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A discrete Hilbert-type inequality in the whole plane

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Abstract

By the use of weight coefficients and technique of real analysis, a discrete Hilbert-type inequality in the whole plane with multi-parameters and a best possible constant factor is given. The equivalent forms, the operator expressions, and a few particular inequalities are considered.

MSC: 26D15; 47A05

Keywords: Hilbert-type inequality; weight coefficient; equivalent form; operator expression

1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $\|a\|_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0$, $\|b\|_q > 0$. We have the following well-known Hardy-Hilbert inequality:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})} \|a\|_p \|b\|_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible (*cf.* [1]). Also we have the following Hilbert-type inequality:

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < pq \|a\|_p \|b\|_q, \tag{2}$$

with the best possible constant factor pq (*cf.* [2]). Inequalities (1) and (2) are important in analysis and its applications (*cf.* [2–4]).

In 2011, Yang gave an extension of (2) as follows (*cf.* [5]): If $0 < \lambda_1, \lambda_2 \leq 1$, $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \geq 0$, $0 < \|a\|_{p,\varphi} = \{\sum_{m=1}^\infty m^{p(1-\lambda_1)-1} a_m^p\}^{\frac{1}{p}} < \infty$, $0 < \|b\|_{q,\psi} = \{\sum_{n=1}^\infty n^{q(1-\lambda_2)-1} b_n^q\}^{\frac{1}{q}} < \infty$, then

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(\max\{m,n\})^\lambda} < \frac{\lambda}{\lambda_1 \lambda_2} \|a\|_{p,\varphi} \|b\|_{q,\psi}, \tag{3}$$

where the constant factor $\frac{\lambda}{\lambda_1 \lambda_2}$ is the best possible.

For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, inequality (3) reduces to (2). Some other results relate to (1)-(3) are provided by [6–23].

In this paper, by the use of weight coefficients and the technique of real analysis, an extension of (3) in the whole plane is given as follows: For $0 < \lambda_1, \lambda_2 \le 1$, $\lambda_1 + \lambda_2 = \lambda$, $a_m, b_n \ge 0$, $0 < \sum_{|m|=1}^{\infty} |m|^{p(1-\lambda_1)-1} a_m^p < \infty$, $0 < \sum_{|n|=1}^{\infty} |n|^{q(1-\lambda_2)-1} b_n^q < \infty$, we have

$$\sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} \frac{1}{(\max\{|m|,|n|\})^{\lambda}} a_m b_n < \frac{2\lambda}{\lambda_1 \lambda_2} \left[\sum_{|m|=1}^{\infty} |m|^{p(1-\lambda_1)-1} a_m^p \right]^{\frac{1}{p}} \left[\sum_{|n|=1}^{\infty} |n|^{q(1-\lambda_2)-1} b_n^q \right]^{\frac{1}{q}}. \tag{4}$$

Moreover, a generation of (4) with multi-parameters and a best possible constant factor is proved. The equivalent forms, the operator expressions and a few particular inequalities are also considered.

2 Some lemmas

In the following, we agree that $\mathbf{N} = \{1, 2, \ldots\}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta \in (0, \pi)$, $\lambda_1, \lambda_2 > -\eta$, $\lambda_1 + \lambda_2 = \lambda$, and for $|x|, |y| > 0$,

$$k(x,y) := \frac{(\min\{|x| + x\cos\alpha, |y| + y\cos\beta\})^{\eta}}{(\max\{|x| + x\cos\alpha, |y| + y\cos\beta\})^{\lambda+\eta}}. \tag{5}$$

Lemma 1 (*cf.* [24]) *Suppose that $g(t)$ (> 0) is decreasing in $\mathbf{R}_+$ and strictly decreasing in $[n_0, \infty)$ $(n_0 \in \mathbf{N})$, satisfying $\int_0^{\infty} g(t)\,dt \in \mathbf{R}_+$. We have*

$$\int_1^{\infty} g(t)\,dt < \sum_{n=1}^{\infty} g(n) < \int_0^{\infty} g(t)\,dt. \tag{6}$$

Definition 1 Define the following weight coefficients:

$$\omega(\lambda_2, m) := \sum_{|n|=1}^{\infty} k(m,n) \frac{(|m| + m\cos\alpha)^{\lambda_1}}{(|n| + n\cos\beta)^{1-\lambda_2}}, \quad |m| \in \mathbf{N}, \tag{7}$$

$$\varpi(\lambda_1, n) := \sum_{|m|=1}^{\infty} k(m,n) \frac{(|n| + n\cos\beta)^{\lambda_2}}{(|m| + m\cos\alpha)^{1-\lambda_1}}, \quad |n| \in \mathbf{N}, \tag{8}$$

where $\sum_{|j|=1}^{\infty} \cdots = \sum_{j=-1}^{-\infty} \cdots + \sum_{j=1}^{\infty} \cdots$ $(j = m, n)$.

Lemma 2 *If $\lambda_2 \le 1 - \eta$, then for $k_\beta(\lambda_1) := \frac{2(\lambda+2\eta)\csc^2\beta}{(\lambda_1+\eta)(\lambda_2+\eta)}$, we have*

$$k_\beta(\lambda_1)\bigl(1 - \theta(\lambda_2, m)\bigr) < \omega(\lambda_2, m) < k_\beta(\lambda_1), \quad |m| \in \mathbf{N}, \tag{9}$$

where

$$\theta(\lambda_2, m) := \frac{(\lambda_1+\eta)(\lambda_2+\eta)}{\lambda+2\eta} \int_0^{\frac{1+\cos\beta}{|m|+m\cos\alpha}} \frac{(\min\{1,u\})^{\eta} u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du = O\biggl(\frac{1}{(|m| + m\cos\alpha)^{\eta+\lambda_2}}\biggr) \in (0,1), \quad |m| \in \mathbf{N}. \tag{10}$$

Proof For $|x| > 0$, we set

$$k^{(1)}(x,y) := \frac{[\min\{|x| + x\cos\alpha, y(\cos\beta - 1)\}]^{\eta}}{[\max\{|x| + x\cos\alpha, y(\cos\beta - 1)\}]^{\lambda+\eta}}, \quad y < 0,$$
$$k^{(2)}(x,y) := \frac{[\min\{|x| + x\cos\alpha, y(1 + \cos\beta)\}]^{\eta}}{[\max\{|x| + x\cos\alpha, y(1 + \cos\beta)\}]^{\lambda+\eta}}, \quad y > 0,$$

from which we have

$$k^{(1)}(x,-y) = \frac{[\min\{|x| + x\cos\alpha, y(1 - \cos\beta)\}]^{\eta}}{[\max\{|x| + x\cos\alpha, y(1 - \cos\beta)\}]^{\lambda+\eta}}, \quad y > 0.$$

We obtain

$$\begin{aligned}
\omega(\lambda_2, m) &= \sum_{n=-1}^{-\infty} k^{(1)}(m,n) \frac{(|m| + m\cos\alpha)^{\lambda_1}}{[n(\cos\beta - 1)]^{1-\lambda_2}} \\
&\quad + \sum_{n=1}^{\infty} k^{(2)}(m,n) \frac{(|m| + m\cos\alpha)^{\lambda_1}}{[n(1 + \cos\beta)]^{1-\lambda_2}} \\
&= \frac{(|m| + m\cos\alpha)^{\lambda_1}}{(1 - \cos\beta)^{1-\lambda_2}} \sum_{n=1}^{\infty} \frac{k^{(1)}(m,-n)}{n^{1-\lambda_2}} \\
&\quad + \frac{(|m| + m\cos\alpha)^{\lambda_1}}{(1 + \cos\beta)^{1-\lambda_2}} \sum_{n=1}^{\infty} \frac{k^{(2)}(m,n)}{n^{1-\lambda_2}}. \qquad (11)
\end{aligned}$$

For fixed $|m| \in \mathbf{N}$, $\lambda_2 \leq 1 - \eta$, we find that

$$\begin{aligned}
\frac{k^{(1)}(m,-y)}{y^{1-\lambda_2}} &= \frac{[\min\{|m| + m\cos\alpha, y(1 - \cos\beta)\}]^{\eta}}{y^{1-\lambda_2}[\max\{|m| + m\cos\alpha, y(1 - \cos\beta)\}]^{\lambda+\eta}} \\
&= \begin{cases} \frac{(1-\cos\beta)^{\eta}}{(|m|+m\cos\alpha)^{\lambda+\eta}} \frac{1}{y^{1-(\lambda_2+\eta)}}, & 0 < y < \frac{|m|+m\cos\alpha}{1-\cos\beta}, \\ \frac{(|m|+m\cos\alpha)^{\eta}}{(1-\cos\beta)^{\lambda+\eta}} \frac{1}{y^{1+(\lambda_1+\eta)}}, & y \geq \frac{|m|+m\cos\alpha}{1-\cos\beta} \end{cases}
\end{aligned}$$

is decreasing for $y > 0$ and strictly decreasing for $y \geq \frac{|m|+m\cos\alpha}{1-\cos\beta}$. Under the same assumptions, it is evident that

$$\begin{aligned}
\frac{k^{(2)}(m,y)}{y^{1-\lambda_2}} &= \frac{[\min\{|m| + m\cos\alpha, y(1 + \cos\beta)\}]^{\eta}}{y^{1-\lambda_2}[\max\{|m| + m\cos\alpha, y(1 + \cos\beta)\}]^{\lambda+\eta}} \\
&= \begin{cases} \frac{(1+\cos\beta)^{\eta}}{(|m|+m\cos\alpha)^{\lambda+\eta}} \frac{1}{y^{1-(\lambda_2+\eta)}}, & 0 < y < \frac{|m|+m\cos\alpha}{1+\cos\beta}, \\ \frac{(|m|+m\cos\alpha)^{\eta}}{(1+\cos\beta)^{\lambda+\eta}} \frac{1}{y^{1+(\lambda_1+\eta)}}, & y \geq \frac{|m|+m\cos\alpha}{1+\cos\beta} \end{cases}
\end{aligned}$$

is decreasing for $y > 0$ and strictly decreasing for $y \geq \frac{|m|+m\cos\alpha}{1+\cos\beta}$.

By (11) and (6), we have

$$\begin{aligned}
\omega(\lambda_2, m) &< \frac{(|m| + m\cos\alpha)^{\lambda_1}}{(1 - \cos\beta)^{1-\lambda_2}} \int_0^{\infty} \frac{k^{(1)}(m,-y)}{y^{1-\lambda_2}}\, dy \\
&\quad + \frac{(|m| + m\cos\alpha)^{\lambda_1}}{(1 + \cos\beta)^{1-\lambda_2}} \int_0^{\infty} \frac{k^{(2)}(m,y)}{y^{1-\lambda_2}}\, dy.
\end{aligned}$$

Setting $u = \frac{y(1-\cos\beta)}{|m|+m\cos\alpha}$ $(\frac{y(1+\cos\beta)}{|m|+m\cos\alpha})$ in the above first (second) integral, by simplifications, we find

$$\begin{aligned}
\omega(\lambda_2, m) &< \left(\frac{1}{1-\cos\beta} + \frac{1}{1+\cos\beta}\right)\int_0^\infty \frac{(\min\{1,u\})^\eta u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du \\
&= 2\csc^2\beta\left(\int_0^1 u^{\eta+\lambda_2-1}\,du + \int_1^\infty \frac{u^{\lambda_2-1}}{u^{\lambda+\eta}}\,du\right) \\
&= \frac{2(\lambda+2\eta)\csc^2\beta}{(\lambda_1+\eta)(\lambda_2+\eta)} = k_\beta(\lambda_1).
\end{aligned}$$

Still by (11) and (6), we have

$$\begin{aligned}
\omega(\lambda_2, m) &> \frac{(|m|+m\cos\alpha)^{\lambda_1}}{(1-\cos\beta)^{1-\lambda_2}}\int_1^\infty \frac{k^{(1)}(m,-y)}{y^{1-\lambda_2}}\,dy \\
&\quad + \frac{(|m|+m\cos\alpha)^{\lambda_1}}{(1+\cos\beta)^{1-\lambda_2}}\int_1^\infty \frac{k^{(2)}(m,y)}{y^{1-\lambda_2}}\,dy \\
&\geq \frac{1}{1-\cos\beta}\int_{\frac{1+\cos\beta}{|m|+m\cos\alpha}}^\infty \frac{(\min\{1,u\})^\eta u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du \\
&\quad + \frac{1}{1+\cos\beta}\int_{\frac{1+\cos\beta}{|m|+m\cos\alpha}}^\infty \frac{(\min\{1,u\})^\eta u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du \\
&= k_\beta(\lambda_1)\bigl(1-\theta(\lambda_2,m)\bigr) > 0.
\end{aligned}$$

We obtain for $|m| + m\cos\alpha \geq 1 + \cos\beta$

$$\begin{aligned}
0 < \theta(\lambda_2, m) &= \frac{(\lambda_1+\eta)(\lambda_2+\eta)}{\lambda+2\eta}\int_0^{\frac{1+\cos\beta}{|m|+m\cos\alpha}} \frac{(\min\{1,u\})^\eta u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du \\
&= \frac{(\lambda_1+\eta)(\lambda_2+\eta)}{\lambda+2\eta}\int_0^{\frac{1+\cos\beta}{|m|+m\cos\alpha}} u^{\eta+\lambda_2-1}\,du \\
&= \frac{\lambda_1+\eta}{\lambda+2\eta}\left(\frac{1+\cos\beta}{|m|+m\cos\alpha}\right)^{\eta+\lambda_2}.
\end{aligned}$$

Then we have (9) and (10). □

In the same way, we have the following.

Lemma 3 *If* $\lambda_1 \leq 1-\eta$, *then for* $k_\alpha(\lambda_1) = \frac{2(\lambda+2\eta)\csc^2\alpha}{(\lambda_1+\eta)(\lambda_2+\eta)}$, *we have*

$$k_\alpha(\lambda_1)\bigl(1-\vartheta(\lambda_1,n)\bigr) < \varpi(\lambda_1,n) < k_\alpha(\lambda_1), \quad |n| \in \mathbf{N}, \tag{12}$$

where

$$\begin{aligned}
\vartheta(\lambda_1, n) &:= \frac{(\lambda_1+\eta)(\lambda_2+\eta)}{\lambda+2\eta}\int_0^{\frac{1+\cos\alpha}{|n|+n\cos\beta}} \frac{(\min\{1,u\})^\eta u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda+\eta}}\,du \\
&= O\left(\frac{1}{(|n|+n\cos\beta)^{\eta+\lambda_1}}\right) \in (0,1), \quad |n| \in \mathbf{N}.
\end{aligned} \tag{13}$$

Lemma 4 *If* $\theta \in (0,\pi)$, *then for* $\rho > 0$, $H_\rho(\theta) := \sum_{|n|=1}^{\infty} \frac{1}{(|n|+n\cos\theta)^{1+\rho}}$, *we have*

$$H_\rho(\theta) = \left[\frac{1}{(1+\cos\theta)^{1+\rho}} + \frac{1}{(1-\cos\theta)^{1+\rho}}\right]\frac{1+\rho O(1)}{\rho} \quad (\rho \to 0^+). \tag{14}$$

Proof We have

$$\begin{aligned} H_\rho(\theta) &= \sum_{n=-1}^{-\infty} \frac{1}{[n(\cos\theta - 1)]^{1+\rho}} + \sum_{n=1}^{\infty} \frac{1}{[n(\cos\theta + 1)]^{1+\rho}} \\ &= \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right] \sum_{n=1}^{\infty} \frac{1}{n^{1+\rho}}. \end{aligned}$$

By (6), we find

$$\begin{aligned} H_\rho(\theta) &= \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]\left(1 + \sum_{n=2}^{\infty} \frac{1}{n^{1+\rho}}\right) \\ &< \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]\left(1 + \int_1^\infty \frac{dy}{y^{1+\rho}}\right) \\ &= \frac{1}{\rho}\left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right](1+\rho), \\ H_\rho(\theta) &> \left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]\int_1^\infty \frac{dy}{y^{1+\rho}} \\ &= \frac{1}{\rho}\left[\frac{1}{(1-\cos\theta)^{1+\rho}} + \frac{1}{(1+\cos\theta)^{1+\rho}}\right]. \end{aligned}$$

Hence we have (14). □

3 Main results

Theorem 1 *If* $\lambda_1, \lambda_2 \le 1-\eta$, $a_m, b_n \ge 0$ $(|m|, |n| \in \mathbf{N})$,

$$0 < \sum_{|m|=1}^{\infty} \bigl(|m| + m\cos\alpha\bigr)^{p(1-\lambda_1)-1} a_m^p < \infty,$$

$$0 < \sum_{|n|=1}^{\infty} \bigl(|n| + n\cos\beta\bigr)^{q(1-\lambda_2)-1} b_n^q < \infty,$$

$$k_{\alpha,\beta}(\lambda_1) := k_\beta^{\frac{1}{p}}(\lambda_1) k_\alpha^{\frac{1}{q}}(\lambda_1) = \frac{2(\lambda + 2\eta)\csc^{\frac{2}{p}}\beta \csc^{\frac{2}{q}}\alpha}{(\lambda_1+\eta)(\lambda_2+\eta)}, \tag{15}$$

then we have the following equivalent inequalities:

$$\begin{aligned} I &:= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty} k(m,n) a_m b_n \\ &< k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=1}^{\infty} \bigl(|m| + m\cos\alpha\bigr)^{p(1-\lambda_1)-1} a_m^p\right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=1}^{\infty} \bigl(|n| + n\cos\beta\bigr)^{q(1-\lambda_2)-1} b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{16}$$

$$\begin{aligned} J &:= \left[\sum_{|n|=1}^{\infty}\bigl(|n| + n\cos\beta\bigr)^{p\lambda_2-1}\left(\sum_{|m|=1}^{\infty}k(m,n)a_m\right)^p\right]^{\frac{1}{p}} \\ &< k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=1}^{\infty}\bigl(|m| + m\cos\alpha\bigr)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}. \end{aligned} \tag{17}$$

In particular, for $\alpha = \beta = \frac{\pi}{2}$, *we have the following equivalent inequalities*:

$$\begin{aligned} &\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{(\min\{|m|,|n|\})^{\eta}}{(\max\{|m|,|n|\})^{\lambda+\eta}}a_m b_n \\ &\quad< \frac{2(\lambda+2\eta)}{(\lambda_1+\eta)(\lambda_2+\eta)}\left[\sum_{|m|=1}^{\infty}|m|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{18}$$

$$\begin{aligned} &\left[\sum_{|n|=1}^{\infty}|n|^{p\lambda_2-1}\left(\sum_{|m|=1}^{\infty}\frac{(\min\{|m|,|n|\})^{\eta}}{(\max\{|m|,|n|\})^{\lambda+\eta}}a_m\right)^p\right]^{\frac{1}{p}} \\ &\quad< \frac{2(\lambda+2\eta)}{(\lambda_1+\eta)(\lambda_2+\eta)}\left[\sum_{|m|=1}^{\infty}|m|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}. \end{aligned} \tag{19}$$

Proof By Hölder's inequality (*cf.* [25]) and (8), we have

$$\begin{aligned} \left(\sum_{|m|=1}^{\infty}k(m,n)a_m\right)^p &= \left[\sum_{|m|=1}^{\infty}k(m,n)\frac{(|m|+m\cos\alpha)^{(1-\lambda_1)/q}a_m}{(|n|+n\cos\beta)^{(1-\lambda_2)/p}}\frac{(|n|+n\cos\beta)^{(1-\lambda_2)/p}}{(|m|+m\cos\alpha)^{(1-\lambda_1)/q}}\right]^p \\ &\le \sum_{|m|=1}^{\infty}k(m,n)\frac{(|m|+m\cos\alpha)^{(1-\lambda_1)p/q}}{(|n|+n\cos\beta)^{1-\lambda_2}}a_m^p \\ &\quad\times\left[\sum_{|m|=1}^{\infty}k(m,n)\frac{(|n|+n\cos\beta)^{(1-\lambda_2)q/p}}{(|m|+m\cos\alpha)^{1-\lambda_1}}\right]^{p-1} \\ &= \frac{(\varpi(\lambda_1,n))^{p-1}}{(|n|+n\cos\beta)^{p\lambda_2-1}}\sum_{|m|=1}^{\infty}k(m,n)\frac{(|m|+m\cos\alpha)^{(1-\lambda_1)p/q}}{(|n|+n\cos\beta)^{1-\lambda_2}}a_m^p. \end{aligned}$$

By (12), we have

$$\begin{aligned} J &< k_{\alpha}^{\frac{1}{q}}(\lambda_1)\left[\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)\frac{(|m|+m\cos\alpha)^{(1-\lambda_1)(p-1)}}{(|n|+n\cos\beta)^{1-\lambda_2}}a_m^p\right]^{\frac{1}{p}} \\ &= k_{\alpha}^{\frac{1}{q}}(\lambda_1)\left[\sum_{|m|=1}^{\infty}\sum_{|n|=1}^{\infty}k(m,n)\frac{(|m|+m\cos\alpha)^{(1-\lambda_1)(p-1)}}{(|n|+n\cos\beta)^{1-\lambda_2}}a_m^p\right]^{\frac{1}{p}} \\ &= k_{\alpha}^{\frac{1}{q}}(\lambda_1)\left[\sum_{|m|=1}^{\infty}\omega(\lambda_2,m)\bigl(|m|+m\cos\alpha\bigr)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}. \end{aligned} \tag{20}$$

By (9), we have (17).

By Hölder's inequality (*cf.* [25]), we have

$$
\begin{aligned}
I &= \sum_{|n|=1}^{\infty}\left[\left(|n| + n\cos\beta\right)^{\lambda_2-\frac{1}{p}}\sum_{|m|=1}^{\infty}k(m,n)a_m\right]\left(|n| + n\cos\beta\right)^{\frac{1}{p}-\lambda_2}b_n \\
&\le J\left[\sum_{|n|=1}^{\infty}\left(|n| + n\cos\beta\right)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \qquad (21)
\end{aligned}
$$

Then by (17), we have (16).

On the other hand, assuming that (16) is valid, we set

$$
b_n := \left(|n| + n\cos\beta\right)^{p\lambda_2-1}\left(\sum_{|m|=1}^{\infty}k(m,n)a_m\right)^{p-1}, \quad |n| \in \mathbf{N}.
$$

Then it follows that

$$
J = \left[\sum_{|n|=1}^{\infty}\left(|n| + n\cos\beta\right)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{p}}.
$$

By (20), we find $J < \infty$. If $J = 0$, then (17) is evidently valid; if $J > 0$, then by (16), we have

$$
\begin{aligned}
0 &< \sum_{|n|=1}^{\infty}\left(|n| + n\cos\beta\right)^{q(1-\lambda_2)-1}b_n^q = J^p = I \\
&< k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=1}^{\infty}\left(|m| + m\cos\alpha\right)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_{|n|=1}^{\infty}\left(|n| + n\cos\beta\right)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \\
J &= \left[\sum_{|n|=1}^{\infty}\left(|n| + n\cos\beta\right)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{p}} \\
&< k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=1}^{\infty}\left(|m| + m\cos\alpha\right)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}},
\end{aligned}
$$

namely, (17) follows, which is equivalent to (16). □

Theorem 2 *As regards the assumptions of Theorem* 1, *the constant factor* $k_{\alpha,\beta}(\lambda_1)$ *in* (16) *and* (17) *is the best possible.*

Proof For any $\varepsilon \in (0, q(\lambda_2 + \eta))$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ $(> -\eta)$, $\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ $(\in (-\eta, 1-\eta))$, and

$$
\begin{aligned}
\widetilde{a}_m &:= \left(|m| + m\cos\alpha\right)^{(\lambda_1-\frac{\varepsilon}{p})-1} = \left(|m| + m\cos\alpha\right)^{\widetilde{\lambda}_1-\varepsilon-1} \quad \left(|m| \in \mathbf{N}\right), \\
\widetilde{b}_n &:= \left(|n| + n\cos\beta\right)^{(\lambda_2-\frac{\varepsilon}{q})-1} = \left(|n| + n\cos\beta\right)^{\widetilde{\lambda}_2-1} \quad \left(|n| \in \mathbf{N}\right).
\end{aligned}
$$

Then by (14) and (9), we find

$$\begin{aligned}
\widetilde{I}_1 &:= \left[\sum_{|m|=1}^{\infty}\big(|m| + m\cos\alpha\big)^{p(1-\lambda_1)-1}\widetilde{a}_m^p\right]^{\frac{1}{p}} \\
&\quad\times\left[\sum_{|n|=1}^{\infty}\big(|n| + n\cos\beta\big)^{q(1-\lambda_2)-1}\widetilde{b}_n^q\right]^{\frac{1}{q}} \\
&= \left[\sum_{|m|=1}^{\infty}\frac{1}{(|m| + m\cos\alpha)^{1+\varepsilon}}\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}\frac{1}{(|n| + n\cos\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]^{\frac{1}{p}}\big(1+\varepsilon O_1(1)\big)^{\frac{1}{p}} \\
&\quad\times\left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}}\big(1+\varepsilon O_2(1)\big)^{\frac{1}{q}}, \\
\widetilde{I} &:= \sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)\widetilde{a}_m\widetilde{b}_n \\
&= \sum_{|m|=1}^{\infty}\sum_{|m|=1}^{\infty}k(m,n)\frac{(|m| + m\cos\alpha)^{\widetilde{\lambda}_1-\varepsilon-1}}{(|n| + n\cos\beta)^{1-\widetilde{\lambda}_2}} \\
&= \sum_{|m|=1}^{\infty}\frac{\omega(\widetilde{\lambda}_2,m)}{(|m| + m\cos\alpha)^{\varepsilon+1}} \geq k_\beta(\widetilde{\lambda}_1)\sum_{|m|=1}^{\infty}\frac{1-\theta(\widetilde{\lambda}_2,m)}{(|m| + m\cos\alpha)^{\varepsilon+1}} \\
&= k_\beta(\widetilde{\lambda}_1)\left[\sum_{|m|=1}^{\infty}\frac{1}{(|m| + m\cos\alpha)^{\varepsilon+1}} - \sum_{|m|=1}^{\infty}\frac{1}{O((|m| + m\cos\alpha)^{(\frac{\varepsilon}{p}+\lambda_2+\eta)+1})}\right] \\
&= \frac{k_\beta(\widetilde{\lambda}_1)}{\varepsilon}\left\{\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]\big(1+\varepsilon O_1(1)\big) - \varepsilon O(1)\right\}.
\end{aligned}$$

If there exists a constant $k \leq k_{\alpha,\beta}(\lambda_1)$, such that (16) is valid when replacing $k_{\alpha,\beta}(\lambda_1)$ by k, then in particular, we have $\varepsilon\widetilde{I} < \varepsilon k\widetilde{I}_1$, namely,

$$\begin{aligned}
&k_\beta(\widetilde{\lambda}_1)\left\{\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]\big(1+\varepsilon O_1(1)\big) - \varepsilon O(1)\right\} \\
&\quad< k\left[\frac{1}{(1+\cos\alpha)^{1+\varepsilon}} + \frac{1}{(1-\cos\alpha)^{1+\varepsilon}}\right]^{\frac{1}{p}}\big(1+\varepsilon O_1(1)\big)^{\frac{1}{p}} \\
&\qquad\times\left[\frac{1}{(1+\cos\beta)^{1+\varepsilon}} + \frac{1}{(1-\cos\beta)^{1+\varepsilon}}\right]^{\frac{1}{q}}\big(1+\varepsilon O_2(1)\big)^{\frac{1}{q}}.
\end{aligned}$$

It follows that

$$\frac{4(\lambda + 2\eta)}{(\lambda_1+\eta)(\lambda_2+\eta)}\csc^2\beta\csc^2\alpha \leq 2k\csc^{\frac{2}{p}}\alpha\csc^{\frac{2}{q}}\beta \quad \big(\varepsilon \to 0^+\big),$$

namely,

$$k_{\alpha,\beta}(\lambda_1) = \frac{2(\lambda + 2\eta)\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha}{(\lambda_1 + \eta)(\lambda_2 + \eta)} \le k.$$

Hence, $k = k_{\alpha,\beta}(\lambda_1)$ is the best possible constant factor of (16).

The constant factor $k_{\alpha,\beta}(\lambda_1)$ in (17) is still the best possible. Otherwise, we would reach a contradiction by (21) that the constant factor in (16) is not the best possible. □

4 Operator expressions

We set functions $\Phi(m)$ and $\Psi(n)$ as follows:

$$\Phi(m) := \big(|m| + m\cos\alpha\big)^{p(1-\lambda_1)-1} \quad \big(|m| \in \mathbf{N}\big),$$
$$\Psi(n) := \big(|n| + n\cos\beta\big)^{q(1-\lambda_2)-1} \quad \big(|n| \in \mathbf{N}\big),$$

from which we have

$$\Psi^{1-p}(n) = \big(|n| + n\cos\beta\big)^{p\lambda_2-1} \quad \big(|n| \in \mathbf{N}\big).$$

We also set the following weight normed spaces:

$$l_{p,\Phi} := \left\{ a = \{a_m\}_{|m|=1}^{\infty}; \|a\|_{p,\Phi} = \left\{ \sum_{|m|=1}^{\infty} \Phi(m)|a_m|^p \right\}^{\frac{1}{p}} < \infty \right\},$$
$$l_{q,\Psi} := \left\{ b = \{b_n\}_{|n|=1}^{\infty}; \|b\|_{q,\Psi} = \left\{ \sum_{|n|=1}^{\infty} \Psi(n)|b_n|^q \right\}^{\frac{1}{q}} < \infty \right\},$$
$$l_{p,\Psi^{1-p}} := \left\{ c = \{c_n\}_{|n|=1}^{\infty}; \|c\|_{p,\Psi^{1-p}} = \left\{ \sum_{|n|=1}^{\infty} \Psi^{1-p}(n)|c_n|^p \right\}^{\frac{1}{p}} < \infty \right\}.$$

Then for $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\Phi}$, $c = \{c_n\}_{|n|=1}^{\infty}$, $c_n = \sum_{|m|=1}^{\infty} k(m,n)a_m$, in view of (17), we have $\|c\|_{p,\Psi^{1-p}} < k_{\alpha,\beta}(\lambda_1)\|a\|_{p,\Phi} < \infty$, namely, $c \in l_{p,\Psi^{1-p}}$.

Definition 2 Define a Hilbert-type operator $T : l_{p,\Phi} \to l_{p,\Psi^{1-p}}$ as follows: For any $a = \{a_m\}_{|m|=1}^{\infty} \in l_{p,\Phi}$, there exists a unique representation $c = Ta \in l_{p,\Psi^{1-p}}$. We also define the formal inner product of Ta and $b = \{b_n\}_{|n|=1}^{\infty} \in l_{q,\Psi}$ ($b_n \ge 0$) as follows:

$$(Ta, b) := \sum_{|n|=1}^{\infty} \sum_{|m|=1}^{\infty} k(m,n)a_m b_n. \tag{22}$$

Then for $a_m \ge 0$ ($|m| \in \mathbf{N}$), we may rewrite (16) and (17) as follows:

$$(Ta, b) < k_{\alpha,\beta}(\lambda_1)\|a\|_{p,\Phi}\|b\|_{q,\Psi}, \tag{23}$$
$$\|Ta\|_{p,\Psi^{1-p}} < k_{\alpha,\beta}(\lambda_1)\|a\|_{p,\Phi}. \tag{24}$$

We define the norm of operator T as follows:

$$\|T\| := \sup_{a\ (\neq\theta)\in l_{p,\Phi}} \frac{\|Ta\|_{p,\Psi^{1-p}}}{\|a\|_{p,\Phi}}. \tag{25}$$

Then $\|Ta\|_{p,\Psi^{1-p}} \leq \|T\| \cdot \|a\|_{p,\Phi}$. Since by Theorem 2, the constant factor $k_{\alpha,\beta}(\lambda_1)$ in (24) is the best possible, we have

$$\|T\| = k_{\alpha,\beta}(\lambda_1) = \frac{2(\lambda + 2\eta)\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha}{(\lambda_1+\eta)(\lambda_2+\eta)}. \tag{26}$$

Remark 1 (i) For $\eta = 0$, (16) reduces to the following inequality:

$$\begin{aligned}&\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{1}{(\max\{|m|+m\cos\alpha,|n|+n\cos\beta\})^{\lambda}}a_m b_n\\ &\quad< \frac{2\lambda}{\lambda_1\lambda_2}\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha\left[\sum_{|m|=1}^{\infty}\bigl(|m|+m\cos\alpha\bigr)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\\ &\qquad\times\left[\sum_{|n|=1}^{\infty}\bigl(|n|+n\cos\beta\bigr)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{27}$$

In particular, for $\alpha = \beta = \frac{\pi}{2}$, (27) reduces to (4). If $a_{-m} = a_m$, $b_{-n} = b_n$ ($m, n \in \mathbf{N}$), then (4) reduces to (3). Hence, (16) is an extension of (4) with multi-parameters.

(ii) For $\eta = -\lambda$, $-1 \leq \lambda_1, \lambda_2 < 0$ in (16), we have

$$\begin{aligned}&\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{1}{(\min\{|m|+m\cos\alpha,|n|+n\cos\beta\})^{\lambda}}a_m b_n\\ &\quad< \frac{2(-\lambda)}{\lambda_1\lambda_2}\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha\left[\sum_{|m|=1}^{\infty}\bigl(|m|+m\cos\alpha\bigr)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\\ &\qquad\times\left[\sum_{|n|=1}^{\infty}\bigl(|n|+n\cos\beta\bigr)^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{28}$$

In particular, for $\alpha = \beta = \frac{\pi}{2}$, we have

$$\begin{aligned}&\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{1}{(\min\{|m|,|n|\})^{\lambda}}a_m b_n\\ &\quad< \frac{2(-\lambda)}{\lambda_1\lambda_2}\left[\sum_{|m|=1}^{\infty}|m|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}.\end{aligned} \tag{29}$$

(iii) For $\lambda = 0$ in (16), we have $\lambda_2 = -\lambda_1$, $|\lambda_1| < \eta$ ($\eta > 0$) and

$$\begin{aligned}
&\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\left(\frac{\min\{|m|+m\cos\alpha, |n|+n\cos\beta\}}{\max\{|m|+m\cos\alpha, |n|+n\cos\beta\}}\right)^{\eta} a_m b_n \\
&\quad < \frac{4\eta}{\eta^2-\lambda_1^2}\csc^{\frac{2}{p}}\beta\csc^{\frac{2}{q}}\alpha\left[\sum_{|m|=1}^{\infty}\bigl(|m|+m\cos\alpha\bigr)^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}} \\
&\qquad \times\left[\sum_{|n|=1}^{\infty}\bigl(|n|+n\cos\beta\bigr)^{q(1+\lambda_1)-1}b_n^q\right]^{\frac{1}{q}}. \qquad (30)
\end{aligned}$$

In particular, for $\alpha = \beta = \frac{\pi}{2}$, we have

$$\begin{aligned}
&\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\left(\frac{\min\{|m|, |n|\}}{\max\{|m|, |n|\}}\right)^{\eta} a_m b_n \\
&\quad < \frac{4\eta}{\eta^2-\lambda_1^2}\left[\sum_{|m|=1}^{\infty}|m|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}. \qquad (31)
\end{aligned}$$

The above particular inequalities are all with the best possible constant factors.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. DX and QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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