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Contractions without non-trivial invariant subspaces satisfying a positivity condition

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Abstract

An operator $A \in B(\mathcal{H})$, the algebra of bounded linear transformations on a complex infinite dimensional Hilbert space $\mathcal{H}$, belongs to class $\mathcal{A}(n)$ (resp., $\mathcal{A}(*-n)$) if $|A|^2 \leq |A^{n+1}|^{\frac{2}{n+1}}$ (resp., $|A^*|^2 \leq |A^{n+1}|^{\frac{2}{n+1}}$) for some integer $n \geq 1$, and an operator $A \in B(\mathcal{H})$ is called n-paranormal, denoted $A \in \mathcal{P}(n)$ (resp., $*-n$-paranormal, denoted $A \in \mathcal{P}(*-n)$) if $\|Ax\|^{n+1} \leq \|A^{n+1}x\|\|x\|^n$ (resp., $\|A^*x\|^{n+1} \leq \|A^{n+1}x\|\|x\|^n$) for some integer $n \geq 1$ and all $x \in \mathcal{H}$. In this paper, we prove that if $A \in \{\mathcal{A}(n) \cup \mathcal{P}(n)\}$ (resp., $A \in \{\mathcal{A}(*-n) \cup \mathcal{P}(*-n)\}$) is a contraction without a non-trivial invariant subspace, then A, $|A^{n+1}|^{\frac{2}{n+1}} - |A|^2$ and $|A^{n+1}|^2 - \frac{n+1}{n}|A|^2 + 1$ (resp., A, $|A^{n+1}|^{\frac{2}{n+1}} - |A^*|^2$ and $|A^{n+2}|^2 - \frac{n+1}{n}|A|^2 + 1 \geq 0$) are proper contractions.

MSC: 47B20; 47A10

Keywords: class $\mathcal{A}(n)$ operator; class $\mathcal{A}(*-n)$ operator; $\mathcal{P}(n)$ operator; class $\mathcal{P}(*-n)$ operator; contraction; proper contraction; strongly stable

1 Introduction

Let $B(\mathcal{H})$ denote the algebra of bounded linear operators on an infinite dimensional complex Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. An operator $A \in B(\mathcal{H})$ is a class $\mathcal{A}$ (resp., $*$-class $\mathcal{A}$) operator, [1] and [2], if $|A|^2 \leq |A^2|$ (resp., $|A^*|^2 \leq |A^2|$). As a generalization of class $\mathcal{A}$, Yuan and Gao [3] have introduced the class of $\mathcal{A}(n)$ operators as follows: An operator $A \in B(\mathcal{H})$ belongs to class $\mathcal{A}(n)$ (resp., $\mathcal{A}(*-n)$) if $|A|^2 \leq |A^{n+1}|^{\frac{2}{n+1}}$ (resp., $|A^*|^2 \leq |A^{n+1}|^{\frac{2}{n+1}}$) for some integer $n \geq 1$. An operator $A \in B(\mathcal{H})$ is called n-paranormal, denoted $A \in \mathcal{P}(n)$ (resp., $*-n$-paranormal, denoted $A \in \mathcal{P}(*-n)$) if $\|Ax\|^{n+1} \leq \|A^{n+1}x\|\|x\|^n$ (resp., $\|A^*x\|^{n+1} \leq \|A^{n+1}x\|\|x\|^n$) for some integer $n \geq 1$ and all $x \in \mathcal{H}$ is a generalization of the class of paranormal (resp., $*$-paranormal) operators (see [4]).

Recall [5] that a contraction $A \in B(\mathcal{H})$ (*i.e.*, an operator $A \in B(\mathcal{H})$ such that $\|A\| \leq 1$; equivalently, such that $\|Ax\| \leq \|x\|$ for every $x \in \mathcal{H}$) is said to be a proper contraction if $\|Ax\| < \|x\|$ for every non-zero $x \in \mathcal{H}$. A strict contraction (*i.e.*, a contraction A such that $\|A\| < 1$) is a proper contraction, but a proper contraction is not necessarily a strict contraction. Kubrusly and Levan [5] have proved that if a hyponormal ($\|Ax\| \geq \|A^*x\|$) contraction A has no non-trivial invariant subspace, then

(a) A is a proper contraction and
(b) its self-commutator $[A^*,A] = A^*A - AA^*$ is a strict contraction.

Class $\mathcal{A}$ operators A satisfy the property that if A is a contraction with no non-trivial invariant subspace, then the non-negative operator $D_0 = |A^2| - |A|^2$ is a proper contrac-

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tion, and hence of the class C_{00} of contractions [6]. Since $\langle |A^2|x,x\rangle \leq \langle |A^2|^2x,x\rangle^{\frac{1}{2}}\|x\|$ (by the Hölder-McCarthy inequality: if $T \in B(\mathcal{H})$ is a non-negative (*i.e.*, ≥ 0) operator, then $\langle T^\lambda x,x\rangle \leq \langle Tx,x\rangle^\lambda \|x\|^{2(1-\lambda)}$ for all $0 < \lambda \leq 1$), if $A \in \mathcal{A}$, then $\|Ax\|^2 \leq \|A^2x\|\|x\|$ for all $x \in \mathcal{H}$. Thus class $\mathcal{A}$ operators are paranormal. Paranormal operators $A \in B(\mathcal{H})$ are characterized by the positivity condition $|A^2|^2 - 2\lambda|A|^2 + \lambda^2 \geq 0$ for all real $\lambda > 0$. Choosing $\lambda = 1$, it follows that class $\mathcal{A}$ (also, paranormal) operators A satisfy the positivity property $D_1 = |A^2|^2 - 2|A|^2 + 1 \geq 0$. If we now choose A to be a contraction without a non-trivial invariant subspace, then D_1 (along with A) is a proper contraction [5].

Positivity properties of the type satisfied by class $\mathcal{A}$ operators are satisfied by other classes of Hilbert space operators, some of them generalizations of the class $\mathcal{A}$ and others distinct from class $\mathcal{A}$.

It is easily seen (we prove so in Section 2) that class $\mathcal{A}(n)$ and class $\mathcal{P}(n)$ satisfy the positivity property that

$$\left|A^{n+1}\right|^2 - \frac{n+1}{n}|A|^2 + 1 \geq 0$$

and class $\mathcal{A}(*-n)$ and class $\mathcal{P}(*-n)$ satisfy the positivity properties

$$\left|A^{n+1}\right|^2 - \frac{n+1}{n}\left|A^*\right|^2 + 1 \geq 0$$

and

$$\left|A^{n+2}\right|^2 - \frac{n+1}{n}|A|^2 + 1 \geq 0.$$

We prove in the following that there is a method as regards the 'proper contraction property satisfied by the operators D_0 and D_1 above'. We prove that if an $A \in \{\mathcal{A}(n) \cup \mathcal{P}(n)\}$ (resp., $A \in \{\mathcal{A}(*-n) \cup \mathcal{P}(*-n)\}$) is a contraction without a non-trivial invariant subspace, then A, $|A^{n+1}|^{\frac{2}{n+1}} - |A|^2$ and $|A^{n+1}|^2 - \frac{n+1}{n}|A|^2 + 1$ (resp., A, $|A^{n+1}|^{\frac{2}{n+1}} - |A^*|^2$ and $|A^{n+1}|^2 - \frac{n+1}{n}|A|^2 + 1 \geq 0$) are proper contractions.

2 Results

We begin with the following lemma. Let $A \in B(\mathcal{H})$.

Lemma 2.1

(i) $A \in \mathcal{P}(n) \cup \mathcal{P}(*-n)$ *if and only if*

$$\left|A^{n+1}\right|^2 - (n+1)\lambda^n|B|^2 + n\lambda^{n+1} \geq 0, \quad \textit{all } \lambda > 0,$$

where $B = A$ *if* $A \in \mathcal{P}(n)$ *and* $B = A^*$ *if* $A \in \mathcal{P}(*-n)$.

(ii) *If* $A \in \mathcal{P}(*-n)$, *then* $A \in \mathcal{P}(n+1)$.

Proof (i) If we let $\alpha = \|A^{n+1}x\|^2$ and $\beta = \beta_1 = \cdots = \beta_n = \lambda^{n+1}\|x\|^2$ for real $\lambda > 0$, then the generalized arithmetic-geometric inequality $\alpha\beta_1\beta_2\cdots\beta_n \leq (\frac{\alpha+\beta_1+\beta_2+\cdots+\beta_n}{n+1})^{n+1}$ [7], p.17, says that

$$\lambda^{n(n+1)}\left\|A^{n+1}x\right\|^2\|x\|^{2n} \leq \left(\frac{\|A^{n+1}x\|^2 + n\lambda^{n+1}\|x\|^2}{n+1}\right)^{n+1}.$$

By definition $A \in \{\mathcal{P}(n) \cup \mathcal{P}(* - n)\}$ if and only if

$$\|Bx\|^{n+1} \leq \|A^{n+1}x\| \|x\|^n$$

(where $B = A$ if $A \in \mathcal{P}(n)$ and $B = A^*$ if $A \in \mathcal{P}(* - n)$). Thus

$$(n+1)\lambda^n \|Bx\|^2 \leq (n+1)\lambda^n \|A^{n+1}x\|^{\frac{2}{n+1}} \|x\|^{\frac{2n}{n+1}} \leq \|A^{n+1}x\|^2 + n\lambda^{n+1}\|x\|^2$$

for all $\lambda > 0$ and all $x \in \mathcal{H}$. Equivalently, if $A \in \mathcal{P}(n) \cup \mathcal{P}(* - n)$, then $|A^{n+1}|^2 - (n+1)\lambda^n |B|^2 + n\lambda^{n+1} \geq 0$ for all $\lambda > 0$.

To see the sufficiency, let $\lambda \to 0$ in

$$\|Bx\|^2 \leq \frac{1}{(n+1)\lambda^n} \|A^{n+1}x\|^2 + \frac{n}{n+1}\lambda \|x\|^2, \quad x \in \mathcal{H},$$

if $\|A^{n+1}x\| = 0$ (when it is seen that $\|Bx\| = 0$) and let $\lambda = (\frac{\|A^{n+1}x\|}{\|x\|})^{\frac{2}{n+1}}$ otherwise (when it follows that $\|Bx\| \leq (\|A^{n+1}x\| \|x\|^n)^{\frac{1}{n+1}}$, $x \in \mathcal{H}$).

(ii) If $A \in \mathcal{P}(* - n)$, then, for all $x \in \mathcal{H}$,

$$\begin{aligned} \|Ax\|^{2(n+1)} &= \langle A^*Ax, x\rangle^{n+1} \leq \|A^*Ax\|^{n+1} \|x\|^{n+1} \leq \|A^{n+2}x\| \|Ax\|^n \|x\|^{n+1} \\ &\implies \quad \|Ax\|^{n+2} \leq \|A^{n+2}x\| \|x\|^{n+1}, \end{aligned}$$

i.e., $A \in \mathcal{P}(n+1)$. □

It is immediate from Lemma 2.1 that the operators $A \in \mathcal{P}(n)$ (resp., $A \in \mathcal{P}(* - n)$) satisfy the positivity property, henceforth denoted property $Q_\lambda(n)$ (resp., property $Q_\lambda(* - n)$), that

$$|A^{n+1}|^2 - (n+1)\lambda^n |B|^2 + n\lambda^{n+1} \geq 0$$

for all $\lambda > 0$. (Here, as above, $B = A$ if $A \in \mathcal{P}(n)$ and $B = A^*$ if $A \in \mathcal{P}(* - n)$.) We prove that the operators $A \in \mathcal{A}(n)$ (resp., $A \in \mathcal{A}(* - n)$) also satisfy property $Q_\lambda(n)$ (resp., $Q_\lambda(* - n)$). The following lemma, the Hölder-McCarthy inequality, is well known.

Lemma 2.2 *If $A \in B(\mathcal{H})$, then the following properties hold*:

(1) $\langle A^\lambda x, x\rangle \geq \langle Ax, x\rangle^\lambda \|x\|^{2(1-\lambda)}$ *for any $\lambda > 1$ and any vector x.*

(2) $\langle A^\lambda x, x\rangle \leq \langle Ax, x\rangle^\lambda \|x\|^{2(1-\lambda)}$ *for any $\lambda \in (0, 1]$ and any vector x.*

Lemma 2.3 *The operators $A \in \mathcal{A}(n)$ (resp., $A \in \mathcal{A}(* - n)$) satisfy property $Q_\lambda(n)$ (resp., property $Q_\lambda(* - n)$).*

Proof The proof is a simple consequence of an application of Lemma 2.2: If $A \in \mathcal{A}(n) \cup \mathcal{A}(* - n)$ and the operator B is defined as above, then, for all $x \in \mathcal{H}$,

$$\langle |B|^2 x, x\rangle \leq \langle |A^{n+1}|^{\frac{2}{n+1}} x, x\rangle \leq \langle |A^{n+1}|^2 x, x\rangle^{\frac{1}{n+1}} \|x\|^{\frac{2n}{n+1}},$$

i.e., $A \in \mathcal{A}(n)$ implies $A \in \mathcal{P}(n)$ and $A \in \mathcal{A}(* - n)$ implies $A \in \mathcal{P}(* - n)$. Consequently the operators $A \in \mathcal{A}(n)$ satisfy property $Q_\lambda(n)$ and the operators $A \in \mathcal{A}(* - n)$ satisfy property $Q_\lambda(* - n)$. □

It is clear from Lemma 2.2 that the operators $A \in \mathcal{P}(*-n)$ satisfy property $Q_\lambda(n+1)$ (*i.e.*, if $A \in \mathcal{P}(*-n)$, then $|A^{n+2}|^2 - (n+2)\lambda^{n+1}|B|^2 + (n+1)\lambda^{n+2} \geq 0$ for all $\lambda > 0$).

Given an operator $A \in B(\mathcal{H})$, let B denote either A or A^* (exclusive 'or'), and let

$$M = \{x \in \mathcal{H} : \|Bx\| = \|B\|\|x\| = \|A\|\|x\|\}.$$

Lemma 2.4 *If $A \in \{\mathcal{P}(n) \cup \mathcal{P}(*-n)\}$, then M is a closed subspace of $\mathcal{H}$ such that $A(M) \subseteq M$.*

Proof M being the null space of the operator $|B|^2 - \|A\|^2$ is a closed subspace of $\mathcal{H}$. Define the operator B as before by letting $B = A$ whenever $A \in \mathcal{P}(n)$ and $B = A^*$ whenever $A \in \mathcal{P}(*-n)$. Let $x \in M$, and let $A \in \{\mathcal{P}(n) \cup \mathcal{P}(*-n)\}$. Then

$$\begin{aligned}\|Bx\|^2 &\leq \langle |A^{n+1}|^2 x, x\rangle^{\frac{1}{n+1}} \|x\|^{\frac{2n}{n+1}} = \|A^{n+1}x\|^{\frac{2}{n+1}} \|x\|^{\frac{2n}{n+1}} \\ &\leq \|A\|^2\|x\|^2 = \|Bx\|^2,\end{aligned}$$

and hence

$$\begin{aligned}&\|Bx\|^2 = \|A^{n+1}x\|^{\frac{2}{n+1}} \|x\|^{\frac{2n}{n+1}} = \|A\|^2\|x\|^2 \\ \Longleftrightarrow \quad &\|Bx\|^{n+1} = \|A^{n+1}x\|\|x\|^n = \|A\|^{n+1}\|x\|^{n+1}.\end{aligned}$$

But then (for all $x \in M$ and $A \in \mathcal{P}(n) \cup \mathcal{P}(*-n)$)

$$\begin{aligned}\|Bx\|^{n+1} &= \|A^{n+1}x\|\|x\|^n \\ &\leq \|A\|\|A^n x\|\|x\|^n = \|A^n x\|\|x\|^{n-1}\|Bx\| \\ &\cdots \\ &\leq \|Ax\|\|Bx\|^n \leq \|Bx\|^{n+1},\end{aligned}$$

which implies

$$\|Bx\|^m = \|A^m x\|\|x\|^{m-1}, \quad \text{all integers } 1 \leq m \leq n+1.$$

In particular,

$$\|Bx\| = \|Ax\| \quad \text{and} \quad \|A^2x\|\|x\| = \|Bx\|^2.$$

Now if $A \in \mathcal{P}(n)$ (so that $B = A$), then

$$\begin{aligned}&\|A^2x\|\|x\| = \|Ax\|^2 = \|A\|^2\|x\|^2 = \|A\|\|Ax\|\|x\| \\ \Longrightarrow \quad &\|A^2x\| = \|A\|\|Ax\| \quad \Longrightarrow \quad A(M) \subseteq M,\end{aligned}$$

and if $A \in \mathcal{P}(*-n)$ (so that $B = A^*$), then (using Lemma 2.1(ii))

$$\begin{aligned}&\|Ax\| = \|Bx\| = \|A^*x\| = \|A\|\|x\| \\ \Longrightarrow \quad &\{x \in \mathcal{H} : \|Ax\| = \|A\|\|x\|\} = \{x \in \mathcal{H} : \|A^*x\| = \|A\|\|x\|\},\end{aligned}$$

and hence $A(M) \subseteq M$. □

Corollary 2.5 *If a contraction $A \in \{\mathcal{P}(n) \cup \mathcal{P}(* - n)\}$ has no non-trivial invariant subspace, then A is a proper contraction.*

Proof If $A \in \{\mathcal{P}(n) \cup \mathcal{P}(* - n)\}$, then $AM \subseteq M$. Now if A is not a proper contraction, then (it is not a strict contraction, hence) $\|A\| = \|A^*\| = 1$. Hence, since A has no non-trivial invariant subspace, $M = \{0\}$ (for the reason that if $M = \mathcal{H}$, then either A or A^* is an isometry and isometries have non-trivial invariant subspaces). Consequently, $\|Ax\| \leq \|A\|\|x\| < \|x\|$ for all $x \in \mathcal{H}$, *i.e.*, A is a proper contraction. □

We say in the following that an operator $A \in B(\mathcal{H})$ satisfies the positivity condition:

(D_1) if the operator $D_1 = |A^{n+1}|^{\frac{2}{n+1}} - |A|^2 \geq 0$,
(D_2) if the operator $D_2 = |A^{n+1}|^{\frac{2}{n+1}} - |A^*|^2 \geq 0$, and
(D_3) if the operator $D_3 = |A^{n+1}|^2 - \frac{n+1}{n}|A|^2 + 1 \geq 0$.

It is evident from the definition of operators $A \in \mathcal{A}(n)$ (resp., $A \in \mathcal{A}(* - n)$) that $\mathcal{A}(n)$ operators satisfy condition (D_1) (resp., $\mathcal{A}(* - n)$ operators satisfy condition (D_2)). If we choose $0 < \lambda = \frac{1}{\sqrt[n]{n}}$ in $Q_\lambda(n)$, then (since $|A^{n+1}|^2 - \frac{n+1}{n}|A|^2 + \frac{1}{\sqrt[n]{n}} \leq |A^{n+1}|^2 - \frac{n+1}{n}|A|^2 + 1$ for all integers $n \geq 1$) operators $A \in \mathcal{P}(n)$ are seen to satisfy positivity condition (D_3). Again, if we choose $0 < \lambda = \sqrt[n+1]{\frac{n+1}{n(n+2)}}$, then the fact that $\mathcal{P}(* - n)$ operators satisfy property $Q_\lambda(n+1)$ implies that $|A^{n+2}|^2 - \frac{n+1}{n}|A|^2 + 1 \geq |A^{n+2}|^2 - \frac{n+1}{n}|A|^2 + (n+1)(\frac{n+1}{n(n+2)})^{\frac{n+2}{n+1}} \geq 0$; in particular, $\mathcal{P}(* - n)$ contractions A satisfy positivity condition (D_3).

Remark 2.6 An interesting class of operators, which contains many a familiar class of operators (such as p-hyponormal operators, $0 < p \leq 1$, w-hyponormal operators and class $\mathcal{A}$ operators) considered by a large number of authors in the recent past, is that of the class $\mathcal{A}(s,t)$ operators $A \in B(\mathcal{H})$ defined by the positivity condition $|A^*|^{2t} \leq (|A^*|^t|A|^{2s}|A^*|^t)^{\frac{t}{s+t}}$, $0 < s, t$ [8]. Class $\mathcal{A}(s,t)$ operators satisfy the property that $A \in \mathcal{A}(s,t)$ implies $A \in \mathcal{A}(\alpha, \beta)$ for every $\alpha \geq s$ and $\beta \geq t$ ([8], Theorem 4). Hence, if $0 < s, t \leq 1$, then every $A \in \mathcal{A}(s,t)$ is an $\mathcal{A}(1,1) = \mathcal{A}$ operator ([8], Theorem 3). Consequently the operators $A \in \mathcal{A}(s,t)$, $0 < s, t \leq 1$, satisfy positivity conditions (D_1) and (D_3) (with $n = 1$).

Lemma 2.7 *If an operator $A \in B(\mathcal{H})$ is a contraction such that $D_i \geq 0$, $1 \leq i \leq 3$, for an $i = i_0$, then D_{i_0} is a contraction.*

Proof Let $D_i = R_i^2$, let $x \in \mathcal{H}$ and let $R_i^m x = y_i$. Then

$$\begin{aligned}
\langle D_1^{m+1}x, x\rangle &= \langle |A^{n+1}|^{\frac{2}{n+1}} y_1, y_1\rangle - \|Ay_1\|^2 \\
&\leq \|A^{n+1}y_1\|^{\frac{2}{n+1}} \|y_1\|^{\frac{2n}{n+1}} - \|Ay_1\|^2 \\
&\leq \|y_1\|^2 = \langle D_1^m x, x\rangle \quad (\text{case } i_0 = 1), \\
\langle D_2^{m+1}x, x\rangle &= \langle |A^{n+1}|^{\frac{2}{n+1}} y_2, y_2\rangle - \|A^*y_2\|^2 \\
&\leq \|A^{n+1}y_2\|^{\frac{2}{n+1}} \|y_2\|^{\frac{2n}{n+1}} - \|A^*y_2\|^2 \\
&\leq \|y_2\|^2 = \langle D_2^m x, x\rangle \quad (\text{case } i_0 = 2),
\end{aligned}$$

$$\begin{aligned}
\langle D_3^{m+1}x,x\rangle &= \langle |A^{n+1}|^2 y_3, y_3\rangle - \frac{n+1}{n}\|Ay_3\|^2 + \|y_3\|^2 \\
&= \|A^{n+1}y_3\|^2 - \frac{n+1}{n}\|Ay_3\|^2 + \|y_3\|^2 \\
&\leq \|Ay_3\|^2 - \frac{n+1}{n}\|Ay_3\|^2 + \|y_3\|^2 \\
&= \|y_3\|^2 - \frac{1}{n}\|Ay_3\|^2 \\
&\leq \|y_3\|^2 = \langle D_3^m x, x\rangle \quad (\text{case } i_0 = 3).
\end{aligned}$$

Hence, in either of the cases $i_0 = 1, 2$ and 3, D_{i_0} is a contraction. □

We remark here that it is in general false that if $A \in B(\mathcal{H})$ is a $\mathcal{P}(n)$ (or $\mathcal{P}(* - n)$) contraction, then the positive operator $D = |A^{n+1}|^2 - (n+1)\lambda^n|A|^2 + n\lambda^{n+1} \geq 0$ (resp., $|A^{n+1}|^2 - (n+1)\lambda^n|A^*|^2 + n\lambda^{n+1} \geq 0$), all $\lambda > 0$, characterizing $\mathcal{P}(n)$ (resp., $\mathcal{P}(* - n)$) operators is a contraction. Consider for example the forward unilateral shift $U \in B(\mathcal{H})$. Trivially, $\alpha U \in \mathcal{P}(1)$ is a (proper) contraction for all positive $\alpha < 1$. The operator $D = |\alpha^2 U^2|^2 - 2\lambda|\alpha U|^2 + \lambda^2 = \alpha^4 - 2\alpha^2\lambda + \lambda^2 = (\alpha^2 - \lambda)^2 > 1$ for all $\lambda > 1 + \alpha^2$. It is possible that, for contractions $A \in \mathcal{P}(* - n)$, the positive operator $D = |A^{n+1}|^2 - \frac{n+1}{n}|A^*|^2 + 1$ is a contraction. We have, however, not been able to prove this.

The conclusion that D_{i_0} is a contraction in Lemma 2.7 implies that the sequence $\{D_{i_0}^p\}_1^\infty$ being a monotonic decreasing bounded sequence of non-negative operators converges to a projection P_{i_0}.

Lemma 2.8 *If D_i $(i = 1, 2, 3)$ is the non-negative contraction of Lemma* 2.7 *with $\lim_{p\to\infty} D_{i_0}^p = P_{i_0}$ for an $i = i_0$, then $AP_{i_0} = 0$ if $i_0 = 1, 3$ and $A^*P_{i_0} = 0$ if $i_0 = 2$.*

Proof Letting $D_{i_0} = R_{i_0}^2$ and $R_{i_0}^m x = y_{i_0}$ for $x \in \mathcal{H}$ and $1 \leq i_0 \leq 3$, we have

$$\begin{aligned}
\|y_1\|^2 - \|R_1 y_1\|^2 &= \|y_1\|^2 - \langle |A^{n+1}|^{\frac{2}{n+1}} y_1, y_1\rangle + \|Ay_1\|^2 \\
&\geq \|y_1\|^2 - \|A^{n+1}y_1\|^{\frac{2}{n+1}}\|y_1\|^{\frac{2n}{n+1}} + \|Ay_1\|^2 \\
&\geq \|Ay_1\|^2 \quad (\text{case } i_0 = 1), \\
\|y_2\|^2 - \|R_2 y_2\|^2 &= \|y_2\|^2 - \langle |A^{n+1}|^{\frac{2}{n+1}} y_2, y_2\rangle + \|A^* y_2\|^2 \\
&\geq \|y_2\|^2 - \|A^{n+1}y_2\|^{\frac{2}{n+1}}\|y_2\|^{\frac{2n}{n+1}} + \|A^* y_2\|^2 \\
&\geq \|A^* y_1\|^2 \quad (\text{case } i_0 = 2), \quad \text{and} \\
\|y_3\|^2 - \|R_3 y_3\|^2 &= \|y_3\|^2 - \langle |A^{n+1}|^2 y_3, y_3\rangle + \frac{n}{n+1}\|Ay_3\|^2 - \|y_3\|^2 \\
&= -\|A^{n+1}y_3\|^2\|y_3\| + \frac{n+1}{n}\|Ay_3\|^2 \\
&\geq \frac{1}{n}\|Ay_3\|^2 \quad (\text{case } i_0 = 3).
\end{aligned}$$

Let the operator B stand for A if $i_0 = 1$ or 3, and let $B = A^*$ if $i_0 = 2$. Letting $a = 1$ if $i_0 = 1$ or 2, and $a = 1/n$ if $i_0 = 3$, we then have

$$a\sum_{m=0}^{p}\left\|BR_{i_0}^m x\right\|^2 \leq \sum_{m=0}^{p}\left\|R_{i_0}^m x\right\|^2 - \sum_{m=0}^{p}\left\|R_{i_0}^{m+1}x\right\|^2 = \|x\|^2 - \left\|R_{i_0}^{p+1}x\right\|^2 \leq \|x\|^2$$

for every $x \in \mathcal{H}$ and integer $p \geq 0$. The positive integer n being fixed, it follows that $\|BR_{i_0}^p x\| \to 0$ as $p \to \infty$; hence

$$0 = \lim_{p\to\infty} BR_{i_0}^{2p}x = B\lim_{p\to\infty} D_{i_0}^p x = BP_{i_0}$$

for every $x \in \mathcal{H}$. Consequently, $AP_{i_0} = 0$ if $i_0 = 1,3$ and $A^*P_{i_0} = 0$ if $i_0 = 2$. □

Recall that $T \in B(\mathcal{H})$ is a $C_{0\cdot}$-contraction (resp., $C_{1\cdot}$-contraction) if $\|T^n x\|$ converges to 0 for all $x \in \mathcal{H}$ (resp., does not converge to 0 for all non-trivial $x \in \mathcal{H}$); T is of class $C_{\cdot 0}$, or $C_{\cdot 1}$, if T^* is of class $C_{0\cdot}$, respectively $C_{1\cdot}$. All combinations are allowed, leading to the classes C_{00}, C_{01}, C_{10}, and C_{11} of contractions ([9], p.72). We say that a contraction $T \in B(\mathcal{H})$ is *strongly stable* if T^n converges strongly to the 0 operator as $n \to \infty$.

The following theorem is our main result.

Theorem 2.9 *If a contraction $0 \neq A \in B(\mathcal{H})$ has no non-trivial invariant subspace, and if A satisfies the positivity condition (D_i) $(i = 1,2,3)$ for an $i = i_0$, then D_{i_0} is a strongly stable (hence C_{00}) proper contraction.*

Proof Start by recalling that A has a non-trivial invariant subspace if and only if A^* does. The hypotheses imply that the sequence $\{D_{i_0}^p\}$ of non-negative contractions converges to a projection P_{i_0} such that $AP_{i_0} = 0$ whenever $i_0 = 1$ or 3 and $A^*P_{i_0} = 0$ whenever $i_0 = 2$. Equivalently, $P_{i_0}A^* = 0$ whenever $i_0 = 1$ or 3 and $P_{i_0}A = 0$ whenever $i_0 = 2$. Thus $P_{i_0}^{-1}(0)$ is a non-zero invariant subspace for A^* in the case in which $i_0 = 1$ or 3, and $P_2^{-1}(0)$ is a non-zero invariant subspace for A in the case in which $i_0 = 2$. Hence we must have $P_i^{-1}(0) = \mathcal{H}$ for every choice of i_0 $(= 1,2,3)$, and then the sequence $\{D_{i_0}^p\}$ converges strongly to the 0 operator. Since strong stability coincides with proper contractiveness for non-negative operators [10], D_{i_0} is a proper contraction (of the class C_{00} of contractions). □

Remark 2.10 A generalization of the class $\mathcal{A}(n)$ of operators in $B(\mathcal{H})$ is obtained by considering operators $A \in B(\mathcal{H})$ for which

$$D_{m(n)} = A^{*m}\left(\left|A^{n+1}\right|^{\frac{2}{n+1}} - |A|^2\right)A^m \geq 0$$

for integers $m \geq 1$. (Similar generalizations of the classes $\mathcal{A}(* - n)$, $\mathcal{P}(n)$, and $\mathcal{P}(* - n)$ are obtained by considering $A^{*m}(|A^{n+1}|^{\frac{2}{n+1}} - |A^*|^2)A^m \geq 0$, $A^{*m}(|A^{n+1}|^2 - 2\lambda|A|^2 + \lambda^2)A^m \geq 0$, and $A^{*m}(|A^{n+1}|^2 - 2\lambda|A^*|^2 + \lambda^2)A^m \geq 0$, respectively.) Calling this class of operators A the class of m-quasi($\mathcal{A}(n)$) operators, denoted $A \in m$-Q($\mathcal{A}(n)$), it is seen that the operators $A \in m$-Q($\mathcal{A}(n)$) have an upper triangular matrix representation

$$A = \begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} \overline{\operatorname{ran} A^m} \\ A^{*m-1}(0) \end{pmatrix},$$

where $A_1 \in \mathcal{A}(n)$ and A_2^* is $(m+1)$-nilpotent. An argument similar to that used above (*cf.* [3] to see the minor changes in detail that are required) shows that, if A is a contraction, then the sequence $\{D_{m(n)}^p\}$ of positive operators converges strongly to a projection P such that $A^{m+1}P = 0$. The operators $A \in m\text{-Q}(\mathcal{A}(n))$ are not normaloid. If a contraction $A \in m\text{-Q}(\mathcal{A}(n))$ has no non-trivial invariant subspaces, then A is a quasi-affinity (*i.e.*, A is injective and has a dense range), and hence $A \in \mathcal{A}(n)$. Thus Theorem 2.9 has the following analog for contractions $A \in m\text{-Q}(\mathcal{A}(n))$: If a contraction $A \in m\text{-Q}(\mathcal{A}(n))$ has no non-trivial invariant subspace, then A is a proper contraction such that $D_{m(n)} \in C_{00}$ is a strongly stable contraction.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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References

1. Takahashi, K, Jeon, IH, Kim, IH, Uchiyama, M: Quasinilpotent part of class A or (p,k)-quasihyponormal operators. Oper. Theory, Adv. Appl. **187**, 199-210 (2008)
2. Furuta, T: Invitation to Linear Operators. Taylor & Francis, London (2001)
3. Yuan, J, Gao, Z: Weyl spectrum of class $A(n)$ and n-paranormal operators. Integral Equ. Oper. Theory **60**, 289-298 (2008)
4. Ito, M: Several properties on class A including p-hyponormal and log-hyponormal operators. Math. Inequal. Appl. **2**, 569-578 (1999)
5. Kubrusly, CS, Levan, N: Proper contractions and invariant subspaces. Int. J. Math. Math. Sci. **28**, 223-230 (2001)
6. Duggal, BP, Jeon, IH, Kubrusly, CS: Contractions satisfying the absolute value property $|A|^2 \leq |A^2|$. Integral Equ. Oper. Theory **49**, 141-148 (2004)
7. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1967)
8. Ito, M, Yamazaki, T: Relations between two inequalities $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$ and $A^r \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{p+r}}$. Integral Equ. Oper. Theory **44**, 442-450 (2002)
9. Sz.-Nagy, B, Foiaş, C: Harmonic Analysis of Operators on Hilbert Space. North-Holland, Amsterdam (1970)
10. Duggal, BP, Kubrusly, CS, Nhan, N: Paranormal contractions and invariant subspaces. J. Korean Math. Soc. **40**, 933-942 (2003)