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Weak contractive integral inequalities and fixed points in modular metric spaces

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Abstract

Branciari (Int. J. Math. Math. Sci. 29(9):531-536, 2002) gave an interesting supplement of Banach's contraction principle for an integral-type inequality. In this paper, we introduce different notions of generalized ω-weak contractive inequalities of integral type in modular metric spaces and prove the presence and uniqueness of common fixed points for such contractions under ω-weak compatibility of underlying maps. Our results generalize and extend the results of Azadifar *et al.* (J. Inequal. Appl. 2013:483, 2013), Liu *et al.* (Fixed Point Theory Appl. 2013:2672013, 2013), Beygmohammadi and Razani (Int. J. Math. Math. Sci. 2010: Article ID 317107, 2010), and many others. Moreover, an example is provided here to demonstrate the applicability of the obtained results.

Keywords: modular metric space; fixed point; ω-weakly compatible maps

1 Introduction and preliminaries

Banach [5], in 1922, proved a contraction principle, this key principle ensures the existence and uniqueness of fixed point theorem for Banach contraction. Later, this famous principle was extended by many authors to more general contractive conditions in different space (see [1, 6–11]). In 1982, Sessa [12] introduced the notion of weakly commuting maps and derived common fixed point for these maps. The first paper [13] on modular function spaces was published in 1990. After that many authors developed this theory by finding fixed point in modular function spaces. Recently, Chistyakov gave the concept of modular metric spaces in [14, 15].

Definition 1.1 [16] Let X be a nonempty set. A modular metric on X is a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$ satisfying the following axioms:

(1) $u = v$ if and only if $\omega_\lambda(u,v) = 0$, for all $\lambda > 0$;
(2) $\omega_\lambda(u,v) = \omega_\lambda(v,u)$, for all $\lambda > 0$ and $u,v \in X$;
(3) $\omega_{\lambda+\nu}(u,v) \le \omega_\lambda(u,w) + \omega_\nu(w,v)$, for all $\lambda,\nu > 0$ and $u,v,w \in X$.

In the sequel, for a function $\omega : (0,\infty) \times X \times X \to [0,\infty]$, we will write

$$\omega_\lambda(u,v) = \omega(\lambda,u,v), \tag{1.1}$$

for all $\lambda > 0$ and $u,v \in X$ and modular metric space as MMS. For related terminologies see [16]. Afterwards many mathematicians studied fixed point properties for modular metric spaces; see [16–19]. Recently, Azadifar *et al.* [2] defined compatible mappings in modular metric space and obtained a common fixed point theorem of integral type as an extension of Jungck [20, 21].

Definition 1.2 [2] Let X_ω be a MMS produced by the metric modular ω. Two mappings $f, h : X_\omega \to X_\omega$ on X_ω are called ω-compatible if $\omega_\lambda(fhx_n, hfx_n) \to 0$ as $n \to \infty$, whenever $\{x_n\}_{n=1}^\infty$ is a sequence in X_ω such that $hx_n \to q$ and $hx_n \to q$ for some point $q \in X_\omega$ and for $\lambda > 0$.

Further, Mongkolkeha and Kumam [22] obtained a common fixed point theorem for pair of compatible mappings satisfying a generalize weak contraction of integral type in modular spaces. Hussain and Salimi [8] established more general fixed point results for some integral-type contractions in MMS. The main intent of this paper is to establish certain common fixed point theorems for ω-weakly compatible maps under different weak contractive conditions which are more general than the corresponding contractive condition of integral type. Our results are more general and are an extension of [2, 4, 11, 22, 23] in the setting of modular metric spaces.

2 Common fixed point theorems for quasi-type weak contractions of integral type

Here, we define weakly compatible mappings for modular metric space and find of a common fixed point for quasi-type weak contractions of integral type satisfying the condition of weakly compatible in MMS.

Definition 2.1 Let X_ω be a MMS produced by the metric modular ω, f and h be two self-mappings of X_ω. A point $x \in X_\omega$ is called a coincidence point of f and h if and only if $fx = hx$. We will call $q = fx = hx$ a point of coincidence of f and h.

Denote the set of all coincidence points of f and h by $C(f, h)$.

Definition 2.2 Two mappings $f, h : X_\omega \to X_\omega$ are said to be ω-weakly compatible if and only if $fhq = hfq$ for $q \in C(f, h)$.

Note that every ω-compatible map is a ω-weakly compatible, but a ω-weakly compatible map needs to be ω-compatible (see Example 2.2).

Lemma 2.1 [24] *Let f and g be weakly compatible self-maps of a set X. If f and g have a unique point of coincidence $q = fx = gx$, then q is the unique common fixed point of f and g.*

Denote by Φ, Θ, Ψ, and Π the collection of lower semicontinuous functions $\phi : [0, \infty) \to [0, \infty)$ with $\phi(r) > 0$ for all $r > 0$ and $\phi(r) = 0$ if and only if $r = 0$, the collection of Lebesgue integrable functions $\varphi : [0, \infty) \to [0, \infty)$ which is nonnegative, summable, and, for all $\epsilon > 0$, $\int_0^\epsilon \varphi(r)\,dr > 0$, the collection of lower semicontinuous functions $\psi : [0, \infty) \to [0, \infty)$ for which $\psi(r) < r$ for all $r > 0$ and the collection of nondecreasing functions $\pi : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^\infty \pi^n(r) < +\infty$ for all $r > 0$, where π^n is the nth iterate of π, respectively.

Lemma 2.2 [25] *If $\pi \in \Pi$, then the following hold*:

(i) $(\pi^n(r))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \to \infty$ *for all* $r \in (0, +\infty)$;

(ii) $\pi(r) < r$ *for all* $r > 0$;
(iii) $\pi(r) = 0$ *if and only if* $r = 0$.

Lemma 2.3 [10] *Let* $\varphi \in \Theta$ *and* $\{s_n\}_{n\in\mathbb{N}}$ *be a nonnegative sequence with* $s_n \to a$ *as* $n \to \infty$. *Then*

$$\lim_{n\to\infty} \int_0^{s_n} \varphi(r)\,\mathrm{d}r = \int_0^{a} \varphi(r)\,\mathrm{d}r.$$

Now we present the main results of this section.

Theorem 2.1 *Let* X_ω *be a MMS. Suppose* $a, t \in \mathbb{R}^+$ *with* $a > t$ *and* $S, h : X_\omega \to X_\omega$ *are two self-mappings satisfying the following assertions*:

(1) $S(X_\omega) \subseteq h(X_\omega)$, $h(X_\omega)$ *is a complete subspace of* X_ω;
(2) S *and* h *are* ω*-weakly compatible*;
(3) $\int_0^{\omega_{\lambda/a}(Sx,Sy)} \varphi(r)\,\mathrm{d}r \le \int_0^{\mathcal{M}(x,y)} \varphi(r)\,\mathrm{d}r - \phi(\int_0^{\mathcal{M}(x,y)} \varphi(r)\,\mathrm{d}r)$,

where

$$\begin{aligned}\mathcal{M}(x,y) = \max\biggl\{&\omega_{\lambda/t}(hx,hy), \omega_{\lambda/t}(hx,Sx), \omega_{\lambda/t}(hy,Sy), \frac{\omega_{\lambda/t}(hx,Sy) + \omega_{\lambda/t}(hy,Sx)}{2},\\ &\frac{\omega_{\lambda/t}(hx,Sy)\omega_{\lambda/t}(hy,Sx)}{1+\omega_{\lambda/t}(hx,hy)}, \frac{\omega_{\lambda/t}(hx,Sx)\omega_{\lambda/t}(hx,Sy)}{2[1+\omega_{\lambda/t}(hx,hy)]}, \frac{\omega_{\lambda/t}(hy,Sy)\omega_{\lambda/t}(hy,Sx)}{2[1+\omega_{\lambda/t}(hx,hy)]}\biggr\},\end{aligned}$$

$\varphi \in \Theta$ *and* $\phi \in \Phi$. *Then* S *and* h *have a unique common fixed point.*

Proof Choose $a > 2t$ and let $x_0 \in X_\omega$ be an arbitrary point. Since $S(X_\omega) \subseteq h(X_\omega)$, there is a point $x_1 \in X_\omega$ such that $S(x_0) = h(x_1)$. On continuing this, we generate a sequence $\{hx_n\}_{n=1}^{\infty}$ as follows: $Sx_n = hx_{n+1}$ for each n. Suppose for any n, $hx_n \neq hx_{n+1}$, since, otherwise, there exists a point of coincidence of S and h, (3) shows that

$$\begin{aligned}\int_0^{\omega_{\lambda/a}(hx_{n+1},hx_n)} \varphi(r)\,\mathrm{d}r &= \int_0^{\omega_{\lambda/a}(Sx_n,Sx_{n-1})} \varphi(r)\,\mathrm{d}r\\ &\le \int_0^{\mathcal{M}(x_n,x_{n-1})} \varphi(r)\,\mathrm{d}r - \phi\biggl(\int_0^{\mathcal{M}(x_n,x_{n-1})} \varphi(r)\,\mathrm{d}r\biggr),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(x_n,x_{n-1}) = \max\biggl\{&\omega_{\lambda/t}(hx_n,hx_{n-1}), \omega_{\lambda/t}(hx_n,Sx_n), \omega_{\lambda/t}(hx_{n-1},Sx_{n-1}),\\ &\frac{\omega_{\lambda/t}(hx_n,Sx_{n-1}) + \omega_{\lambda/t}(hx_{n-1},Sx_n)}{2}, \frac{\omega_{\lambda/t}(hx_n,Sx_{n-1})\omega_{\lambda/t}(hx_{n-1},Sx_n)}{1+\omega_{\lambda/t}(hx_n,hx_{n-1})},\\ &\frac{\omega_{\lambda/t}(hx_n,Sx_n)\omega_{\lambda/t}(hx_n,Sx_{n-1})}{2[1+\omega_{\lambda/t}(hx_n,hx_{n-1})]}, \frac{\omega_{\lambda/t}(hx_{n-1},Sx_{n-1})\omega_{\lambda/t}(hx_{n-1},Sx_n)}{2[1+\omega_{\lambda/t}(hx_n,hx_{n-1})]}\biggr\}.\end{aligned}$$

Since $hx_n = Sx_{n-1}$, it follows that

$$\begin{aligned}\mathcal{M}(x_n,x_{n-1}) = \max\biggl\{&\omega_{\lambda/t}(hx_{n-1},hx_n), \omega_{\lambda/t}(hx_n,hx_{n+1}), \frac{\omega_{\lambda/t}(hx_{n-1},hx_{n+1})}{2},\\ &\frac{\omega_{\lambda/t}(hx_{n-1},hx_n)\omega_{\lambda/t}(hx_{n-1},hx_{n+1})}{2[1+\omega_{\lambda/t}(hx_n,hx_{n-1})]}\biggr\}.\end{aligned}$$

Moreover,

$$\begin{aligned}\omega_{\lambda/t}(hx_{n-1},hx_{n+1}) &\leq \omega_{\lambda/2t}(hx_{n-1},hx_n)+\omega_{\lambda/2t}(hx_n,hx_{n+1})\\ &\leq \omega_{\lambda/a}(hx_{n-1},hx_n)+\omega_{\lambda/a}(hx_n,hx_{n+1})\end{aligned}$$

and

$$\begin{aligned}\frac{\omega_{\lambda/t}(hx_{n-1},hx_n)\omega_{\lambda/t}(hx_{n-1},hx_{n+1})}{2[1+\omega_{\lambda/t}(hx_n,hx_{n-1})]} &\leq \frac{\omega_{\lambda/t}(hx_{n-1},hx_{n+1})}{2}\\ &\leq \frac{\omega_{\lambda/a}(hx_{n-1},hx_n)+\omega_{\lambda/a}(hx_n,hx_{n+1})}{2}\\ &\leq \max\bigl\{\omega_{\lambda/a}(hx_{n-1},hx_n),\omega_{\lambda/a}(hx_n,hx_{n+1})\bigr\},\end{aligned}$$

then

$$\mathcal{M}(x_n,x_{n-1}) \leq \max\bigl\{\omega_{\lambda/a}(hx_{n-1},hx_n),\omega_{\lambda/a}(hx_n,hx_{n+1})\bigr\}.$$

Now if $\omega_{\lambda/a}(hx_n,hx_{n+1}) > \omega_{\lambda/a}(hx_{n-1},hx_n)$, then

$$\begin{aligned}\int_0^{\omega_{\lambda/a}(hx_{n+1},hx_n)}\varphi(r)\,\mathrm{d}r &\leq \int_0^{\omega_{\lambda/a}(hx_n,hx_{n+1})}\varphi(r)\,\mathrm{d}r-\phi\biggl(\int_0^{\omega_{\lambda/a}(hx_n,hx_{n+1})}\varphi(r)\,\mathrm{d}r\biggr)\\ &< \int_0^{\omega_{\lambda/a}(hx_n,hx_{n+1})}\varphi(r)\,\mathrm{d}r.\end{aligned}$$

This is a contradiction. So, $\mathcal{M}(x_n,x_{n-1}) \leq \omega_{\lambda/a}(hx_{n-1},hx_n)$. Therefore

$$\begin{aligned}\int_0^{\omega_{\lambda/a}(hx_{n+1},hx_n)}\varphi(r)\,\mathrm{d}r &\leq \int_0^{\omega_{\lambda/a}(hx_n,hx_{n-1})}\varphi(r)\,\mathrm{d}r-\phi\biggl(\int_0^{\omega_{\lambda/a}(hx_n,hx_{n-1})}\varphi(r)\,\mathrm{d}r\biggr)\\ &< \int_0^{\omega_{\lambda/a}(hx_n,hx_{n-1})}\varphi(r)\,\mathrm{d}r,\end{aligned}\tag{2.1}$$

it shows that the sequence $\{\int_0^{\omega_{\lambda/a}(hx_{n+1},hx_n)}\varphi(r)\}$ is decreasing and bounded below. Hence, there is $k \geq 0$ such that

$$\lim_{n\to\infty}\int_0^{\omega_{\lambda/a}(hx_{n+1},hx_n)}\varphi(r)\,\mathrm{d}r = k.$$

If $k > 0$, then by Lemma 2.3 and (2.1), we have a contradiction. So, we get

$$\lim_{n\to\infty}\omega_{\lambda/a}(hx_{n+1},hx_n) = 0.$$

Suppose $l < a' < 2t$, since ω_λ is a decreasing function, so $\omega_{\lambda/a'}(hx_{n+1},hx_n) \leq \omega_{\lambda/a}(hx_{n+1},hx_n)$, whenever $a' < 2t \leq a$. On considering the limit as $n \to \infty$ from both sides of this inequality shows that $\omega_{\lambda/a'}(hx_{n+1},hx_n) \to 0$ for $t < a' < 2t$ and $\lambda > 0$. Thus we have $\omega_{\lambda/a}(hx_{n+1},hx_n) \to 0$ as $n \to \infty$ for any $a > t$. Next, we show that $\{hx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. So, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega_{\lambda/a}(hx_{n+1},hx_n) < \frac{\varepsilon}{a}$ for all $n \in \mathbb{N}$

with $n \geq n_0$ and $\lambda > 0$. Suppose $m, n \in \mathbb{N}$ and $m > n$. Observe that, for $\frac{\lambda}{a(m-n)}$, there exists $n_{\frac{\lambda}{(m-n)}} \in \mathbb{N}$ such that

$$\omega_{\frac{\lambda}{a(m-n)}}(hx_{n+1}, hx_n) < \frac{\epsilon}{a(m-n)},$$

for all $n \geq n_{\frac{\lambda}{(m-n)}}$. Now, we have

$$\begin{aligned}
\omega_{\lambda/l}(hx_n, hx_m) &\leq \omega_{\frac{\lambda}{a(m-n)}}(hx_n, hx_{n+1}) + \omega_{\frac{\lambda}{a(m-n)}}(hx_{n+1}, hx_{n+2}) + \cdots + \omega_{\frac{\lambda}{a(m-n)}}(hx_{m-1}, hx_m) \\
&< \frac{\varepsilon}{a(m-n)} + \frac{\varepsilon}{a(m-n)} + \cdots + \frac{\varepsilon}{a(m-n)} \\
&= \varepsilon/a,
\end{aligned}$$

for all $m, n \geq n_{\frac{\lambda}{(m-n)}}$. This shows that $\{hx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. From completeness of $h(X_\omega)$, it follows that there exists $x^* \in X$ such that $\omega_{\lambda/t}(hx_n, x^*) \to 0$ as $n \to \infty$. Consequently, we can find p in X_ω such that $h(p) = x^*$. By (3), we get

$$\begin{aligned}
\int_0^{\omega_{\lambda/a}(hx_n,Sp)} \varphi(r)\,\mathrm{d}r &= \int_0^{\omega_{\lambda/a}(Sx_{n-1},Sp)} \varphi(r)\,\mathrm{d}r \\
&\leq \int_0^{\mathcal{M}(x_{n-1},p)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathcal{M}(x_{n-1},p)} \varphi(r)\,\mathrm{d}r\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}(x_{n-1}, p) = \max\biggl\{&\omega_{\lambda/t}(hx_{n-1}, hp), \omega_{\lambda/t}(hx_{n-1}, Sx_{n-1}), \omega_{\lambda/t}(hp, Sp), \\
&\frac{\omega_{\lambda/t}(hx_{n-1}, Sp) + \omega_{\lambda/t}(hp, Sx_{n-1})}{2}, \frac{\omega_{\lambda/t}(hx_{n-1}, Sp)\omega_{\lambda/t}(hp, Sx_{n-1})}{1 + \omega_{\lambda/t}(hx_{n-1}, hp)} \\
&\frac{\omega_{\lambda/t}(hx_{n-1}, Sx_{n-1})\omega_{\lambda/t}(hx_{n-1}, Sp)}{2[1 + \omega_{\lambda/t}(hx_{n-1}, hp)]}, \frac{\omega_{\lambda/t}(hp, Sp)\omega_{\lambda/t}(hp, Sx_{n-1})}{2[1 + \omega_{\lambda/t}(hx_{n-1}, hp)]}\biggr\}.
\end{aligned}$$

By taking the limit as $n \to \infty$, we have

$$\begin{aligned}
\int_0^{\omega_{\lambda/a}(x^*,Sp)} \varphi(r)\,\mathrm{d}r &\leq \int_0^{\omega_{\lambda/t}(x^*,Sp)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\omega_{\lambda/t}(x^*,Sp)} \varphi(r)\,\mathrm{d}r\right) \\
&< \int_0^{\omega_{\lambda/t}(x^*,Sp)} \varphi(r)\,\mathrm{d}r \\
&\leq \int_0^{\omega_{\lambda/a}(x^*,Sp)} \varphi(r)\,\mathrm{d}r.
\end{aligned}$$

This shows $\omega_{\lambda/a}(Sp, x^*) = 0$ for $\lambda > 0$. Hence $Tp = x^*$ and S and h have the point of coincidence x^*. Suppose that $q \neq x^*$ is another point of coincidence of S and h in X_ω. Then $Tv = hv = q$ for some v in X_ω. By (3), we get

$$\begin{aligned}
\int_0^{\omega_{\lambda/a}(hp,hv)} \varphi(r)\,\mathrm{d}r &= \int_0^{\omega_{\lambda/a}(Sp,Sv)} \varphi(r)\,\mathrm{d}r \\
&\leq \int_0^{\mathcal{M}(p,v)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathcal{M}(p,v)} \varphi(r)\,\mathrm{d}r\right),
\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(p,v) &= \max\biggl\{\omega_{\lambda/t}(hp,hv),\omega_{\lambda/t}(hp,Sp),\omega_{\lambda/t}(hv,Sv),\\&\quad\frac{\omega_{\lambda/t}(hv,Sp)+\omega_{\lambda/t}(hp,Sv)}{2},\frac{\omega_{\lambda/t}(hp,Sv)\omega_{\lambda/t}(hv,Sp)}{1+\omega_{\lambda/t}(hp,hv)},\\&\quad\frac{\omega_{\lambda/t}(hp,Sp)\omega_{\lambda/t}(hp,Sv)}{2[1+\omega_{\lambda/t}(hp,hv)]},\frac{\omega_{\lambda/t}(hv,Sv)\omega_{\lambda/t}(hv,Sp)}{2[1+\omega_{\lambda/t}(hp,hv)]}\biggr\}\\&= \omega_{\lambda/t}(hp,hv).\end{aligned}$$

So,

$$\begin{aligned}\int_0^{\omega_{\lambda/a}(hp,hv)}\varphi(r)\,\mathrm{d}r &\le \int_0^{\omega_{\lambda/t}(hp,hv)}\varphi(r)\,\mathrm{d}r-\phi\biggl(\int_0^{\omega_{\lambda/t}(hp,hv)}\varphi(t)\,\mathrm{d}t\biggr)\\&< \int_0^{\omega_{\lambda/t}(hp,hv)}\varphi(r)\,\mathrm{d}r\le\int_0^{\omega_{\lambda/a}(hp,hv)}\varphi(r)\,\mathrm{d}r.\end{aligned}$$

From this contradiction, we see that S and h have a unique coincidence point x^*. By using Lemma 2.1, we get x^* a unique common fixed point of S and h. □

Here is an example to illustrate Theorem 2.1.

Example 2.1 Let $X_\omega = \{0,1,2,3,4,\ldots\}$ and $\omega_\lambda(x,y) = \frac{d(x,y)}{\lambda}$, where

$$d(x,y)=\begin{cases}x+y, & x\neq y,\\0, & x=y.\end{cases}$$

Define $S,h: X_\omega \to X_\omega$ and $\varphi,\phi:[0,\infty)\to[0,\infty)$ as

$$Sx=0\quad\forall x\in X_\omega,\qquad hx=\begin{cases}x-1, & \text{if } x\neq 0,\\0, & \text{if } x=0,\end{cases}$$

$\varphi(r) = 2r$ and $\phi(r) = \sqrt{r}$, respectively. Then $S(X_\omega)\subseteq h(X_\omega)$ and $h(X_\omega)$ is a complete subspace of X_ω. Note that $x = 0$ is the coincidence point of S and h and

$$Sh(0) = S(0) = h(0) = hS(0).$$

This shows that S and h are ω-weakly compatible maps. Now we verify that S and h satisfy condition (3) of Theorem 2.1. Suppose $a,t\in\mathbb{R}^+$, $a>t$. Then there arise four cases.

Case 1: Assume $x = y = 0$. Then condition (3) holds trivially because $Sx = Sy = hx = hy = 0$.

Case 2: Assume $y = 0$ and $x > 0$. Then

$$\omega_{\lambda/a}(Sx,Sy)=\omega_{\lambda/a}(0,0)=0.$$

This implies that

$$\int_0^{\omega_{\lambda/a}(Sx,Sy)}2r\,\mathrm{d}r=0.$$

Also,

$$\begin{aligned}\mathcal{M}(x,y) &= \max\left\{\omega_{\lambda/t}(x-1,0), \omega_{\lambda/t}(x-1,0), \omega_{\lambda/t}(0,0), \frac{\omega_{\lambda/t}(x-1,0)+\omega_{\lambda/t}(0,0)}{2},\right.\\ &\qquad \left.\frac{\omega_{\lambda/t}(x-1,0)\omega_{\lambda/t}(0,0)}{1+\omega_{\lambda/t}(x-1,0)}, \frac{\omega_{\lambda/t}(x-1,0)\omega_{\lambda/t}(x-1,0)}{2[1+\omega_{\lambda/t}(x-1,0)]}, \frac{\omega_{\lambda/t}(0,0)\omega_{\lambda/t}(0,x-1)}{2[1+\omega_{\lambda/t}(x-1,0)]}\right\}\\ &= \max\left\{\frac{t}{\lambda}(x-1), \frac{t/\lambda(x-1)}{2}, \frac{t^2/\lambda^2(x-1)^2}{2(1+t/\lambda(x-1))}\right\}\\ &= \max\left\{\frac{t}{\lambda}(x-1), \frac{1}{2(1+\frac{1}{t/\lambda(x-1)})}\right\}\\ &= \frac{t}{\lambda}(x-1),\end{aligned}$$

so,

$$\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r = \frac{t^2}{\lambda^2}(x-1)^2.$$

Therefore,

$$\begin{aligned}\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r - \phi\left(\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r\right) &= \frac{t^2}{\lambda^2}(x-1)^2 - \frac{t}{\lambda}(x-1)\\ &= \frac{t}{\lambda}(x-1)\left[\frac{t}{\lambda}(x-1)-1\right].\end{aligned}$$

Since $\frac{t}{\lambda}(x-1) \geq 1$, this shows that

$$\frac{t}{\lambda}(x-1)\left[\frac{t}{\lambda}(x-1)-1\right] \geq 0 = \int_0^{\omega_{\lambda/a}(Sx,Sy)} 2r\,\mathrm{d}r.$$

Thus condition (3) is satisfied in this case.

Case 3: Assume $x > y > 0$. Then we need to consider two subcases:

Subcase 1: If $x = y+1$ or $y = x-1$, then $Sx = Sy = 0$, $hx = x-1$ and $hy = x-2$. This implies that

$$\omega_{\lambda/a}(Sx,Sy) = 0$$

and

$$\begin{aligned}\mathcal{M}(x,y) = \max&\left\{\frac{t}{\lambda}(2x-3), \frac{t}{\lambda}(x-1), \frac{t}{\lambda}(x-2), \frac{t/\lambda(2x-3)}{2}, \frac{t^2/\lambda^2(x-1)(x-2)}{(1+t/\lambda(2x-3))},\right.\\ &\left.\frac{t^2/\lambda^2(x-1)^2}{2(1+t/\lambda(2x-3))}, \frac{t^2/\lambda^2(x-2)^2}{2(1+t/\lambda(2x-3))}\right\}.\end{aligned}$$

Since $2x-3 \geq x-1 \geq x-2$, $\mathcal{M}(x,y) = \frac{t}{\lambda}(2x-3)$. Hence

$$\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r - \phi\left(\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r\right) = \frac{t^2}{\lambda^2}(2x-3)^2 - \frac{t}{\lambda}(2x-3) \geq 0 = \int_0^{\omega_{\lambda/a}(Sx,Sy)} 2r\,\mathrm{d}r.$$

Subcase 2: If $x > y+1$ and $x = 2y$, then $Sx = Sy = 0$, $hx = 2y-1$ and $hy = y-1$. This implies that

$$\omega_{\lambda/a}(Sx,Sy) = 0$$

and

$$\mathcal{M}(x,y) = \max\left\{\frac{t}{\lambda}(3y-2), \frac{t}{\lambda}(2y-1), \frac{t}{\lambda}(y-1), \frac{t/\lambda(3y-2)}{2}, \frac{t^2/\lambda^2(2y-1)(y-1)}{(1+t/\lambda(3y-2))}, \frac{t^2/\lambda^2(2y-1)^2}{2(1+t/\lambda(3y-2))}, \frac{t^2/\lambda^2(y-1)^2}{2(1+t/\lambda(3y-2))}\right\}.$$

Since $3y-2 \geq 2y-1 \geq y-1$, $\mathcal{M}(x,y) = \frac{t}{\lambda}(3y-2)$. Hence

$$\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r - \phi\left(\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r\right) = \frac{t^2}{\lambda^2}(3y-2)^2 - \frac{t}{\lambda}(3y-2) \geq 0 = \int_0^{\omega_{\lambda/a}(Sx,Sy)} 2r\,\mathrm{d}r.$$

Now if $x > 2y$, then $Sx = Sy = 0$, $hx = x-1$ and $hy = y-1$. This implies that $\mathcal{M}(x,y) = \frac{t}{\lambda}(x+y-2)$. Hence

$$\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r - \phi\left(\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r\right) = \frac{t^2}{\lambda^2}(x+y-2)^2 - \frac{t}{\lambda}(x+y-2) \geq 0 = \int_0^{\omega_{\lambda/a}(Sx,Sy)} 2r\,\mathrm{d}r.$$

Thus condition (3) is satisfied in this case.

Case 4: Assume $x = y > 0$. Then $Sx = Sy = 0$ and $hx = hy = x-1$. This implies that

$$\omega_{\lambda/a}(Sx,Sy) = 0$$

and

$$\mathcal{M}(x,y) = \max\left\{0, \frac{t}{\lambda}(x-1), \frac{t^2}{\lambda^2}(x-1)^2, \frac{t^2/\lambda^2(x-1)^2}{2}\right\} = \frac{t^2}{\lambda^2}(x-1)^2.$$

Hence

$$\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r - \phi\left(\int_0^{\mathcal{M}(x,y)} 2r\,\mathrm{d}r\right) = \frac{t^4}{\lambda^4}(x-1)^4 - \frac{t^2}{\lambda^2}(x-1)^2$$
$$\geq 0 = \int_0^{\omega_{\lambda/a}(Sx,Sy)} 2r\,\mathrm{d}r.$$

So, condition (3) is satisfied in this case. Thus all conditions of Theorem 2.1 hold and 0 is a unique common fixed point of S and h.

From Theorem 2.1, we conclude the following results:

Theorem 2.2 *Let X_ω be a MMS. Assume that $a,t\in\mathbb{R}^+$ with $a>t$ and $S,h:X_\omega\to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and*

$$\int_0^{\omega_{\lambda/a}(Sx,Sy)} \varphi(r)\,\mathrm{d}r \leq \int_0^{\mathfrak{m}_1(x,y)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathfrak{m}_1(x,y)} \varphi(r)\,\mathrm{d}r\right), \tag{2.2}$$

for all $x,y\in X_\omega$ and $\lambda>0$, where

$$\mathfrak{m}_1(x,y) = \max\left\{\omega_{\lambda/t}(hx,hy), \omega_{\lambda/t}(hx,Sx), \omega_{\lambda/t}(hy,Sy), \frac{\omega_{\lambda/t}(hx,Sy)+\omega_{\lambda/t}(hy,Sx)}{2}\right\},$$

$\varphi\in\Theta$ and $\phi\in\Phi$. Then T and h have a unique common fixed point.

Theorem 2.3 *Let X_ω be a MMS. Assume that $a,t\in\mathbb{R}^+$, $a>t$ and $S,h:X_\omega\to X_\omega$ are two self mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and*

$$\int_0^{\omega_{\lambda/a}(Sx,Sy)} \varphi(r)\,\mathrm{d}r \leq \int_0^{\omega_{\lambda/t}(hx,hy)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\omega_{\lambda/t}(hx,hy)} \varphi(r)\,\mathrm{d}r\right), \tag{2.3}$$

for all $x,y\in X_\omega$, where $\varphi\in\Theta$ and $\phi\in\Phi$. Then T and h have a unique common fixed point.

Now we give an Example 2.2 which shows that Theorem 2.3 extends significantly Theorem 2.2 and Theorem 4.2 of [2].

Example 2.2 Let $X_\omega=[0,1]$ and $\omega_\lambda(x,y)=\frac{|x-y|}{\lambda}$. Define $S,h:X_\omega\to X_\omega$ and $\varphi,\phi:[0,\infty)\to[0,\infty)$ as

$$Sx=\begin{cases} 1/2 & \text{if } x\in[0,\frac{1}{2}],\\ 3/4 & \text{if } x\in(\frac{1}{2},1],\end{cases} \qquad hx=\begin{cases} 1-x & \text{if } x\in[0,\frac{1}{2}],\\ 0 & \text{if } x\in(\frac{1}{2},1],\end{cases}$$

$\varphi(r)=2r$ and $\phi(r)=\ln(1+r)$, respectively. First of all we verify that S and h satisfies the inequality (2.3). Suppose $a,t\in\mathbb{R}^+$, $a>t$. Then there are two cases.

Case 1: Let $x\in[0,\frac{1}{2}]$. Then

$$\omega_{\lambda/a}(Sx,Sy)=\omega_{\lambda/a}\left(\frac{1}{2},\frac{1}{2}\right)=0.$$

This implies that

$$\int_0^{\omega_{\lambda/a}(Sx,Sy)} 2r\,\mathrm{d}r = 0.$$

Also,

$$\omega_{\lambda/t}(hx,hy) = \omega_{\lambda/t}(1-x,1-y) = \frac{l|x-y|}{\lambda},$$

so,

$$\int_0^{\omega_{\lambda/t}(hx,hy)} 2r\,\mathrm{d}r = \frac{l|x-y|}{\lambda}^2.$$

Therefore,

$$\int_0^{\omega_{\lambda/t}(hx,hy)} 2r\,\mathrm{d}r - \phi\left(\int_0^{\omega_{\lambda/t}(hx,hy)} 2r\,\mathrm{d}r\right) = \frac{t^2|x-y|^2}{\lambda^2} - \ln\left(1+\frac{t^2|x-y|^2}{\lambda^2}\right).$$

Since $\ln(1+x) \leq x$ for all $x \in [0,1]$, this shows that

$$\frac{t^2|x-y|^2}{\lambda^2} - \ln\left(1+\frac{t^2|x-y|^2}{\lambda^2}\right) \geq 0 = \int_0^{\omega_{\lambda/a}(Sx,Sy)} 2r\,\mathrm{d}r.$$

Thus (2.3) is satisfied in this case.

Case 2: Let $x \in (\frac{1}{2},1]$. Then

$$\omega_{\lambda/a}(Sx,Sy) = 0 = \omega_{\lambda/t}(hx,hy).$$

Thus (2.3) is satisfied trivially in this case.

Next, since $x = \frac{1}{2}$ is the coincidence point of S and h and

$$Sh\left(\frac{1}{2}\right) = S\left(\frac{1}{2}\right) = h\left(\frac{1}{2}\right) = hS\left(\frac{1}{2}\right),$$

showing that S and h are ω-weakly compatible maps. Thus all conditions of Theorem 2.3 hold and $\frac{1}{2}$ is a unique common fixed point of S and h.

Further, consider a sequence $\{x_n\} = \{\frac{1}{2} - \frac{1}{n}\}$, $n \geq 2$, in X_ω. We have

$$\begin{aligned}\omega_\lambda(Sx_n,hx_n) &= \omega_\lambda\left(S\left(\frac{1}{2}-\frac{1}{n}\right),h\left(\frac{1}{2}-\frac{1}{n}\right)\right)\\ &= \omega_\lambda\left(\frac{1}{2}+\frac{1}{n},\frac{1}{2}\right) \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

But

$$\omega_\lambda(Shx_n,hSx_n) \nrightarrow 0 \quad \text{as } n\to\infty.$$

Therefore, S and h are not ω-compatible.

Like to the arguments of Theorem 2.1, we state the following results and exclude their proofs.

Theorem 2.4 *Let X_ω be a MMS. Suppose $S, h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and $a, t \in \mathbb{R}^+$ with $a > t$*

$$\int_0^{\omega_{\lambda/a}(Sx,Sy)} \varphi(r)\,\mathrm{d}r \le \int_0^{\mathfrak{m}_2(x,y)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathfrak{m}_2(x,y)} \varphi(r)\,\mathrm{d}r\right), \tag{2.4}$$

for all $x, y \in X_\omega$, where

$$\begin{aligned}\mathfrak{m}_2(x,y) = \max\biggl\{&\omega_{\lambda/t}(hx,hy), \omega_{\lambda/t}(hx,Sx), \omega_{\lambda/t}(hy,Sy), \frac{\omega_{\lambda/t}(hx,Sy) + \omega_{\lambda/t}(hy,Sx)}{2},\\ &\frac{\omega_{\lambda/t}(hy,Sx)\omega_{\lambda/t}(hx,Sy)}{1 + \omega_{\lambda/t}(Sx,Sy)}, \frac{\omega_{\lambda/t}(hy,Sx)\omega_{\lambda/t}(hx,Sx)}{2[1 + \omega_{\lambda/t}(Sx,Sy)]},\\ &\frac{\omega_{\lambda/t}(hx,Sy)\omega_{\lambda/t}(hy,Sy)}{2[1 + \omega_{\lambda/t}(Sx,Sy)]}\biggr\},\end{aligned} \tag{2.5}$$

$\varphi \in \Theta$ and $\phi \in \Phi$. Then there exists a unique common fixed point of S and h.

Theorem 2.5 *Let X_ω be a MMS. Assume that $S, h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and $a, t \in \mathbb{R}^+$ with $a > t$ such that*

$$\int_0^{\omega_{\lambda/a}(Sx,Sy)} \varphi(r)\,\mathrm{d}r \le \int_0^{\mathfrak{m}_3(x,y)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathfrak{m}_3(x,y)} \varphi(r)\,\mathrm{d}r\right), \tag{2.6}$$

for all $x, y \in X_\omega$, where

$$\begin{aligned}\mathfrak{m}_3(x,y) = \max\biggl\{&\omega_{\lambda/t}(hx,hy), \omega_{\lambda/t}(hx,Sx), \omega_{\lambda/t}(hy,Sy),\\ &\frac{\omega_{\lambda/t}(hx,Sy) + \omega_{\lambda/t}(hy,Sx)}{2}, \frac{\omega_{\lambda/t}(hy,Sx)\omega_{\lambda/t}(hx,Sy)}{1 + \omega_{\lambda/t}(hx,hy)},\\ &\min\biggl\{\frac{\omega_{\lambda/t}(hx,Sx)\omega_{\lambda/t}(hx,Sy)}{1 + \omega_{\lambda/t}(hx,hy)}, \frac{\omega_{\lambda/t}(hy,Sy)\omega_{\lambda/t}(hy,Sx)}{1 + \omega_{\lambda/t}(hx,hy)}\biggr\}\biggr\},\end{aligned} \tag{2.7}$$

$\varphi \in \Theta$ and $\phi \in \Phi$. Then there exists a unique common fixed point of S and h.

Theorem 2.6 *Let X_ω be a MMS. Suppose $S, h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and for all $x, y \in X_\omega$, there exist $a, t \in \mathbb{R}^+$ with $a > t$*

$$\int_0^{\omega_{\lambda/a}(Sx,Sy)} \varphi(r)\,\mathrm{d}r \le \int_0^{\mathfrak{m}_4(x,y)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathfrak{m}_4(x,y)} \varphi(r)\,\mathrm{d}r\right), \tag{2.8}$$

where

$$\begin{aligned}\mathfrak{m}_4(x,y) = \max\biggl\{&\omega_{\lambda/t}(hx,hy), \omega_{\lambda/t}(hx,Sx), \omega_{\lambda/t}(hy,Sy),\\ &\frac{\omega_{\lambda/t}(hx,Sy)+\omega_{\lambda/t}(hy,Sx)}{2}, \frac{\omega_{\lambda/t}(hy,Sx)\omega_{\lambda/t}(hx,Sy)}{1+\omega_{\lambda/t}(Sx,Sy)},\\ &\min\biggl\{\frac{\omega_{\lambda/t}(hy,Sx)\omega_{\lambda/t}(hx,Sx)}{1+\omega_{\lambda/t}(Sx,Sy)}, \frac{\omega_{\lambda/t}(hx,Sy)\omega_{\lambda/t}(hy,Sy)}{1+\omega_{\lambda/t}(Sx,Sy)}\biggr\}\biggr\},\end{aligned} \tag{2.9}$$

$\varphi \in \Theta$ and $\phi \in \Phi$. Then there exists a unique common fixed point of S and h.

Theorem 2.7 *Let X_ω be a MMS. Suppose $S, h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and*

$$\int_0^{\pi(\omega_{\lambda/a}(Sx,Sy))} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathcal{M}(x,y))} \varphi(r)\,\mathrm{d}r - \phi\biggl(\int_0^{\pi(\mathcal{M}(x,y))} \varphi(r)\,\mathrm{d}r\biggr), \tag{2.10}$$

for all $x, y \in X_\omega$, $a, t \in \mathbb{R}$ with $a > t$, where $\mathcal{M}(x,y)$ is as in Theorem 2.1, *$\varphi \in \Theta$, $\phi \in \Phi$ and $\pi \in \Pi$. Then there exists a unique common fixed point of S and h.*

Proof Choose $a > 2t$. Let $x_0 \in X_\omega$ be an arbitrary point. Since $S(X_\omega) \subseteq h(X_\omega)$, there is a point x_1 in X_ω such that $S(x_0) = h(x_1)$. By continuing this, we generate a sequence $\{hx_n\}_{n=1}^\infty$ as follows: $Sx_n = hx_{n+1}$ for each n. Suppose for any n, $hx_n \neq hx_{n+1}$, since, otherwise, S and h have a point of coincidence, (2.10) shows that

$$\begin{aligned}\int_0^{\pi(\omega_{\lambda/a}(hx_{n+1},hx_n))} \varphi(r)\,\mathrm{d}r &= \int_0^{\pi(\omega_{\lambda/a}(Sx_n,Sx_{n-1}))} \varphi(r)\,\mathrm{d}r\\ &\le \int_0^{\pi(\mathcal{M}(x_n,x_{n-1}))} \varphi(r)\,\mathrm{d}r - \phi\biggl(\int_0^{\pi(\mathcal{M}(x_n,x_{n-1}))} \varphi(r)\,\mathrm{d}r\biggr),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(x,y) = \max\biggl\{&\omega_{\lambda/t}(hx_n,hx_{n-1}), \omega_{\lambda/t}(hx_n,Sx_n), \omega_{\lambda/t}(hx_{n-1},Sx_{n-1}),\\ &\frac{\omega_{\lambda/t}(hx_n,Sx_{n-1})+\omega_{\lambda/t}(hx_{n-1},Sx_n)}{2}, \frac{\omega_{\lambda/t}(hx_n,Sx_{n-1})\omega_{\lambda/t}(hx_{n-1},Sx_n)}{1+\omega_{\lambda/t}(hx_n,hx_{n-1})},\\ &\frac{\omega_{\lambda/t}(hx_n,Sx_n)\omega_{\lambda/t}(hx_n,Sx_{n-1})}{2[1+\omega_{\lambda/t}(hx_n,hx_{n-1})]}, \frac{\omega_{\lambda/t}(hx_{n-1},Sx_{n-1})\omega_{\lambda/t}(hx_{n-1},Sx_n)}{2[1+\omega_{\lambda/t}(hx_n,hx_{n-1})]}\biggr\}.\end{aligned}$$

As in the proof of Theorem 2.1, we get

$$\mathcal{M}(x_n,x_{n-1}) \le \max\bigl\{\omega_{\lambda/a}(hx_{n-1},hx_n), \omega_{\lambda/a}(hx_n,hx_{n+1})\bigr\}.$$

Now if $\omega_{\lambda/a}(hx_n,hx_{n+1}) > \omega_{\lambda/a}(hx_{n-1},hx_n)$, then

$$\begin{aligned}\int_0^{\pi(\omega_{\lambda/a}(hx_{n+1},hx_n))} \varphi(r)\,\mathrm{d}r &\le \int_0^{\pi(\omega_{\lambda/a}(hx_n,hx_{n+1}))} \varphi(r)\,\mathrm{d}r - \phi\biggl(\int_0^{\pi(\omega_{\lambda/t}(hx_n,hx_{n+1}))} \varphi(r)\,\mathrm{d}r\biggr)\\ &< \int_0^{\pi(\omega_{\lambda/a}(hx_n,hx_{n+1}))} \varphi(r)\,\mathrm{d}r.\end{aligned}$$

This gives a contradiction. So, $\mathcal{M}(x_n, x_{n-1}) \leq \omega_{\lambda/a}(hx_{n-1}, hx_n)$. Therefore

$$\begin{aligned}\int_0^{\pi(\omega_{\lambda/a}(hx_{n+1},hx_n))} \varphi(r)\,\mathrm{d}r &\leq \int_0^{\pi(\omega_{\lambda/a}(hx_n,hx_{n-1}))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\omega_{\lambda/t}(hx_n,hx_{n-1}))} \varphi(r)\,\mathrm{d}r\right) \\ &< \int_0^{\pi(\omega_{\lambda/a}(hx_n,hx_{n-1}))} \varphi(r)\,\mathrm{d}r, \end{aligned} \tag{2.11}$$

which implies that there exists $k \geq 0$ such that

$$\lim_{n\to\infty} \int_0^{\pi(\omega_{\lambda/a}(hx_{n+1},hx_n))} \varphi(r)\,\mathrm{d}r = k.$$

If $k > 0$, then by Lemma 2.3 and (2.11), we get the contradiction. So, we have

$$\lim_{n\to\infty} \pi\left(\omega_{\lambda/a}(hx_{n+1}, hx_n)\right) = 0.$$

Since $\pi \in \Pi$, Lemma 2.2 gives

$$\lim_{n\to\infty} \omega_{\lambda/a}(hx_{n+1}, hx_n) = 0.$$

From this we see that $\{hx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since $h(X_\omega)$ is complete, there exists $x^* \in X$ such that $\omega_{\lambda/t}(hx_n, x^*) \to 0$ as $n \to \infty$. Consequently, we can find p in X_ω such that $h(p) = x^*$ By (2.10), we get

$$\begin{aligned}\int_0^{\pi(\omega_{\lambda/a}(hx_n,Sp))} \varphi(r)\,\mathrm{d}r &= \int_0^{\pi(\omega_{\lambda/a}(Sx_{n-1},Sp))} \varphi(r)\,\mathrm{d}r \\ &\leq \int_0^{\pi(\mathcal{M}(x_{n-1},p))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathcal{M}(x_{n-1},p))} \varphi(r)\,\mathrm{d}r\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(x_{n-1}, p) = \max\Bigg\{ &\omega_{\lambda/t}(hx_{n-1}, hp), \omega_{\lambda/t}(hx_{n-1}, Sx_{n-1}), \omega_{\lambda/t}(hp, Sp), \\ &\frac{\omega_{\lambda/t}(hx_{n-1}, Sp) + \omega_{\lambda/t}(hp, Sx_{n-1})}{2}, \frac{\omega_{\lambda/t}(hx_{n-1}, Sp)\omega_{\lambda/t}(hp, Sx_{n-1})}{1 + \omega_{\lambda/t}(hx_{n-1}, hp)} \\ &\frac{\omega_{\lambda/t}(hx_{n-1}, Sx_{n-1})\omega_{\lambda/t}(hx_{n-1}, Sp)}{2[1 + \omega_{\lambda/t}(hx_{n-1}, hp)]}, \frac{\omega_{\lambda/t}(hp, Sp)\omega_{\lambda/t}(hp, Sx_{n-1})}{2[1 + \omega_{\lambda/t}(hx_{n-1}, hp)]}\Bigg\}.\end{aligned}$$

Taking the limit as $n \to \infty$ and using Lemma 2.3 yields

$$\begin{aligned}\int_0^{\pi(\omega_{\lambda/a}(x^*,Sp))} \varphi(r)\,\mathrm{d}r &\leq \int_0^{\pi(\omega_{\lambda/t}(x^*,Sp))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\omega_{\lambda/t}(x^*,Sp))} \varphi(r)\,\mathrm{d}r\right) \\ &< \int_0^{\pi(\omega_{\lambda/t}(x^*,Sp))} \varphi(r)\,\mathrm{d}r \\ &\leq \int_0^{\pi(\omega_{\lambda/a}(x^*,Sp))} \varphi(r)\,\mathrm{d}r.\end{aligned}$$

This contradiction gives $\pi(\omega_{\lambda/a}(Sp,x^*)) = 0$, by using Lemma 2.2, we get $\omega_{\lambda/a}(Sp,x^*) = 0$ for $\lambda > 0$. Hence $Sp = x^*$. Hence x^* is the point of coincidence of S and h. Assume that there is another point of coincidence q in X_ω such that $q \neq x^*$. Then there exists u in X_ω such that $Su = hu = q$. By (2.10), we get

$$\begin{aligned}\int_0^{\pi(\omega_{\lambda/a}(hp,hu))}\varphi(r)\,\mathrm{d}r &= \int_0^{\pi(\omega_{\lambda/a}(Sp,Su))}\varphi(r)\,\mathrm{d}r \\ &\leq \int_0^{\pi(\mathcal{M}(p,u))}\varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathcal{M}(p,u))}\varphi(r)\,\mathrm{d}r\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}(p,u) = \max\biggl\{&\omega_{\lambda/t}(hp,hu), \omega_{\lambda/t}(hp,Sp), \omega_{\lambda/t}(hu,Su), \\ &\frac{\omega_{\lambda/t}(hu,Sp)+\omega_{\lambda/t}(hp,Su)}{2}, \frac{\omega_{\lambda/t}(hp,Su)\omega_{\lambda/t}(hu,Sp)}{1+\omega_{\lambda/t}(hp,hu)}, \\ &\frac{\omega_{\lambda/t}(hp,Sp)\omega_{\lambda/t}(hp,Su)}{2[1+\omega_{\lambda/t}(hp,hu)]}, \frac{\omega_{\lambda/t}(hu,Su)\omega_{\lambda/t}(hu,Sp)}{2[1+\omega_{\lambda/t}(hp,hu)]}\biggr\} \\ = {}&\omega_{\lambda/t}(hp,hu).\end{aligned}$$

So,

$$\begin{aligned}\int_0^{\pi(\omega_{\lambda/a}(hp,hu))}\varphi(r)\,\mathrm{d}r &\leq \int_0^{\pi(\omega_{\lambda/t}(hp,hu))}\varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\omega_{\lambda/t}(hp,hu))}\varphi(r)\,\mathrm{d}r\right) \\ &< \int_0^{\pi(\omega_{\lambda/t}(hp,hu))}\varphi(r)\,\mathrm{d}r \leq \int_0^{\pi(\omega_{\lambda/a}(hp,hu))}\varphi(r)\,\mathrm{d}r,\end{aligned}$$

which is a contradiction. This proves the uniqueness of the point of coincidence. Thus x^* is a unique coincidence point of S and h. By using Lemma 2.1, we see that S and h have a unique common fixed point. □

From Theorem 2.7, we get the following theorems.

Theorem 2.8 *Let X_ω be a modular metric space. Assume $S,h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and*

$$\int_0^{\pi(\omega_{\lambda/a}(Sx,Sy))}\varphi(r)\,\mathrm{d}r \leq \int_0^{\pi(\mathfrak{m}_1(x,y))}\varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathfrak{m}_1(x,y))}\varphi(r)\,\mathrm{d}r\right), \tag{2.12}$$

where $\mathfrak{m}_1(x,y)$ is as in Theorem 2.2, *$\varphi \in \Theta$, $\phi \in \Phi$, and $\pi \in \Pi$. Then there exists a unique common fixed point of S and h.*

Theorem 2.9 *Let X_ω be a modular metric space. Suppose $a,t \in \mathbb{R}^+$ with $a > t$ and $S,h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and*

$$\int_0^{\pi(\omega_{\lambda/a}(Sx,Sy))}\varphi(r)\,\mathrm{d}r \leq \int_0^{\pi(\omega_{\lambda/t}(hx,hy))}\varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\omega_{\lambda/t}(hx,hy))}\varphi(r)\,\mathrm{d}r\right), \tag{2.13}$$

where $\varphi \in \Theta$, $\phi \in \Phi$ and $\pi \in \Pi$. Then there exists a unique common fixed point of S and h.

Remark 2.1 With $h = I$ (identity map) in Theorems 2.1-2.9, we deduce the fixed point results for one map.

Remark 2.2 In case $\phi(t) = (1-r)t$, where $0 < r < 1$, and $\phi(t) = t - \psi(t)$, where $\psi \in \Psi$, then Theorems 2.1-2.9 reduce to corollaries which elongate and generalize Theorems 2.2-4.3 of [2], Theorem 2.1 of [1], Theorems 2.1 and 2.4 of [4], Theorems 2.1-2.4 of [11], Theorems 2.1 and 3.1 of [22], Theorem 2 of [26] and Theorems 3.1 and 3.4 of [3] in the set-up of modular metric space.

By considering similar argument of Theorem 2.7, we state the following results and exclude their proofs.

Theorem 2.10 *Let X_ω be a MMS. Suppose $S, h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and*

$$\int_0^{\pi(\omega_{\lambda/a}(Sx,Sy))} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathfrak{m}_2(x,y))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathfrak{m}_2(x,y))} \varphi(r)\,\mathrm{d}r\right), \tag{2.14}$$

for all $x, y \in X_\omega$, $a, t \in \mathbb{R}$ with $a > t$, where $\mathfrak{m}_2(x,y)$ is as in Theorem 2.4, *$\varphi \in \Theta$, $\phi \in \Phi$ and $\pi \in \Pi$. Then there exists a unique common fixed point of S and h.*

Theorem 2.11 *Let X_ω be a MMS. Suppose $S, h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and*

$$\int_0^{\pi(\omega_{\lambda/a}(Sx,Sy))} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathfrak{m}_3(x,y))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathfrak{m}_3(x,y))} \varphi(r)\,\mathrm{d}r\right), \tag{2.15}$$

for all $x, y \in X_\omega$, $a, t \in \mathbb{R}$ with $a > t$, where $\mathfrak{m}_3(x,y)$ is as in Theorem 2.5, *$\varphi \in \Theta$, $\phi \in \Phi$ and $\pi \in \Pi$. Then there exists a unique common fixed point of S and h.*

Theorem 2.12 *Let X_ω be a MMS. Suppose $S, h : X_\omega \to X_\omega$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 2.1 *and*

$$\int_0^{\pi(\omega_{\lambda/a}(Sx,Sy))} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathfrak{m}_4(x,y))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathfrak{m}_4(x,y))} \varphi(r)\,\mathrm{d}r\right), \tag{2.16}$$

for all $x, y \in X_\omega$, $a, t \in \mathbb{R}$ with $a > t$, where $\mathfrak{m}_4(x,y)$ is as in Theorem 2.6, *$\varphi \in \Theta$, $\phi \in \Phi$ and $\pi \in \Pi$. Then there exists a unique common fixed point of S and h.*

3 Applications to fuzzy metric spaces

In 1988, Grabiec [27] defined contractive mappings on a fuzzy metric space and extended fixed point theorems of Banach and Edelstein in such spaces. Successively, George and Veeramani [28] slightly modified the notion of a fuzzy metric space introduced by Kramosil and Michálek. For more details see [29–31] and the references therein. In this section we deduce fixed point results in a triangular fuzzy metric space.

Definition 3.1 [28] The 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times (0,\infty)$ satisfying the following conditions: for all $x, y, z \in X$, $s, t > 0$,

(1) $M(x,y,t) > 0$,
(2) $M(x,y,t) = 1$ if and only if $x = y$,
(3) $M(x,y,t) = M(y,x,t)$,
(4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$,
(5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous.

Definition 3.2 [29] Let $(X,M,*)$ be a fuzzy metric space. The fuzzy metric M is called triangular whenever

$$\frac{1}{M(x,y,t)} - 1 \leq \frac{1}{M(x,z,t)} - 1 + \frac{1}{M(z,y,t)} - 1 \tag{3.1}$$

for all $x,y,z \in X$ and all $t > 0$.

Lemma 3.1 [27] *For all $x,y,z \in X$, $(X,M,*)$ is non-deceasing on $(0,\infty)$.*

Lemma 3.2 [8] *Let $(X,M,*)$ be a triangular fuzzy metric space. Define*

$$\omega_\lambda(x,y) = \frac{1}{M(x,y,\lambda)} - 1 \tag{3.2}$$

for all $x,y,z \in X$ and all $\lambda > 0$. Then ω_λ is a modular metric on X.

Definition 3.3 [32] Two self-mappings S and h of a fuzzy metric space $(X,M,*)$ are called weakly compatible if they commute at their coincidence points.

As an application of Lemma 3.2 and the results proved above, we deduce the following new fixed point theorems in triangular fuzzy metric spaces.

Theorem 3.1 *Let $(X,M,*)$ be a triangular fuzzy metric space. Suppose $a,t \in \mathbb{R}^+$ with $a > t$ and $S,h : X \to X$ are two self-mappings satisfying the following assertions*:
(1) *$S(X) \subseteq h(X)$, $h(X)$ is a complete subspace of X*;
(2) *S and h are weakly compatible mappings*;
(3) $\int_0^{\frac{1}{M(Sx,Sy,\lambda/a)}-1} \varphi(r)\,\mathrm{d}r \leq \int_0^{\mathcal{N}(x,y)} \varphi(r)\,\mathrm{d}r - \phi(\int_0^{\mathcal{N}(x,y)} \varphi(r)\,\mathrm{d}r)$,

where

$$\begin{aligned}\mathcal{N}(x,y) = \max\bigg\{ & \frac{1}{M(hx,hy,\lambda/t)} - 1, \frac{1}{M(hx,Sx,\lambda/t)} - 1, \\ & \frac{1}{M(hy,Sy,\lambda/t)} - 1, \frac{1}{2M(hx,Sy,\lambda/t)} + \frac{1}{2M(hy,Sx,\lambda/t)} - 1, \\ & M(hx,hy,\lambda/t)\left(\frac{1}{M(hx,Sy,\lambda/t)} - 1\right)\left(\frac{1}{M(hy,Sx,\lambda/t)} - 1\right), \\ & \frac{M(hx,hy,\lambda/t)}{2}\left(\frac{1}{M(hx,Sx,\lambda/t)} - 1\right)\left(\frac{1}{M(hx,Sy,\lambda/t)} - 1\right), \\ & \frac{M(hx,hy,\lambda/t)}{2}\left(\frac{1}{M(hy,Sy,\lambda/t)} - 1\right)\left(\frac{1}{M(hy,Sx,\lambda/t)} - 1\right)\bigg\},\end{aligned}$$

$\varphi \in \Theta$ and $\phi \in \Phi$. Then S and h have a unique common fixed point.

Theorem 3.2 *Let $(X, M, *)$ be a triangular fuzzy metric space. Assume that $a, t \in \mathbb{R}^+$ with $a > t$ and $S, h : X \to X$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\frac{1}{M(Sx,Sy,\lambda/a)}-1} \varphi(r)\,\mathrm{d}r \le \int_0^{\mathfrak{n}_1(x,y)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathfrak{n}_1(x,y)} \varphi(r)\,\mathrm{d}r\right), \tag{3.3}$$

for all $x, y \in X$ and $\lambda \ge 0$, where

$$\begin{aligned}\mathfrak{n}_1(x,y) = \max\bigg\{ &\frac{1}{M(hx,hy,\lambda/t)} - 1, \frac{1}{M(hx,Sx,\lambda/t)} - 1, \frac{1}{M(hy,Sy,\lambda/t)} - 1, \\ &\frac{1}{2M(hx,Sy,\lambda/t)} + \frac{1}{2M(hy,Sx,\lambda/t)} - 1\bigg\},\end{aligned}$$

$\varphi \in \Theta$ and $\phi \in \Phi$. Then S and h have a unique common fixed point.

Theorem 3.3 *Let $(X, M, *)$ be a triangular fuzzy metric space. Assume that $a, t \in \mathbb{R}^+$ with $a > t$ and $S, h : X \to X$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\frac{1}{M(Sx,Sy,\lambda/a)}-1} \varphi(r)\,\mathrm{d}r \le \int_0^{\frac{1}{M(hx,hy,\lambda/a)}-1} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\frac{1}{M(hx,hy,\lambda/a)}-1} \varphi(r)\,\mathrm{d}r\right), \tag{3.4}$$

for all $x, y \in X$ and $\lambda \ge 0$, where $\varphi \in \Theta$ and $\phi \in \Phi$. Then S and h have a unique common fixed point.

Theorem 3.4 *Let $(X, M, *)$ be a triangular fuzzy metric space. Suppose $a, t \in \mathbb{R}^+$ with $a > t$ and $S, h : X \to X$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\frac{1}{M(Sx,Sy,\lambda/a)}-1} \varphi(r)\,\mathrm{d}r \le \int_0^{\mathfrak{n}_2(x,y)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathfrak{n}_2(x,y)} \varphi(r)\,\mathrm{d}r\right), \tag{3.5}$$

where

$$\begin{aligned}\mathfrak{n}_2(x,y) = \max\bigg\{ &\frac{1}{M(hx,hy,\lambda/t)} - 1, \frac{1}{M(hx,Sx,\lambda/t)} - 1, \\ &\frac{1}{M(hy,Sy,\lambda/t)} - 1, \frac{1}{2M(hx,Sy,\lambda/t)} + \frac{1}{2M(hy,Sx,\lambda/t)} - 1, \\ &M(Sx,Sy,\lambda/t)\left(\frac{1}{M(hy,Sx,\lambda/t)} - 1\right)\left(\frac{1}{M(hx,Sy,\lambda/t)} - 1\right), \\ &\frac{M(Sx,Sy,\lambda/t)}{2}\left(\frac{1}{M(hy,Sx,\lambda/t)} - 1\right)\left(\frac{1}{M(hx,Sx,\lambda/t)} - 1\right), \\ &\frac{M(Sx,Sy,\lambda/t)}{2}\left(\frac{1}{M(hx,Sy,\lambda/t)} - 1\right)\left(\frac{1}{M(hy,Sy,\lambda/t)} - 1\right)\bigg\},\end{aligned}$$

$\varphi \in \Theta$ and $\phi \in \Phi$. Then S and h have a unique common fixed point.

Theorem 3.5 *Let $(X, M, *)$ be a triangular fuzzy metric space. Suppose $a, t \in \mathbb{R}^+$ with $a > t$ and $S, h : X \to X$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\frac{1}{M(Sx,Sy,\lambda/a)}-1} \varphi(r)\,\mathrm{d}r \leq \int_0^{\mathfrak{n}_3(x,y)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathfrak{n}_3(x,y)} \varphi(r)\,\mathrm{d}r\right), \tag{3.6}$$

where

$$\begin{aligned} \mathfrak{n}_3(x,y) = \max\biggl\{ & \frac{1}{M(hx,hy,\lambda/t)} - 1, \frac{1}{M(hx,Sx,\lambda/t)} - 1, \\ & \frac{1}{M(hy,Sy,\lambda/t)} - 1, \frac{1}{2M(hx,Sy,\lambda/t)} + \frac{1}{2M(hy,Sx,\lambda/t)} - 1, \\ & M(hx,hy,\lambda/t)\left(\frac{1}{M(hx,Sy,\lambda/t)} - 1\right)\left(\frac{1}{M(hy,Sx,\lambda/t)} - 1\right), \\ & \min\biggl\{ \frac{M(hx,hy,\lambda/t)}{2}\left(\frac{1}{M(hx,Sx,\lambda/t)} - 1\right)\left(\frac{1}{M(hx,Sy,\lambda/t)} - 1\right), \\ & \frac{M(hx,hy,\lambda/t)}{2}\left(\frac{1}{M(hy,Sy,\lambda/t)} - 1\right)\left(\frac{1}{M(hy,Sx,\lambda/t)} - 1\right)\biggr\}\biggr\}, \end{aligned}$$

$\varphi \in \Theta$ and $\phi \in \Phi$. Then S and h have a unique common fixed point.

Theorem 3.6 *Let $(X, M, *)$ be a triangular fuzzy metric space. Suppose $a, t \in \mathbb{R}^+$ with $a > t$ and $S, h : X \to X$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\frac{1}{M(Sx,Sy,\lambda/a)}-1} \varphi(r)\,\mathrm{d}r \leq \int_0^{\mathfrak{n}_4(x,y)} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\mathfrak{n}_4(x,y)} \varphi(r)\,\mathrm{d}r\right), \tag{3.7}$$

where

$$\begin{aligned} \mathfrak{n}_4(x,y) = \max\biggl\{ & \frac{1}{M(hx,hy,\lambda/t)} - 1, \frac{1}{M(hx,Sx,\lambda/t)} - 1, \\ & \frac{1}{M(hy,Sy,\lambda/t)} - 1, \frac{1}{2M(hx,Sy,\lambda/t)} + \frac{1}{2M(hy,Sx,\lambda/t)} - 1, \\ & M(Sx,Sy,\lambda/t)\left(\frac{1}{M(hy,Sx,\lambda/t)} - 1\right)\left(\frac{1}{M(hx,Sy,\lambda/t)} - 1\right), \\ & \min\biggl\{ \frac{M(Sx,Sy,\lambda/t)}{2}\left(\frac{1}{M(hy,Sx,\lambda/t)} - 1\right)\left(\frac{1}{M(hx,Sx,\lambda/t)} - 1\right), \\ & \frac{M(Sx,Sy,\lambda/t)}{2}\left(\frac{1}{M(hx,Sy,\lambda/t)} - 1\right)\left(\frac{1}{M(hy,Sy,\lambda/t)} - 1\right)\biggr\}\biggr\}, \end{aligned}$$

$\varphi \in \Theta$ and $\phi \in \Phi$. Then S and h have a unique common fixed point.

Theorem 3.7 *Let $(X, M, *)$ be a triangular fuzzy metric space. Suppose $a, t \in \mathbb{R}^+$ with $a > t$ and $S, h : X \to X$ are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1

and

$$\int_0^{\pi(\frac{1}{M(Sx,Sy,\lambda/a)}-1)} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathcal{N}(x,y))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathcal{N}(x,y))} \varphi(r)\,\mathrm{d}r\right), \tag{3.8}$$

for all $x, y \in X$, $a, t \in \mathbb{R}$ *with* $a > t$, *where* $\mathcal{N}(x,y)$ *is as in Theorem* 3.1, $\varphi \in \Theta$, $\phi \in \Phi$ *and* $\pi \in \Pi$. *Then there exists a unique common fixed point of S and h.*

Theorem 3.8 *Let* $(X, M, *)$ *be a triangular fuzzy metric space. Suppose* $a, t \in \mathbb{R}^+$ *with* $a > t$ *and* $S, h : X \to X$ *are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\pi(\frac{1}{M(Sx,Sy,\lambda/a)}-1)} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathfrak{n}_1(x,y))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathfrak{n}_1(x,y))} \varphi(r)\,\mathrm{d}r\right), \tag{3.9}$$

for all $x, y \in X$, $a, t \in \mathbb{R}$ *with* $a > t$, *where* $\mathfrak{n}_1(x,y)$ *is as in Theorem* 3.2, $\varphi \in \Theta$, $\phi \in \Phi$ *and* $\pi \in \Pi$. *Then there exists a unique common fixed point of S and h.*

Theorem 3.9 *Let* $(X, M, *)$ *be a triangular fuzzy metric space. Suppose* $a, t \in \mathbb{R}^+$ *with* $a > t$ *and* $S, h : X \to X$ *are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\pi(\frac{1}{M(Sx,Sy,\lambda/a)}-1)} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathfrak{n}_2(x,y))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathfrak{n}_2(x,y))} \varphi(r)\,\mathrm{d}r\right), \tag{3.10}$$

for all $x, y \in X$, $a, t \in \mathbb{R}$ *with* $a > t$, *where* $\mathfrak{n}_2(x,y)$ *is as in Theorem* 3.4, $\varphi \in \Theta$, $\phi \in \Phi$, *and* $\pi \in \Pi$. *Then there exists a unique common fixed point of S and h.*

Theorem 3.10 *Let* $(X, M, *)$ *be a triangular fuzzy metric space. Suppose* $a, t \in \mathbb{R}^+$ *with* $a > t$ *and* $S, h : X \to X$ *are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\pi(\frac{1}{M(Sx,Sy,\lambda/a)}-1)} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathfrak{n}_3(x,y))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathfrak{n}_3(x,y))} \varphi(r)\,\mathrm{d}r\right), \tag{3.11}$$

for all $x, y \in X$, $a, t \in \mathbb{R}$ *with* $a > t$, *where* $\mathfrak{n}_3(x,y)$ *is as in Theorem* 3.5, $\varphi \in \Theta$, $\phi \in \Phi$ *and* $\pi \in \Pi$. *Then there exists a unique common fixed point of S and h.*

Theorem 3.11 *Let* $(X, M, *)$ *be a triangular fuzzy metric space. Suppose* $a, t \in \mathbb{R}^+$ *with* $a > t$ *and* $S, h : X \to X$ *are two self-mappings satisfying the conditions* (1) *and* (2) *of Theorem* 3.1 *and*

$$\int_0^{\pi(\frac{1}{M(Sx,Sy,\lambda/a)}-1)} \varphi(r)\,\mathrm{d}r \le \int_0^{\pi(\mathfrak{n}_4(x,y))} \varphi(r)\,\mathrm{d}r - \phi\left(\int_0^{\pi(\mathfrak{n}_4(x,y))} \varphi(r)\,\mathrm{d}r\right), \tag{3.12}$$

for all $x, y \in X$, $a, t \in \mathbb{R}$ *with* $a > t$, *where* $\mathfrak{n}_4(x,y)$ *is as in Theorem* 3.6, $\varphi \in \Theta$, $\phi \in \Phi$, *and* $\pi \in \Pi$. *Then there exists a unique common fixed point of S and h.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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