \\ &\geq f_1(\tau)\widetilde{\Omega}_q\big(\Pi_p^{+,*}K\big)^{-\frac{p(n+q)}{n(n-q)}} + f_2(\tau)\widetilde{\Omega}_q\big(\Pi_p^{-,*}K\big)^{-\frac{p(n+q)}{n(n-q)}}. \end{aligned} \tag{4.10}$$

But (2.9) shows that, for any $Q \in \mathcal{S}_{\mathrm{os}}^n$, $\widetilde{V}_q(-K,Q) = \widetilde{V}_q(K,Q)$. This and (2.11) give $\widetilde{\Omega}_q(-K) = \widetilde{\Omega}_q(K)$. From this we see that

$$\widetilde{\Omega}_q\big(\Pi_p^{-,*}K\big) = \widetilde{\Omega}_q\big(-\Pi_p^{+,*}K\big) = \widetilde{\Omega}_q\big(\Pi_p^{+,*}K\big). \tag{4.11}$$

This, together with (4.10) and (3.3), yields

$$\widetilde{\Omega}_q\big(\Pi_p^{\tau,*}K\big)^{-\frac{p(n+q)}{n(n-q)}} \geq \widetilde{\Omega}_q\big(\Pi_p^{\pm,*}K\big)^{-\frac{p(n+q)}{n(n-q)}},$$

that is, for $0 < q < n$ and $p \geq 1$,

$$\widetilde{\Omega}_q\big(\Pi_p^{\tau,*}K\big) \leq \widetilde{\Omega}_q\big(\Pi_p^{\pm,*}K\big).$$

This is the right-hand side inequality (1.15).

According to the equality condition of inequality (4.8), equality holds in the right-hand side inequality of (1.15) if and only if $\Pi_p^{+,*}K$ and $\Pi_p^{-,*}K$ are dilates. This and (4.11) give $\Pi_p^{+,*}K = \Pi_p^{-,*}K$, that is, $\Pi_p^{+}K = \Pi_p^{-}K$. From this, by Theorem 3.A, it follows that if K is not origin-symmetric and p is not an odd integer, then equality holds in the right-hand side inequality of (1.15) if and only if $\tau = \pm 1$.

On the other hand, by (4.6) and inequality (4.8), noticing that

$$\widetilde{\Omega}_q\big(\Pi_p^{-\tau,*}K\big) = \widetilde{\Omega}_q\big(-\Pi_p^{\tau,*}K\big) = \widetilde{\Omega}_q\big(\Pi_p^{\tau,*}K\big), \tag{4.12}$$

we obtain that, for $0 < q < n$, $p \geq 1$ and $\tau \in [-1,1]$,

$$\widetilde{\Omega}_q\big(\Pi_p^{*}K\big) \leq \widetilde{\Omega}_q\big(\Pi_p^{\tau,*}K\big).$$

This yields the left-hand side inequality of (1.15).

According to the equality condition of (4.8) and using (4.12), we know that equality holds in the left-hand side inequality of (1.15) if and only if $\Pi_p^{\tau}K = \Pi_p^{-\tau}K$. This, combined with Theorem 3.3, implies that if K is not origin-symmetric and p is not an odd integer, then equality holds in the left-hand side inequality of (1.15) if and only if $\tau = 0$. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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