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Approximation of functions in Besov space by deferred Cesàro mean

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Abstract

In this paper we study the degree of approximation of functions (signals) in a Besov space by trigonometric polynomials using deferred Cesàro mean. We also deduce a few corollaries of our main result and compare them with the existing results.

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Keywords: Besov space; degree of approximation; deferred Cesàro mean

1 Introduction

During the last few decades, various investigators such as Alexits [1], Chandra [2, 3], Das *et al.* [4, 5], Leindler [6, 7], Mittal *et al.* [8–10], Mohapatra and Chandra [11], Prössdorf [12], Quade [13], *etc.* have studied the approximation properties of functions in Lipschitz and Hölder spaces using different summability methods. Here it is difficult to mention all the relevant published research papers in this area. However, some of the well-known results regarding the Lipschitz and Hölder norms are presented in survey papers [14–16] in an elegant way. Besov spaces are a much more general tool in describing the smoothness properties of functions and contain a large number of fundamental spaces such as Sobolev spaces, Hölder spaces, Lipschitz spaces, *etc.* [17]. This has motivated us to work on the degree of approximation of functions in Besov spaces.

We recall a few definitions and some notation from DeVore and Lorentz [18] that are necessary before introducing our results. Let $C_{2\pi} := C[0,2\pi]$ denote the Banach space of all 2π-periodic continuous functions (signals) f defined on $[0,2\pi]$ under the supremum norm, and $L_p := L^p[0,2\pi] := \{f : [0,2\pi] \to \mathbb{R}; \int_0^{2\pi} |f(x)|^p\,dx < \infty\}$, $p \geq 1$, be the space of all 2π-periodic integrable functions. The L_p-norm of a function f is defined by

$$\|f\|_p := \begin{cases} (\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx)^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{0<x\leq 2\pi} |f(x)|, & p = \infty. \end{cases}$$

The kth-order modulus of smoothness of a signal $f \in L_p$, $0 < p \leq \infty$, is defined by

$$\omega_k(f,t)_p := \sup_{0<h\leq t} \left\| \triangle_h^k(f,\cdot) \right\|_p, \quad t > 0,$$

where $\triangle_h^k(f,x) = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} f(x+ih)$, $k \in \mathbb{N}$. For $p = \infty$, $k = 1$, and a continuous function f, the modulus of smoothness $\omega_k(f,t)_p$ reduces to the well-known modulus of continuity $\omega(f,t)$, and for $0 < p < \infty$ and $k = 1$, $\omega_k(f,t)_p$ becomes the integral modulus of continuity $\omega(f,t)_p$.

Lipschitz spaces If a signal $f \in C_{2\pi}$ and $\omega(f,t) = O(t^\alpha)$, $0 < \alpha \leq 1$, then $f \in \operatorname{Lip}\alpha$. If a signal $f \in L_p$, $0 < p < \infty$, and $\omega(f,t)_p = O(t^\alpha)$, $0 < \alpha \leq 1$, then $f \in \operatorname{Lip}(\alpha,p)$. For $p = \infty$, the class $\operatorname{Lip}(\alpha,p)$ reduces to the class $\operatorname{Lip}\alpha$.

Let $\alpha > 0$ be given, and let k denote the smallest integer $k > \alpha$, that is, $k = [\alpha] + 1$. For $f \in L_p$, if

$$\omega_k(f,t)_p = O\bigl(t^\alpha\bigr), \quad t > 0, \tag{1.1}$$

then the signal f belongs to the generalized Lipschitz space $\operatorname{Lip}^*(\alpha,p)$. Then the seminorm is $|f|_{\operatorname{Lip}^*(\alpha,p)} = \sup_{t>0}(t^{-\alpha}\omega_k(f,t)_p)$. Thus, $\operatorname{Lip}(\alpha,p) \subseteq \operatorname{Lip}^*(\alpha,p)$.

Hölder spaces For $0 < \alpha \leq 1$, let $H_\alpha = \{f \in C_{2\pi} : \omega(f,t) = O(t^\alpha)\}$. It is well known that H_α is a Banach space with norm

$$\|f\|_\alpha = \|f\|_C + \sup_{t>0}\bigl(t^{-\alpha}\omega(t)\bigr) \quad \text{for } 0 < \alpha \leq 1 \text{ and } \|f\|_0 = \|f\|_C,$$

and $H_\alpha \subseteq H_\beta \subseteq C_{2\pi}$ for $0 < \beta \leq \alpha \leq 1$. The metric induced by the norm $\|\cdot\|_\alpha$ on H_α is called the Hölder metric.

For $0 < \alpha \leq 1$ and $0 < p \leq \infty$, let $H_{\alpha,p} := H_{\alpha,p}[0,2\pi] = \{f \in L_p : \omega(f,t)_p = O(t^\alpha)\}$ with the norm $\|\cdot\|_{\alpha,p}$ defined as follows:

$$\|f\|_{\alpha,p} = \|f\|_p + \sup_{t>0}\bigl(t^{-\alpha}\omega(f,t)_p\bigr) \quad \text{for } 0 < \alpha \leq 1 \text{ and } \|f\|_{0,p} = \|f\|_p.$$

Then $H_{\alpha,p}$ is a Banach space for $p \geq 1$ and a complete p-normed space (Maddox [19], p.87) for $0 < p < 1$. Also, $H_{\alpha,p} \subseteq H_{\beta,p} \subseteq L_p$ for $0 < \beta \leq \alpha \leq 1$.

Besov space Let $\alpha > 0$ be given, and let $k = [\alpha] + 1$. For $0 < p,q \leq \infty$, the Besov space $B^\alpha_q(L_p)$ is the collection of all the signals (2π-periodic functions) $f \in L_p$ such that

$$|f|_{B^\alpha_q(L_p)} := \bigl\|\omega_k(f,\cdot)\bigr\|_{\alpha,q} = \begin{cases} (\int_0^\pi [t^{-\alpha}\omega_k(f,t)_p]^q \frac{dt}{t})^{1/q}, & 0 < q < \infty, \\ \sup_{t>0}(t^{-\alpha}\omega_k(f,t)_p), & q = \infty, \end{cases} \tag{1.2}$$

is finite (Wojtaszczyk [20], p.237). It is known that (1.2) is a seminorm if $1 \leq p,q \leq \infty$ and a quasi-seminorm in other cases (DeVore and Lorentz [18], p.55). The (quasi-)norm for $B^\alpha_q(L_p)$ is

$$\|f\|_{B^\alpha_q(L_p)} := \|f\|_p + |f|_{B^\alpha_q(L_p)} = \|f\|_p + \bigl\|\omega_k(f,\cdot)\bigr\|_{\alpha,q}. \tag{1.3}$$

Note 1

(i) In particular, for $q = \infty$, $B^\alpha_\infty(L_p) = \operatorname{Lip}^*(\alpha,p)$.
(ii) When $0 < \alpha < 1$, the space $B^\alpha_\infty(L_p)$ reduces to the space $H_{\alpha,p}$ (Das *et al.* [4]).
(iii) By taking $p = \infty = q$ and $0 < \alpha < 1$, the Besov space reduces to the space H_α (Prössdorf [12]).
(iv) In this paper, we consider the cases where $p \geq 1$ and $1 < q \leq \infty$.

2 Preliminaries

Deferred Cesàro mean (*DCM*) Let $\sum u_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. The *DCM* of sequence $\{s_n\}$ is defined by [21], p.414,

$$\mathcal{D}(a_n, b_n; s_n) = \frac{s_{a_n+1} + s_{a_n+2} + \cdots + s_{b_n}}{b_n - a_n}, \tag{2.1}$$

where $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative integers satisfying

$$a_n < b_n \quad \text{and} \quad \lim_{n\to\infty} b_n = \infty. \tag{2.2}$$

In the notation of matrix transformation,

$$\mathcal{D}(a_n, b_n; s_n) = \sum_{k=0}^{\infty} a_{n,k} s_k, \quad \text{where } a_{n,k} = \begin{cases} \frac{1}{b_n - a_n}, & a_n < k \leq b_n, \\ 0 & \text{otherwise.} \end{cases}$$

This method is regular [21] under condition (2.2). If $a_n = n-1$ and $b_n = n$, then $\mathcal{D}(a_n, b_n; s_n)$ is the identity transformation, and if $a_n = 0$ and $b_n = n$, then $\mathcal{D}(a_n, b_n; s_n)$ is the Cesàro transformation (of order 1) of s_n, that is, σ_n.

It is known that [21]

$$(C, 1) \subset \mathcal{D}(a_n, b_n) \quad \text{if and only if} \quad \frac{a_n}{b_n - a_n} = O(1).$$

Also, note that

$$\mathcal{D}(n-1, n+k-1; s_n) = \sigma_{n,k} = \left(1 + \frac{n}{k}\right)\sigma_{n+k-1} - \frac{n}{k}\sigma_{n-1}, \tag{2.3}$$

which is called the delayed arithmetic mean (*DAM*) of sequence $\{s_n\}$ [22], p.79. Some of its interesting properties can also be found in [22, 23]. Putting $k = n, 2n, 3n, \ldots$ in (2.3) gives a variety of *DAM*. For $k = 2n$, $\sigma_{n,k}$ is called the second-type *DAM* [24], p.566.

For a given signal $f \in L_p$, let

$$s_n(f; x) \equiv \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{n} u_k(f; x) \tag{2.4}$$

denote the partial sums, called trigonometric polynomials of degree (or order) n, of the first $(n+1)$ terms of the trigonometric Fourier series of f.

Let $\mathcal{D}_n(f) := \mathcal{D}(a_n, b_n, s_n(f; x))$ denote *DCM* of $s_n(f; x)$, again a trigonometric polynomial. Then by ordinary calculations [24], p.568, using (2.1) we get

$$\mathcal{D}_n(f) = \frac{1}{2\pi} \int_0^{\pi} \frac{\sin[((b_n + a_n + 2)/2)u] \sin[((b_n - a_n)/2)u]}{(b_n - a_n)\sin^2(u/2)} \bigl[f(x+u) + f(x-u)\bigr]\, du.$$

Let $b_n = (2j+1)a_n + 2j$, where $j \in \mathbb{N}$ [24], p.567. Then

$$\mathcal{D}_n(f) = \frac{1}{j(a_n+1)\pi} \int_0^{\pi} \frac{\sin[(j+1)(a_n+1)u] \sin[j(a_n+1)u]}{4\sin^2(u/2)} \bigl[f(x+u) + f(x-u)\bigr]\, du.$$

Using the identity

$$\frac{2}{j(a_n+1)\pi}\int_0^{\pi}\frac{\sin[(j+1)(a_n+1)u]\sin[j(a_n+1)u]}{4\sin^2(u/2)}\,du=1,$$

we get

$$l_n(x):=\mathcal{D}_n(f)-f(x)=\frac{1}{\pi}\int_0^{\pi}K_n^{\mathcal{D}}(u)\phi_x(u)\,du, \tag{2.5}$$

where

$$\begin{aligned}K_n^{\mathcal{D}}(u)&=\frac{2}{j(a_n+1)}\frac{\sin[(j+1)(a_n+1)u]\sin[j(a_n+1)u]}{4\sin^2(u/2)},\\ \phi_x(u)&=f(x+u)+f(x-u)-2f(x).\end{aligned}$$

We write

$$\begin{aligned}\Phi(x,t,u)&=\begin{cases}\phi_{x+t}(u)-\phi_x(u), & 0<\alpha<1,\\ \phi_{x+t}(u)+\phi_{x-t}(u)-2\phi_x(u), & 1\leq\alpha<2,\end{cases}\\ \mathcal{L}_n(x,t)&=\begin{cases}l_n(x+t)-l_n(x), & 0<\alpha<1,\\ l_n(x+t)+l_n(x-t)-2l_n(x), & 1\leq\alpha<2.\end{cases}\end{aligned}$$

By elementary computations we get

$$\mathcal{L}_n(x,t)=\frac{1}{\pi}\int_0^{\pi}K_n^{\mathcal{D}}(u)\Phi(x,t,u)\,du\quad\text{and}\quad\omega_k(l_n,t)_p=\left\|\mathcal{L}_n(\cdot,t)\right\|_p.$$

We need the following lemmas in the proof of our main result.

Lemma 1 ([25]) *Let* $1\leq p\leq\infty$ *and* $0<\alpha<2$. *If* $f\in L_p$, *then for* $0<t,u\leq\pi$,

(i) $\|\Phi(\cdot,t,u)\|_p\leq 4\omega_k(f,t)_p$,
(ii) $\|\Phi(\cdot,t,u)\|_p\leq 4\omega_k(f,u)_p$,
(iii) $\|\phi_{\cdot}(u)\|_p\leq 2\omega_k(f,u)_p$,
where $k=[\alpha]+1$.

In view of our observation [26], p.6, we replace the ordinary kernel $K_n(u)$ by the deferred kernel $K_n^{\mathcal{D}}(u)$ in Lemma 4.2 of [26].

Lemma 2 ([25]) *Let* $0\leq\beta<\alpha<2$. *If* $f\in B_q^{\alpha}(L_p)$, $p\geq 1$, $1<q<\infty$, *then*

$$\begin{aligned}\text{(i)}\quad&\int_0^{\pi}\left|K_n^{\mathcal{D}}(u)\right|\left(\int_0^{u}\frac{\|\Phi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{1/q}du\\ &\quad=O(1)\left\{\int_0^{\pi}\left(u^{\alpha-\beta}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)}du\right\}^{1-(1/q)},\\ \text{(ii)}\quad&\int_0^{\pi}\left|K_n^{\mathcal{D}}(u)\right|\left(\int_u^{\pi}\frac{\|\Phi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right)^{1/q}du\\ &\quad=O(1)\left\{\int_0^{\pi}\left(u^{\alpha-\beta+(1/q)}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)}du\right\}^{1-(1/q)}.\end{aligned}$$

The proofs run similarly to that of Lemma 2 of [25], p.22.

Lemma 3 ([25]) *Let* $0 \leq \beta < \alpha < 2$. *If* $f \in B_q^\alpha(L_p)$, $p \geq 1$, $q = \infty$, *then*

$$\sup_{0<t,u\leq\pi} \bigl(t^{-\beta} \|\Phi(\cdot,t,u)\|_p\bigr) = O\bigl(u^{\alpha-\beta}\bigr).$$

Lemma 4 *For* $0 < u < \pi$, $|K_n^{\mathcal{D}}(u)| = \begin{cases} O(a_n+1), \\ O((a_n+1)^{-1}u^{-2}). \end{cases}$

Proof In view of [24], p.568, and $j^{-1} = (a_n+1)/(b_n-a_n) = O(1)$, we get

$$\frac{\sin[(j+1)(a_n+1)u]\sin[j(a_n+1)u]}{4\sin^2(u/2)} = O\bigl((a_n+1)^2\bigr) \quad \text{for } 0 < u < \pi$$

$$\Rightarrow \quad \bigl|K_n^{\mathcal{D}}(u)\bigr| = \left|\frac{2}{j(a_n+1)} \frac{\sin[(j+1)(a_n+1)u]\sin[j(a_n+1)u]}{4\sin^2(u/2)}\right| = O(a_n+1).$$

This completes the proof of the first part of Lemma 4.

The proof of the second part follows from the facts that $|\sin ku| \leq 1$ and $|\sin(u)| \geq 2u/\pi$ for $0 \leq u \leq \pi/2$. □

3 Main result and discussion

It is well known that the theory of approximations by trigonometric polynomials, which is originated from a theorem of Weierstrass, has become an exciting interdisciplinary field of study for the past 130 years [9]. These approximations have assumed important new dimensions due to their wide applications in signal analysis [27] in general and in digital signal processing [28] in particular, in view of the classical Shannon sampling theorem [29], p.373.

Recently, Nayak *et al.* [24, 30] studied the rate of convergence of Fourier series in the generalised Hölder metric by *DCM* and second-type *DAM*. Here we study the degree of approximation of a function in the Besov space by trigonometric polynomials using *DCM*. We prove the following:

Theorem 1 *If* $0 \leq \beta < \alpha < 2$ *and* $f \in B_q^\alpha(L_p)$, $p \geq 1$, $1 < q \leq \infty$, *then*

$$\|l_n(\cdot)\|_{B_q^\beta(L_p)} = O(1) \begin{cases} (a_n+1)^{-1}, & \alpha-\beta-q^{-1} > 1, \\ (a_n+1)^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-q^{-1} < 1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta-q^{-1} = 1. \end{cases} \tag{3.1}$$

Now we deduce a few corollaries of Theorem 1 for *DAM* of second type. If $j = 1$ and $a_n = n-1$, then $\mathcal{D}(a_n, b_n; s_n)$ reduces to $\sigma_{n,2n}$, and we obtain the following:

Corollary 1 *If* $0 \leq \beta < \alpha < 2$ *and* $f \in B_q^\alpha(L_p)$, $p \geq 1$, $1 < q \leq \infty$, *then*

$$\|\sigma_{n,2n}(f;\cdot) - f(\cdot)\|_{B_q^\beta(L_p)} = O(1) \begin{cases} n^{-1}, & \alpha-\beta-q^{-1} > 1, \\ n^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-q^{-1} < 1, \\ n^{-1}[\log(n)]^{1-q^{-1}}, & \alpha-\beta-q^{-1} = 1. \end{cases}$$

We note that the estimates in Corollary 1 are similar to that of [31], p.26, for the ordinary Cesàro mean. Now in view of Note 1, we get the following:

Corollary 2 *If* $0 \leq \beta < \alpha < 2$ *and* $f \in \mathrm{Lip}^*(\alpha,p)$, $p \geq 1$, *then*

$$\|\sigma_{n,2n}(f;\cdot) - f(\cdot)\|_{B^{\beta}_{\infty}(L_p)} = O(1) \begin{cases} n^{-1}, & \alpha - \beta > 1, \\ n^{-\alpha+\beta}, & \alpha - \beta < 1, \\ n^{-1}\log n, & \alpha - \beta = 1. \end{cases}$$

We further deduce the following results from Corollary 2.

Corollary 3 ([24], p.573) *If* $0 \leq \beta < \alpha < 1$ *and* $f \in H_{\alpha,p}$, $p \geq 1$, *then*

$$\|\sigma_{n,2n}(f;\cdot) - f(\cdot)\|_{\beta,p} = O(n^{-\alpha+\beta}).$$

Taking $p = \infty$ in Corollary 3, we have the following:

Corollary 4 *If* $0 \leq \beta < \alpha < 1$ *and* $f \in H_{\alpha}$, *then*

$$\|\sigma_{n,2n}(f;\cdot) - f(\cdot)\|_{\beta} = O(n^{-\alpha+\beta}).$$

This result can be compared with that of Prössdorf [12]. For $\beta = 0$, we get the following:

Corollary 5 *If* $0 < \alpha < 1$ *and* $f \in \mathrm{Lip}(\alpha,p)$, $p \geq 1$, *then*

$$\|\sigma_{n,2n}(f;\cdot) - f(\cdot)\|_{p} = O(n^{-\alpha}).$$

Corollary 6 *If* $\alpha = p = 1$, *that is,* $f \in \mathrm{Lip}(1,1)$, *then*

$$\|\sigma_{n,2n}(f;\cdot) - f(\cdot)\|_{1} = O(n^{-1}\log n).$$

We note that the estimates in Corollaries 5 and 6 are analogous to the results of Quade [13].

4 Proof of main result

The proof of Theorem 1 is divided into two sections.

4.1 The proof for $1 < q < \infty$, $p \geq 1$, $0 \leq \beta < \alpha < 2$

Replacing α by β in (1.3), we have

$$\|l_n(\cdot)\|_{B^{\beta}_{q}(L_p)} = \|l_n(\cdot)\|_p + \|\omega_k(l_n,\cdot)\|_{\beta,q}. \tag{4.1}$$

Using the generalized Minkowski inequality [22], p.19, and Lemma 1(iii), from (2.5) we have

$$\|l_n(\cdot)\|_p \leq \frac{1}{\pi}\int_0^{\pi} \|\phi_{\cdot}(u)\|_p \left|K_n^{\mathcal{D}}(u)\right| du \leq \frac{2}{\pi}\int_0^{\pi} \omega_k(f,u)_p \left|K_n^{\mathcal{D}}(u)\right| du. \tag{4.2}$$

Using Hölder's inequality and definition (1.2) of the Besov space, we get

$$\begin{aligned}
\|l_n(\cdot)\|_p &\le \frac{2}{\pi}\left\{\int_0^\pi \left(\left|K_n^{\mathcal{D}}(u)\right|u^{\alpha+q^{-1}}\right)^{q/(q-1)}\,du\right\}^{1-q^{-1}} \left\{\int_0^\pi \left(\frac{\omega_k(f,u)_p}{u^{\alpha+q^{-1}}}\right)^q du\right\}^{q^{-1}} \\
&= O(1)\left\{\int_0^\pi \left(\left|K_n^{\mathcal{D}}(u)\right|u^{\alpha+q^{-1}}\right)^{q/(q-1)}\,du\right\}^{1-q^{-1}}, \\
\|l_n(\cdot)\|_p &= O(1)\left\{\left(\int_0^{\pi/(a_n+1)} + \int_{\pi/(a_n+1)}^{\pi}\right)\left(\left|K_n^{\mathcal{D}}(u)\right|u^{\alpha+q^{-1}}\right)^{q/(q-1)}\,du\right\}^{1-q^{-1}} \\
&:= O(1)[I+J], \quad \text{say.}
\end{aligned} \tag{4.3}$$

By the first part of Lemma 4,

$$\begin{aligned}
I &= \left\{\int_0^{\pi/(a_n+1)} \left(\left|K_n^{\mathcal{D}}(u)\right|u^{\alpha+q^{-1}}\right)^{q/(q-1)}\,du\right\}^{1-q^{-1}} \\
&= O(a_n+1)\left\{\int_0^{\pi/(a_n+1)} u^{\frac{q}{q-1}(\alpha+q^{-1})}\,du\right\}^{1-q^{-1}} \\
&= O(a_n+1)\left\{\int_0^{\pi/(a_n+1)} u^{\frac{q}{q-1}(\alpha+1)-1}\,du\right\}^{1-q^{-1}} = O\left((a_n+1)^{-\alpha}\right).
\end{aligned} \tag{4.4}$$

Now using the second part of Lemma 4, we have

$$\begin{aligned}
J &= \left\{\int_{\pi/(a_n+1)}^{\pi} \left(\left|K_n^{\mathcal{D}}(u)\right|u^{\alpha+q^{-1}}\right)^{q/(q-1)}\,du\right\}^{1-q^{-1}} \\
&= O\left((a_n+1)^{-1}\right)\left\{\int_{\pi/(a_n+1)}^{\pi} u^{\frac{q}{q-1}(\alpha+q^{-1}-2)}\,du\right\}^{1-q^{-1}} \\
&= O\left((a_n+1)^{-1}\right)\left\{\int_{\pi/(a_n+1)}^{\pi} u^{\frac{q}{q-1}(\alpha-1)-1}\,du\right\}^{1-q^{-1}} \\
&= O(1)\begin{cases} (a_n+1)^{-1}, & \alpha>1, \\ (a_n+1)^{-\alpha}, & \alpha<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha=1. \end{cases}
\end{aligned} \tag{4.5}$$

Thus, combining (4.3)-(4.5), we have

$$\|l_n(\cdot)\|_p = O(1)\begin{cases} (a_n+1)^{-1}, & \alpha>1, \\ (a_n+1)^{-\alpha}, & \alpha<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha=1. \end{cases} \tag{4.6}$$

By repeated application of the generalized Minkowski inequality as in [26], p.9, and Lemma 2 for the second term on the right-hand side of (4.1) we have

$$\begin{aligned}
\|\omega_k(l_n,\cdot)\|_{\beta,q} &= \left\{\int_0^\pi \left(\frac{\omega_k(l_n,t)_p}{t^\beta}\right)^q \frac{dt}{t}\right\}^{q^{-1}} = \left\{\int_0^\pi \left(\frac{\|\mathcal{L}_n(\cdot,t)\|_p}{t^\beta}\right)^q \frac{dt}{t}\right\}^{q^{-1}} \\
&\le \frac{1}{\pi}\int_0^\pi \left|K_n^{\mathcal{D}}(u)\right|du\left\{\int_0^u \frac{\|\Phi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{q^{-1}}
\end{aligned}$$

$$\begin{aligned}
&+\frac{1}{\pi}\int_0^\pi \left|K_n^{\mathcal{D}}(u)\right| du\left\{\int_u^\pi \frac{\|\Phi(\cdot,t,u)\|_p^q}{t^{\beta q}}\frac{dt}{t}\right\}^{q^{-1}}\\
&= O(1)\left\{\int_0^\pi \left(u^{\alpha-\beta}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)} du\right\}^{1-(1/q)}\\
&+ O(1)\left\{\int_0^\pi \left(u^{\alpha-\beta+(1/q)}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)} du\right\}^{1-(1/q)}\\
:=\ & O(1)(I_1+J_1), \quad \text{say,}
\end{aligned} \tag{4.7}$$

since $(x+y)^r \le x^r + y^r$ for positive x, y and $0 < r \le 1$ (for $r = 1 - q^{-1} < 1$). Now

$$\begin{aligned}
I_1 &= \left\{\int_0^\pi \left(u^{\alpha-\beta}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)} du\right\}^{1-q^{-1}}\\
&\le \left\{\int_0^{\pi/(a_n+1)} \left(u^{\alpha-\beta}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)} du\right\}^{1-q^{-1}} + \left\{\int_{\pi/(a_n+1)}^\pi \left(u^{\alpha-\beta}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)} du\right\}^{1-q^{-1}}\\
&:= I_{11} + I_{12}, \quad \text{say.}
\end{aligned} \tag{4.8}$$

Using the first part of Lemma 4, we have

$$\begin{aligned}
I_{11} &= O(a_n+1)\left\{\int_0^{\pi/(a_n+1)} u^{\frac{q}{q-1}(\alpha-\beta)} du\right\}^{1-q^{-1}}\\
&= O(a_n+1)\left\{\int_0^{\pi/(a_n+1)} u^{\frac{q}{q-1}(\alpha-\beta+1-(1/q))-1} du\right\}^{1-q^{-1}}\\
&= O\bigl((a_n+1)^{-\alpha+\beta+(1/q)}\bigr).
\end{aligned} \tag{4.9}$$

By the second part of Lemma 4 we have

$$\begin{aligned}
I_{12} &= O\bigl((a_n+1)^{-1}\bigr)\left\{\int_{\pi/(a_n+1)}^\pi u^{\frac{q}{q-1}(\alpha-\beta-2)} du\right\}^{1-q^{-1}}\\
&= O\bigl((a_n+1)^{-1}\bigr)\left\{\int_{\pi/(a_n+1)}^\pi u^{\frac{q}{q-1}(\alpha-\beta-(1/q)-1)-1} du\right\}^{1-q^{-1}}\\
&= O(1)\begin{cases} (a_n+1)^{-1}, & \alpha-\beta-q^{-1} > 1,\\ (a_n+1)^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-q^{-1} < 1,\\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta-q^{-1} = 1.\end{cases}
\end{aligned} \tag{4.10}$$

Now collecting (4.8)-(4.10) and using a similar argument as in (4.6), we have

$$I_1 = O(1)\begin{cases} (a_n+1)^{-1}, & \alpha-\beta-q^{-1} > 1,\\ (a_n+1)^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-q^{-1} < 1,\\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta-q^{-1} = 1.\end{cases} \tag{4.11}$$

Using the earlier argument as in (4.8), we have

$$J_1 = \left\{\int_0^\pi \left(u^{\alpha-\beta+(1/q)}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)} du\right\}^{1-(1/q)},$$

$$\begin{aligned}
J_1 &\leq \left\{\int_0^{\pi/(a_n+1)} \left(u^{\alpha-\beta+(1/q)}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)}\,du\right\}^{1-q^{-1}} \\
&\quad + \left\{\int_{\pi/(a_n+1)}^{\pi} \left(u^{\alpha-\beta+(1/q)}\left|K_n^{\mathcal{D}}(u)\right|\right)^{q/(q-1)}\,du\right\}^{1-q^{-1}} \\
&:= J_{11} + J_{12}, \quad \text{say.}
\end{aligned} \tag{4.12}$$

By the first part of Lemma 4 we get

$$\begin{aligned}
J_{11} &= O(a_n+1)\left\{\int_0^{\pi/(n+1)} u^{\frac{q}{q-1}(\alpha-\beta+(1/q))}\,du\right\}^{1-q^{-1}} \\
&= O(a_n+1)\left\{\int_0^{\pi/(a_n+1)} u^{\frac{q}{q-1}(\alpha-\beta+1)-1}\,du\right\}^{1-q^{-1}} \\
&= O\left(\frac{1}{(a_n+1)^{\alpha-\beta}}\right).
\end{aligned} \tag{4.13}$$

Using Lemma 4 and computing similarly as in I_{12}, we have

$$\begin{aligned}
J_{12} &= O\big((a_n+1)^{-1}\big)\left\{\int_{\pi/(a_n+1)}^{\pi} u^{\frac{q}{q-1}(\alpha-\beta+(1/q)-2)}\,du\right\}^{1-q^{-1}} \\
&= O\big((a_n+1)^{-1}\big)\left\{\int_{\pi/(a_n+1)}^{\pi} u^{\frac{q}{q-1}(\alpha-\beta-1)-1}\,du\right\}^{1-q^{-1}} \\
&= O(1)\begin{cases} (a_n+1)^{-1}, & \alpha-\beta>1, \\ (a_n+1)^{-\alpha+\beta}, & \alpha-\beta<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta=1. \end{cases}
\end{aligned} \tag{4.14}$$

Now collecting (4.12)-(4.14) and using the earlier argument as in I_1, we have

$$J_1 = O(1)\begin{cases} (a_n+1)^{-1}, & \alpha-\beta>1, \\ (a_n+1)^{-\alpha+\beta}, & \alpha-\beta<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta=1. \end{cases} \tag{4.15}$$

Combining (4.7), (4.11), and (4.15), we get

$$\left\|\omega_k(l_n,\cdot)\right\|_{\beta,q} = O(1)\begin{cases} (a_n+1)^{-1}, & \alpha-\beta-q^{-1}>1, \\ (a_n+1)^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-q^{-1}<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta-q^{-1}=1. \end{cases} \tag{4.16}$$

From (4.1), (4.6), and (4.16) we have

$$\left\|l_n(\cdot)\right\|_{B_q^{\beta}(L_p)} = O(1)\begin{cases} (a_n+1)^{-1}, & \alpha-\beta-q^{-1}>1, \\ (a_n+1)^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-q^{-1}<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta-q^{-1}=1. \end{cases} \tag{4.17}$$

This completes the proof of our Theorem 1 for $p \geq 1$, $1 < q < \infty$, and $0 \leq \beta < \alpha < 2$.

4.2 The proof for $q = \infty$ and $0 \leq \beta < \alpha < 2$

$$\left\| l_n(\cdot) \right\|_{B^{\beta}_{\infty}(L_p)} = \left\| l_n(\cdot) \right\|_p + \left\| \omega_k(l_n, \cdot) \right\|_{\beta,\infty}. \tag{4.18}$$

Using condition (1.1) in (4.2), we have

$$\begin{aligned} \left\| l_n(\cdot) \right\|_p &\leq \frac{2}{\pi} \int_0^{\pi} \omega_k(f,u)_p \left| K_n^{\mathcal{D}}(u) \right| du \\ &= O(1) \left\{ \int_0^{\pi/(a_n+1)} u^{\alpha} \left| K_n^{\mathcal{D}}(u) \right| du + \int_{\pi/(a_n+1)}^{\pi} u^{\alpha} \left| K_n^{\mathcal{D}}(u) \right| du \right\} \\ &:= O(1)[I_2 + J_2], \quad \text{say.} \end{aligned} \tag{4.19}$$

Using Lemma 4, we get

$$I_2 = \int_0^{\pi/(a_n+1)} u^{\alpha} \left| K_n^{\mathcal{D}}(u) \right| du = O(a_n+1) \int_0^{\pi/(a_n+1)} u^{\alpha}\, du = O\bigl((a_n+1)^{-\alpha}\bigr), \tag{4.20}$$

$$\begin{aligned} J_2 &= \int_{\pi/(n+1)}^{\pi} u^{\alpha} \left| K_n^{\mathcal{D}}(u) \right| du = O\bigl((a_n+1)^{-1}\bigr) \int_{\pi/(n+1)}^{\pi} u^{\alpha-2}\, du \\ &= O(1) \begin{cases} (a_n+1)^{-1}, & \alpha > 1, \\ (a_n+1)^{-\alpha}, & \alpha < 1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha = 1. \end{cases} \end{aligned} \tag{4.21}$$

Combining (4.19)-(4.21), we get

$$\left\| l_n(\cdot) \right\|_p = O(1) \begin{cases} (a_n+1)^{-1}, & \alpha > 1, \\ (a_n+1)^{-\alpha}, & \alpha < 1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha = 1. \end{cases} \tag{4.22}$$

Using the generalized Minkowski inequality and Lemma 3, we have

$$\begin{aligned} \left\| \omega_k(l_n, \cdot) \right\|_{\beta,\infty} &= \sup_{t>0} \bigl(t^{-\beta} \omega_k(l_n, t)_p \bigr) = \sup_{t>0} \bigl(t^{-\beta} \left\| \mathcal{L}_n(\cdot, t) \right\|_p \bigr) \\ &= \sup_{t>0} \left[t^{-\beta} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| \frac{1}{\pi} \int_0^{\pi} K_n^{\mathcal{D}}(u) \Phi(x,t,u)\, du \right|^p dx \right)^{1/p} \right] \\ &\leq \sup_{t>0} \left[\frac{t^{-\beta}}{\pi} \left(\frac{1}{2\pi} \right)^{1/p} \int_0^{\pi} \left\{ \int_0^{2\pi} \left| K_n^{\mathcal{D}}(u) \right|^p \left| \Phi(x,t,u) \right|^p dx \right\}^{1/p} du \right] \\ &= \sup_{t>0} \left[\frac{t^{-\beta}}{\pi} \int_0^{\pi} \left\| \Phi(\cdot,t,u) \right\|_p \left| K_n^{\mathcal{D}}(u) \right| du \right] \\ &= \frac{1}{\pi} \int_0^{\pi} \Bigl(\sup_{t>0} t^{-\beta} \left\| \Phi(\cdot,t,u) \right\|_p \Bigr) \left| K_n^{\mathcal{D}}(u) \right| du \\ &= O(1) \int_0^{\pi} u^{\alpha-\beta} \left| K_n(u) \right| du \\ &= O(1) \left[\int_0^{\pi/(a_n+1)} u^{\alpha-\beta} \left| K_n^{\mathcal{D}}(u) \right| du + \int_{\pi/(a_n+1)}^{\pi} u^{\alpha-\beta} \left| K_n^{\mathcal{D}}(u) \right| du \right] \\ &:= O(1)[I_3 + J_3], \quad \text{say.} \end{aligned} \tag{4.23}$$

Using Lemma 4, we get

$$I_3 = \int_0^{\pi/(a_n+1)} u^{\alpha-\beta} \bigl|K_n^{\mathcal{D}}(u)\bigr|\,du = O\bigl((a_n+1)^{\beta-\alpha}\bigr), \tag{4.24}$$

$$\begin{aligned} J_3 &= \int_{\pi/(a_n+1)}^{\pi} u^{\alpha-\beta} \bigl|K_n^{\mathcal{D}}(u)\bigr|\,du = O\bigl((a_n+1)^{-1}\bigr) \int_{\pi/(a_n+1)}^{\pi} u^{\alpha-\beta-2}\,du \\ &= O(1) \begin{cases} (a_n+1)^{-1}, & \alpha-\beta>1, \\ (a_n+1)^{-\alpha+\beta}, & \alpha-\beta<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta=1. \end{cases} \end{aligned} \tag{4.25}$$

Combining (4.23)-(4.25), we have

$$\bigl\|\omega_k(l_n,\cdot)\bigr\|_{\beta,\infty} = O(1) \begin{cases} (a_n+1)^{-1}, & \alpha-\beta>1, \\ (a_n+1)^{-\alpha+\beta}, & \alpha-\beta<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta=1. \end{cases} \tag{4.26}$$

From (4.18), (4.22), and (4.26) we have

$$\bigl\|l_n(\cdot)\bigr\|_{B^{\beta}_{\infty}(L_p)} = O(1) \begin{cases} (a_n+1)^{-1}, & \alpha-\beta-q^{-1}>1, \\ (a_n+1)^{-\alpha+\beta+q^{-1}}, & \alpha-\beta-q^{-1}<1, \\ (a_n+1)^{-1}[\log(a_n+1)]^{1-q^{-1}}, & \alpha-\beta-q^{-1}=1. \end{cases} \tag{4.27}$$

This completes the proof of Theorem 1 for $q=\infty$.

Combining Sections 4.1 and 4.2 completes the proof of Theorem 1.

5 Conclusions

It is known that Besov spaces serve as generalizations of more elementary function spaces and are effective at measuring the smoothness properties of functions. As mentioned by DeVore and Popov [32], p.397,

> "There are two definitions of Besov spaces that are currently in use. One uses the Fourier transform, and the second uses the modulus of smoothness of a function f. These two definitions are equivalent only under certain restrictions on the parameters. The Besov spaces defined by the modulus of smoothness occur more naturally in many areas of analysis including approximation theory."

In this paper we compute the error estimates of a function f in a Besov space by *DCM* of partial sums of the trigonometric Fourier series of f. We also deduce a few corollaries of our main result for the second-type *DAM* in a Besov space and other function spaces such as Lipschiz and Hölder spaces as particular cases and compare these results with earlier known results.

As in [24], p.574, we have used more general trigonometric polynomials (*i.e.*, the second-type *DAM* $\sigma_{n,2n}$) in Corollaries 1-6; however, we can obtain similar estimates using other types of *DAM* such as $D(n-1,(2j+1)n-1)$ (or $\sigma_{n,2jn}$).

Remark 1 Recently, Değer and Küçükaslan [33] generalized the concept of *DCM* and studied approximation of a function using deferred Nörlund mean/deferred Riesz mean in Hölder metric, which may be the future interest of investigators in this direction.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

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