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General L_p-mixed chord integrals of star bodies

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Abstract

The notion of general mixed chord integrals of star bodies was introduced by Feng and Wang. In this paper, we extend the concept of the general mixed chord integrals to general L_p-mixed chord integrals of star bodies. Based on this new notion, we study their extremum values and obtain an Aleksandrov-Frenchel type and a cyclic inequality for general L_p-mixed chord integrals of star bodies, respectively. Further, as applications, we establish two Brunn-Minkowski type inequalities for L_p-radial bodies. Finally, we get an interesting identical equality on combining L_p-radial bodies.

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1 Introduction and main results

The setting for this paper is n-dimensional Euclidean spaces $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in $\mathbb{R}^n$. $\mathcal{K}_0^n$ denotes the set of convex bodies containing the origin in their interiors. The n-dimensional volume of a body K is written as $V(K)$. The unit sphere in $\mathbb{R}^n$ is denoted $\mathcal{S}^{n-1}$ and B denotes the standard unit ball in $\mathbb{R}^n$.

If K is a compact star shaped (about the origin) set in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \to [0, \infty)$, is defined by (see [1, 2])

$$\rho(K, u) = \max\{\lambda \geq 0, \lambda u \in K\}, \quad u \in \mathcal{S}^{n-1}.$$

If ρ_K is positive and continuous, then K is called a star body (about the origin) and $\mathcal{S}^n$ denotes the set of star bodies in $\mathbb{R}^n$. We will use $\mathcal{S}_0^n$ to denote the subset of $\mathcal{S}^n$ containing the origin in their interiors. Two star bodies K and L are said to be dilated of one another if $\rho_K(u)/\rho_L(u)$ is independent of $u \in \mathcal{S}^{n-1}$.

The notion of mixed chord integrals of star bodies was defined by Lu (see [3]). Feng and Wang (see [4]) generalized the definition of mixed chord integrals to the general mixed chord integrals of star bodies: For $K_1, \ldots, K_n \in \mathcal{S}_0^n$ and $\tau \in (-1, 1)$, the general mixed chord integral $C^{(\tau)}(K_1, \ldots, K_n)$ is defined by

$$C^{(\tau)}(K_1, \ldots, K_n) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c^{(\tau)}(K_1, u) \cdots c^{(\tau)}(K_n, u) \, du,$$

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where $c^{(\tau)}(K,\cdot) = f_1(\tau)\rho(K,\cdot) + f_2(\tau)\rho(-K,\cdot)$ and the functions $f_1(\tau)$ and $f_2(\tau)$ are defined as follows:

$$f_1(\tau) = \frac{(1+\tau)^2}{2(1+\tau^2)}, \qquad f_2(\tau) = \frac{(1-\tau)^2}{2(1+\tau^2)}.$$

The general mixed chord integral is a map $\underbrace{\mathcal{S}_o^n \times \cdots \times \mathcal{S}_o^n}_{n} \to \mathbb{R}$. It is positive, continuous, multilinear with respect to radial Minkowski linear combinations, positively homogeneous and monotone under set inclusion. Star bodies $K_1, \ldots, K_n$ are said to have a similar chord if there exist constants $\lambda_1, \ldots, \lambda_n > 0$ such that $\lambda_1 c^{(\tau)}(K_1, u) = \cdots = \lambda_n c^{(\tau)}(K_n, u)$ for all $u \in S^{n-1}$. Feng and Wang (see [4]) established the following inequalities.

Theorem 1.A *If $\tau \in (-1,1)$ and $K_1, \ldots, K_n \in \mathcal{S}_o^n$, then*

$$C^{(\tau)}(K_1, \ldots, K_n)^n \leq V(K_1) \cdots V(K_n),$$

with equality if and only if $K_1, \ldots, K_n$ are dilates of each other centered.

Theorem 1.B *If $\tau \in (-1,1)$ and $K_1, \ldots, K_n \in \mathcal{S}_o^n$, then for $1 < m \leq n$,*

$$C^{(\tau)}(K_1, \ldots, K_n)^m \leq \prod_{i=1}^{m} C^{(\tau)}(K_1, \ldots, K_{n-m}, K_{n-i+1}, K_{n-i+1}, \ldots, K_{n-i+1}),$$

with equality if and only if $K_{n-m+1}, \ldots, K_n$ are all of similar general chord.

Theorem 1.C *If $\tau \in (-1,1)$ and $K, L \in \mathcal{S}_o^n$, then for $i < j < k$,*

$$C_i^{(\tau)}(K,L)^{k-j} C_k^{(\tau)}(K,L)^{j-i} \geq C_j^{(\tau)}(K,L)^{k-i},$$

with equality if and only if K and L have a similar general chord.

Here $C_i^{(\tau)}(K,L) = C_i^{(\tau)}(K, n-i; L, i)$ in which K appears $n-i$ times and L appears i times.

Motivated by Lu and Feng and Wang (see [3, 4]), in this paper, we define the general L_p-mixed chord integrals as follows: For $K_1, \ldots, K_n \in \mathcal{S}_0^n$, $p > 0$, and $\tau \in [-1, 1]$, the general L_p-mixed chord integral $C_p^{(\tau)}(K_1, \ldots, K_n)$ of $K_1, \ldots, K_n$ is defined by

$$C_p^{(\tau)}(K_1, \ldots, K_n) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K_1, u) \cdots c_p^{(\tau)}(K_n, u) \, du. \tag{1.1}$$

Here $c_p^{(\tau)}(K, \cdot)$ is defined by

$$c_p^{(\tau)}(K, u) = \left(f_1(\tau)\rho^p(K, u) + f_2(\tau)\rho^p(-K, u) \right)^{\frac{1}{p}}$$

for any $u \in \mathcal{S}^{n-1}$, and $f_1(\tau)$ and $f_2(\tau)$ are chosen as (see [5])

$$f_1(\tau) = \frac{(1+\tau)^p}{(1+\tau)^p + (1-\tau)^p}, \qquad f_2(\tau) = \frac{(1-\tau)^p}{(1+\tau)^p + (1-\tau)^p}. \tag{1.2}$$

Obviously, $f_1(\tau)$ and $f_2(\tau)$ satisfy

$$f_1(\tau) + f_2(\tau) = 1;$$

$$f_1(-\tau) = f_2(\tau), \qquad f_2(-\tau) = f_1(\tau).$$

The general L_p-mixed chord integral is also a map $\underbrace{\mathcal{S}_o^n \times \cdots \times \mathcal{S}_o^n}_{n} \to \mathbb{R}$ and it is positive, continuous, multilinear with respect to L_p-radial Minkowski linear combinations, positively homogeneous and monotone under set inclusion. If there exist constants $\lambda_1, \ldots, \lambda_n > 0$ such that $\lambda_1 c_p^{(\tau)}(K_1, u) = \cdots = \lambda_n c_p^{(\tau)}(K_n, u)$ for all $u \in \mathcal{S}^{n-1}$, star bodies $K_1, \ldots, K_n$ are said to have a similar general L_p-chord.

Let $\underbrace{K_1 = \cdots = K_{n-i}}_{n-i} = K$ and $\underbrace{K_{n-i+1} = \cdots = K_n}_{i} = L$ ($i = 0, 1, \ldots, n$) in (1.1), we denote $C_{p,i}^{(\tau)}(K, L)$ in which K appears $n - i$ times and L appears i times. From this, if i is any real, we define the following.

For $K, L \in \mathcal{S}_o^n$, $p > 0$, $\tau \in [-1, 1]$, and i any real, the ith general L_p-mixed chord integral, $C_{p,i}^{(\tau)}(K, L)$, of K and L is given by

$$C_{p,i}^{(\tau)}(K, L) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K, u)^{n-i} c_p^{(\tau)}(L, u)^i \, du. \tag{1.3}$$

Further, if $L = B$ in (1.3), we write $C_{p,i}^{(\tau)}(K, B)$ as $C_{p,i}^{(\tau)}(K)$ and notice that $c_p^{(\tau)}(B, u) = 1$, we have

$$C_{p,i}^{(\tau)}(K) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K, u)^{n-i} \, du, \tag{1.4}$$

where $C_{p,i}^{(\tau)}(K)$ is called the ith general L_p-mixed chord integral of K. If $i = 0$, we write $C_{p,0}^{(\tau)}(K) = C_p^{(\tau)}(K)$, which we call the general L_p-chord integral of K, that is,

$$C_p^{(\tau)}(K) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K, u)^n \, du.$$

The general L_p-mixed chord integrals of K belong to a new and rapidly evolving asymmetric L_p Brunn-Minkowski theory that has its origins in the work of Ludwig *et al.* (see [5–11]). For the further research of asymmetric L_p Brunn-Minkowski theory, also see [12–28].

In this paper, motivated by Feng and Wang (see [4]), we deduce several inequalities for the general L_p-mixed chord integrals.

Theorem 1.1 *If $\tau \in [-1, 1]$ and $K_1, \ldots, K_n \in \mathcal{S}_o^n$, then for $p < n$,*

$$C_p^{(\tau)}(K_1, \ldots, K_n)^n \leq V(K_1) \cdots V(K_n); \tag{1.5}$$

for $p > n$,

$$C_p^{(\tau)}(K_1, \ldots, K_n)^n \geq V(K_1) \cdots V(K_n), \tag{1.6}$$

with equality if and only if $K_1, \ldots, K_n$ are dilates of each other.

Theorem 1.2 *If $K \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, then for $i < n - p$,*

$$C_{p,i}^{(0)}(K) \leq C_{p,i}^{(\tau)}(K) \leq C_{p,i}^{(\pm 1)}(K); \tag{1.7}$$

for $n - p < i < n$ or $i > n$,

$$C_{p,i}^{(0)}(K) \leq C_{p,i}^{(\tau)}(K) \leq C_{p,i}^{(\pm 1)}(K). \tag{1.8}$$

If K is not origin-symmetric, equality holds in the left inequality if and only if $\tau = 0$ and equality holds in the right inequality if and only if $\tau = \pm 1$. When $i = n$, (1.7) and (1.8) are identical.

Theorem 1.3 *If $K_1, \ldots, K_n \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, then for $1 < m \leq n$,*

$$C_p^{(\tau)}(K_1, \ldots, K_n)^m \leq \prod_{i=1}^m C_p^{(\tau)}(K_1, \ldots, K_{n-m}, K_{n-i+1}, K_{n-i+1}, \ldots, K_{n-i+1}), \tag{1.9}$$

with equality if and only if $K_{n-m+1}, \ldots, K_n$ all have a similar general L_p-chord.

Theorem 1.4 *If $K, L \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, then, for $i < j < k$,*

$$C_{p,j}^{(\tau)}(K, L)^{k-i} \leq C_{p,i}^{(\tau)}(K, L)^{k-j} C_{p,k}^{(\tau)}(K, L)^{j-i}, \tag{1.10}$$

with equality if and only if K and L have a similar general L_p-chord.

Further, as applications, for the L_p-radial combination, we give two Brunn-Minkowski type inequalities for the general L_p-mixed chord integrals as follows.

Theorem 1.5 *If $K, L \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, $\lambda, \mu \geq 0$ (not both zero), then for $i \leq n - p$,*

$$C_{p,i}^{(\tau)}(\lambda \circ K \, \tilde{\mp}_p \, \mu \circ L)^{\frac{p}{n-i}} \leq \lambda C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + \mu C_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}; \tag{1.11}$$

for $n - p < i < n$ or $i > n$,

$$C_{p,i}^{(\tau)}(\lambda \circ K \, \tilde{\mp}_p \, \mu \circ L)^{\frac{p}{n-i}} \geq \lambda C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} + \mu C_{p,i}^{(\tau)}(L)^{\frac{p}{n-i}}, \tag{1.12}$$

with equality in each inequality if and only if K and L have a similar general L_p-chord. Here and in the following theorem $\lambda \circ K \, \tilde{\mp}_p \, \mu \circ L$ denotes the L_p-radial Minkowski combination of K and L.

Theorem 1.6 *If $K, L \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, then for $i \leq n - p \leq j \leq n$,*

$$\left(\frac{C_{p,i}^{(\tau)}(K \, \tilde{\mp}_p \, L)}{C_{p,j}^{(\tau)}(K \, \tilde{\mp}_p \, L)} \right)^{\frac{p}{j-i}} \leq \left(\frac{C_{p,i}^{(\tau)}(K)}{C_{p,j}^{(\tau)}(K)} \right)^{\frac{p}{j-i}} + \left(\frac{C_{p,i}^{(\tau)}(L)}{C_{p,j}^{(\tau)}(L)} \right)^{\frac{p}{j-i}}; \tag{1.13}$$

for $j \geq n \geq i \geq n - p$, inequality (1.13) is reversed, with equality in each inequality if and only if K and L have a similar general L_p-chord.

Finally, we get an interesting identical equality for the L_p-radial bodies.

Theorem 1.7 *If $K \in \mathcal{S}_o^n$ and $p > 0$, then for $\forall \tau \in [-1, 1]$, we have*

$$C_{p,i}^{(\tau)}(\bar{\triangle}_p K) = C_{p,i}(K). \tag{1.14}$$

The proofs of Theorems 1.1-1.7 are given in Section 3 of this paper. In the following section, we summarize some associated properties of the general L_p-mixed chord integrals needed in the proof of main results.

2 Preliminaries

The dual quermassintegral was defined by Lutwak (see [29]) as follows. For $K \in \mathcal{S}_o^n$ and any real i, the dual quermassintegral, $\widetilde{W}_i(K)$, of K is defined by

$$\widetilde{W}_i(K) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} \rho(K, u)^{n-i} \, du, \tag{2.1}$$

obviously, (2.1) implies the volume $V(K)$ of K is

$$V(K) = \widetilde{W}_0(K) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} \rho(K, u)^n \, du. \tag{2.2}$$

If E is a nonempty set in $\mathbb{R}^n$, then the polar set of E, E^*, is defined by (see [1])

$$E^* = \{ x \in \mathbb{R}^n : x \cdot y \le 1, \text{any } y \in E \}.$$

For $K \in \mathcal{S}_{os}^n$, an extension of the Blaschke-Santaló inequality takes the following form (see [30]).

Theorem 2.A *If $K \in \mathcal{S}_{os}^n$, then*

$$V(K)V(K^*) \le \omega_n^2, \tag{2.3}$$

with equality if and only if K is an ellipsoid. Here $\omega_n = V(B)$.

For $K, L \in \mathcal{S}_o^n$, $p > 0$, and $\lambda, \mu \ge 0$ (not both zero), the Firey L_p-radial combination, $\lambda \circ K \tilde{+}_p \mu \circ L$, of K and L is defined by (see [6, 31])

$$\rho(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot)^p = \lambda \rho(K, \cdot)^p + \mu \rho(L, \cdot)^p, \tag{2.4}$$

here $\lambda \circ K$ denotes the L_p-radial scalar multiplication and $\lambda \circ K = \lambda^{\frac{1}{p}} K$.

In (2.4), let $\lambda = \mu = \frac{1}{2}$ and let $L = -K$, we have the definition of L_p-radial body $\bar{\triangle}_p K$ as follows (see [31]):

$$\bar{\triangle}_p K = \frac{1}{2} \circ K \tilde{+}_p \frac{1}{2} \circ (-K). \tag{2.5}$$

3 Proofs of main results

In this section, we prove Theorems 1.1-1.7. The proofs of Theorems 1.1 and 1.2 require the following lemma.

Lemma 3.1 *If $K \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, then for $i < n - p$,*

$$C_{p,i}^{(\tau)}(K) \leq \widetilde{W}_i(K); \tag{3.1}$$

for $n - p < i < n$ or $i > n$,

$$C_{p,i}^{(\tau)}(K) \leq \widetilde{W}_i(K), \tag{3.2}$$

with equality if and only if K is centered. When $i = n$, (3.1) and (3.2) are identical.

Proof From definitions (1.1) and (1.4), and using Minkowski's inequality (see [32]), for $i < n - p$, we have

$$
\begin{aligned}
C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}} &= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K, u)^{n-i} \, du \right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(f_1(\tau) \rho^p(K, u) + f_2(\tau) \rho^p(-K, u) \right)^{\frac{n-i}{p}} \, du \right]^{\frac{p}{n-i}} \\
&\leq \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(f_1(\tau) \rho^p(K, u) \right)^{\frac{n-i}{p}} \, du \right]^{\frac{p}{n-i}} \\
&\quad + \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(f_2(\tau) \rho^p(-K, u) \right)^{\frac{n-i}{p}} \, du \right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(\rho^p(K, u) \right)^{\frac{n-i}{p}} \, du \right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \rho(K, u)^{n-i} \, du \right]^{\frac{p}{n-i}}
\end{aligned}
$$

i.e.

$$C_{p,i}^{(\tau)}(K) \leq \frac{1}{n} \int_{\mathcal{S}^{n-1}} \rho(K, u)^{n-i} \, du = \widetilde{W}_i(K).$$

Similarly, for $n - p < i < n$ or $i < n$, by Minkowski's inequality we know (3.2) holds. From the equality condition of Minkowski's inequality, we see that equalities hold in (3.1) and (3.2) if and only if K and $-K$ are dilates of one another, that is, K is centered. $\square$

In Lemma 3.1, let $i = 0$ and notice (2.2), we have the following corollary immediately.

Corollary 3.1 *If $K \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, then for $p < n$,*

$$C_p^{(\tau)}(K) \leq V(K); \tag{3.3}$$

for $p > n$,

$$C_p^{(\tau)}(K) \geq V(K), \tag{3.4}$$

with equality if and only if K is centered.

Theorem 2.A leads to the following corollary.

Corollary 3.2 *If $K \in \mathcal{S}_{os}^n$ and $\tau \in [-1,1]$, $p > 0$, then for $p < n$ and $n \geq 1$,*

$$C_p^{(\tau)}(K)C_p^{(\tau)}\big(K^*\big) \leq \omega_n^2, \tag{3.5}$$

with equality in each inequality if and only if K is an ellipsoid centered at the origin.

Proof of Theorem 1.1 Using Hölder's inequality (see [32]), we have for $p < n$ and $n \geq 1$,

$$
\begin{aligned}
C_p^{(\tau)}(K_1,\ldots,K_n) &= \left[\frac{1}{n}\int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K_1,u)\cdots c_p^{(\tau)}(K_n,u)\,du\right]^{n\cdot\frac{1}{n}} \\
&\leq \left[\frac{1}{n}\int_{\mathcal{S}^{n-1}}\big(c_p^{(\tau)}(K_1,u)\big)^n\,du\right]^{\frac{1}{n}}\times\cdots\times\left[\frac{1}{n}\int_{\mathcal{S}^{n-1}}\big(c_p^{(\tau)}(K_n,u)\big)^n\,du\right]^{\frac{1}{n}} \\
&= C_p^{(\tau)}(K_1)^{\frac{1}{n}}\cdots C_p^{(\tau)}(K_n)^{\frac{1}{n}} \\
&\leq V(K_1)^{\frac{1}{n}}\cdots V(K_n)^{\frac{1}{n}}
\end{aligned}
$$

i.e. $C_p^{(\tau)}(K_1,\ldots,K_n)^n \leq V(K_1)\cdots V(K_n)$.

Similarly, we can obtain $C_p^{(\tau)}(K_1,\ldots,K_n)^n \geq V(K_1)\cdots V(K_n)$ when $p > n$. The equality conditions of (3.3) and (3.4) tell us that equalities hold in Theorem 1.1 if and only if $K_1,\ldots,K_n$ are dilates of each other. □

Proof of Theorem 1.2 From the proof of Lemma 3.1, for $i < n - p$, we get

$$C_{p,i}^{(\tau)}(K) \leq \frac{1}{n}\int_{\mathcal{S}^{n-1}}\big(\rho^p(K,u)\big)^{\frac{n-i}{p}}\,du \quad\text{or}\quad \frac{1}{n}\int_{\mathcal{S}^{n-1}}\big(\rho^p(-K,u)\big)^{\frac{n-i}{p}}\,du. \tag{3.6}$$

Notice (1.2), $f_1(+1) = 1$, $f_2(+1) = 0$, we have

$$C_{p,i}^{(+1)}(K) = \frac{1}{n}\int_{\mathcal{S}^{n-1}}\big(\rho^p(K,u)\big)^{\frac{n-i}{p}}\,du.$$

Similarly, we obtain

$$C_{p,i}^{(-1)}(K) = \frac{1}{n}\int_{\mathcal{S}^{n-1}}\big(\rho^p(-K,u)\big)^{\frac{n-i}{p}}\,du.$$

Together with (3.6), it follows that

$$C_{p,i}^{(\tau)}(K) \leq C_{p,i}^{(\pm1)}(K),$$

which is just the right inequality of (1.7).

According the equality condition of Lemma 3.1, we know that if $\tau \neq \pm 1$, equality holds in the right inequality of (1.7) if and only if K is centered, that means that, if K is not origin-symmetric, equality holds in the right inequality if and only if $\tau = \pm 1$.

Now we prove the left inequality of (1.7). When $\tau = 0$, we have $f_1(\tau) = f_2(\tau) = \frac{1}{2}$, thus, by Minkowski's inequality,

$$
\begin{aligned}
C_{p,i}^{(0)}(K)^{\frac{p}{n-i}} &= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(\frac{1}{2} \rho^p(K,u) + \frac{1}{2} \rho^p(-K,u) \right)^{\frac{n-i}{p}} du \right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(\frac{1}{2} f_1(\tau) \rho^p(K,u) + \frac{1}{2} f_2(\tau) \rho^p(-K,u) \right. \right. \\
&\qquad \left. \left. + \frac{1}{2} f_1(\tau) \rho^p(-K,u) + \frac{1}{2} f_2(\tau) \rho^p(K,u) \right)^{\frac{n-i}{p}} du \right]^{\frac{p}{n-i}} \\
&\leq \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(\frac{1}{2} f_1(\tau) \rho^p(K,u) + \frac{1}{2} f_2(\tau) \rho^p(-K,u) \right)^{\frac{n-i}{p}} du \right]^{\frac{p}{n-i}} \\
&\qquad + \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(\frac{1}{2} f_1(\tau) \rho^p(-K,u) + \frac{1}{2} f_2(\tau) \rho^p(K,u) \right)^{\frac{n-i}{p}} du \right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(f_1(\tau) \rho^p(K,u) + f_2(\tau) \rho^p(-K,u) \right)^{\frac{n-i}{p}} du \right]^{\frac{p}{n-i}} \\
&= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K,u)^{n-i} du \right]^{\frac{p}{n-i}} \\
&= C_{p,i}^{(\tau)}(K)^{\frac{p}{n-i}}
\end{aligned}
$$

i.e. $C_{p,i}^{(0)}(K) \leq C_{p,i}^{(\tau)}(K)$, which just is the left inequality of (1.7).

The equality condition of Minkowski's inequality tells us that if $\tau \neq 0$, equality holds in the left inequality of (1.7) if and only if K is centered, then if K is not origin-symmetric, equality holds in the right inequality if and only if $\tau = 0$.

For the case $n - p < i < n$ or $i > n$, the proof is similar. □

The proof of Theorem 1.3 requires the following inequality (see [29]).

Lemma 3.2 *If $f_0, f_1, \ldots, f_m$ are strictly positive continuous functions defined on $\mathcal{S}^{n-1}$ and $\lambda_1, \ldots, \lambda_m$ are positive constants the sum of whose reciprocals is unity, then*

$$
\int_{\mathcal{S}^{n-1}} f_0(u) f_1(u) \cdots f_m(u) \, du \leq \prod_{i=1}^{m} \left[\int_{\mathcal{S}^{n-1}} f_0(u) f_i^{\lambda_i}(u) \, du \right]^{\frac{1}{\lambda_i}}, \tag{3.7}
$$

with equality if and only if there exist positive constants $\alpha_1, \ldots, \alpha_m$ such that $\alpha_1 f_1^{\lambda_1}(u) = \cdots = \alpha_m f_m^{\lambda_m}(u)$ for all $u \in \mathcal{S}^{n-1}$.

Proof of Theorem 1.3 For $K_1, \ldots, K_n \in \mathcal{S}_o^n$, in (3.7), take $\lambda_1 = \lambda_2 = \cdots = \lambda_m$, and

$$
f_0 = c_p^{(\tau)}(K_1, u) \cdots c_p^{(\tau)}(K_{n-m}, u) \quad (f_0 = 1 \text{ if } m = n),
$$

$$
f_i = c_p^{(\tau)}(K_{n-i+1}, u) \quad (1 \leq i \leq m),
$$

then it follows that

$$\int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K_1, u) \cdots c_p^{(\tau)}(K_n, u) \, du$$

$$\leq \prod_{i=1}^{m} \left[\int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K_1, u) \cdots c_p^{(\tau)}(K_{n-m}, u) c_p^{(\tau)}(K_{n-i+1}, u)^m \, du \right]^{\frac{1}{m}}.$$

By definition (1.1), this yields

$$C_p^{(\tau)}(K_1, \ldots, K_n)^m \leq \prod_{i=1}^{m} C_p^{(\tau)}(K_1, \ldots, K_{n-m}, K_{n-i+1}, K_{n-i+1}, \ldots, K_{n-i+1}).$$

From the equality condition of inequality (3.7), we can see that equality holds in Theorem 1.3 if and only if $K_{n-m+1}, \ldots, K_n$ all have a similar general L_p-chord. □

When $m = n$ in Theorem 1.3 and use (1.4), we have the following corollary.

Corollary 3.3 *If $K_1, \ldots, K_n \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, then*

$$C_p^{(\tau)}(K_1, \ldots, K_n)^n \leq C_p^{(\tau)}(K_1) C_p^{(\tau)}(K_2) \cdots C_p^{(\tau)}(K_n),$$

with equality if and only if $K_1, \ldots, K_n$ all have a similar general L_p-chord.

Proof of Theorem 1.4 For $i < j < k$, by Hölder's inequality (see [32]), we have

$$C_{p,i}^{(\tau)}(K, L)^{\frac{k-j}{k-i}} C_{p,k}^{(\tau)}(K, L)^{\frac{j-i}{k-i}}$$

$$= \left(\frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K, u)^{n-i} c_p^{(\tau)}(L, u)^i \, du \right)^{\frac{k-j}{k-i}} \left(\frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K, u)^{n-k} c_p^{(\tau)}(L, u)^k \, du \right)^{\frac{j-i}{k-i}}$$

$$\geq \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K, u)^{\frac{(n-i)(k-j)+(n-k)(j-i)}{k-i}} c_p^{(\tau)}(L, u)^{\frac{i(k-j)+k(j-i)}{k-i}} \, du$$

$$= \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K)^{n-j} c_p^{(\tau)}(L, u)^j \, du$$

$$= C_{p,j}^{(\tau)}(K, L),$$

that is,

$$C_{p,j}^{(\tau)}(K, L)^{k-i} \leq C_{p,i}^{(\tau)}(K, L)^{k-j} C_{p,k}^{(\tau)}(K, L)^{j-i}.$$

From the equality condition of Hölder's inequality, we know that equality holds in (1.10) if and only if K and L have a similar general L_p-chord. □

Letting $i = 0$, $k = n$ in Theorem 1.4, we have the following fact.

Corollary 3.4 *If $K, L \in \mathcal{S}_o^n$ and $\tau \in [-1, 1]$, $p > 0$, then for $0 \leq j \leq n$,*

$$C_{p,j}^{(\tau)}(K, L)^n \leq C_p^{(\tau)}(K)^{n-j} C_p^{(\tau)}(L)^j. \tag{3.8}$$

For $j < 0$ or $j > n$, the inequality (3.8) is reversed, with equality if and only if K and L have a similar general L_p-chord.

Let $j = 1$ in Corollary 3.4, we obtain the following dual Minkowski inequality for the general L_p-mixed chord integrals.

Corollary 3.5 *If $K, L \in \mathcal{S}_o^n$ and $\tau \in [-1, 1], p > 0$, then*

$$C_{p,1}^{(\tau)}(K, L)^n \le C_p^{(\tau)}(K)^{n-1} C_p^{(\tau)}(L), \tag{3.9}$$

with equality if and only if K and L have a similar general L_p-chord.

Now we prove the associated inequalities based on L_p-radial combinations.

Proof of Theorem 1.5 For $i \le n - p$, it follows from Minkowski's inequality (see [32]) that

$$C_{p,i}^{(\tau)}(\lambda \circ K \,\tilde{+}_p\, \mu \circ L)^{\frac{p}{n-i}}$$

$$= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(\lambda \circ K \,\tilde{+}_p\, \mu \circ L, u)^{n-i}\, du \right]^{\frac{p}{n-i}}$$

$$= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(f_1(\tau) \rho^p(\lambda \circ K \,\tilde{+}_p\, \mu \circ L, u) + f_2(\tau) \rho^p(\lambda \circ K \,\tilde{+}_p\, \mu \circ L, -u) \right)^{\frac{n-i}{p}}\, du \right]^{\frac{p}{n-i}}$$

$$= \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(\lambda f_1(\tau) \rho^p(K, u) + \lambda f_2(\tau) \rho^p(-K, u) \right. \right.$$

$$\left. \left. + \mu f_1(\tau) \rho^p(L, u) + \mu f_2(\tau) \rho^p(-L, u) \right)^{\frac{n-i}{p}}\, du \right]^{\frac{p}{n-i}}$$

$$\le \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(\lambda f_1(\tau) \rho^p(K, u) + \lambda f_2(\tau) \rho^p(-K, u) \right)^{\frac{n-i}{p}}\, du \right]^{\frac{p}{n-i}}$$

$$+ \left[\frac{1}{n} \int_{\mathcal{S}^{n-1}} \left(\mu f_1(\tau) \rho^p(L, u) + \mu f_2(\tau) \rho^p(-L, u) \right)^{\frac{n-i}{p}}\, du \right]^{\frac{p}{n-i}}$$

$$= \lambda C_{p,i}^{(\tau)}(K)^{\frac{1}{n-i}} + \mu C_{p,i}^{(\tau)}(L)^{\frac{1}{n-i}}.$$

Similarly, we can prove the case of $n - p < i < n$ or $i > n$. From the equality conditions of Minkowski's inequality, we know that equality holds in Theorem 1.5 if and only if K and L have a similar general L_p-chord. $\qquad\square$

In order to prove Theorem 1.6, we require the following lemma which is an extended form of Beckenbach's inequality (see [33]) obtained by Dresher (see [34]) through the means of moment-space techniques.

Lemma 3.3 (The Beckenbach-Dresher inequality) *If $p \ge 1 \ge r \ge 0, f, g \ge 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_E (f + g)^p\, d\phi}{\int_E (f + g)^r\, d\phi} \right)^{\frac{1}{p-r}} \le \left(\frac{\int_E f^p\, d\phi}{\int_E f^r\, d\phi} \right)^{\frac{1}{p-r}} + \left(\frac{\int_E g^p\, d\phi}{\int_E g^r\, d\phi} \right)^{\frac{1}{p-r}}, \tag{3.10}$$

with equality if and only if the functions f and g are positively proportional.

Moreover, the inverse Beckenbach-Dresher inequality was obtained by Li and Zhao (see [35]).

Lemma 3.4 (The Inverse Beckenbach-Dresher inequality) *If $r \leq 0 \leq p \leq 1, f, g \geq 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_E (f+g)^p \, d\phi}{\int_E (f+g)^r \, d\phi} \right)^{\frac{1}{p-r}} \geq \left(\frac{\int_E f^p \, d\phi}{\int_E f^r \, d\phi} \right)^{\frac{1}{p-r}} + \left(\frac{\int_E g^p \, d\phi}{\int_E g^r \, d\phi} \right)^{\frac{1}{p-r}}, \tag{3.11}$$

with equality if and only if the functions f and g are positively proportional.

Proof of Theorem 1.6 By definitions (1.4) and (2.4), we have

$$\left(\frac{C_{p,i}^{(\tau)}(K \tilde{+}_p L)}{C_{p,j}^{(\tau)}(K \tilde{+}_p L)} \right)^{\frac{p}{j-i}}$$

$$= \left(\frac{\frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K \tilde{+}_p L, u)^{n-i} \, du}{\frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(K \tilde{+}_p L, u)^{n-j} \, du} \right)^{\frac{p}{j-i}}$$

$$= \left(\frac{\frac{1}{n} \int_{\mathcal{S}^{n-1}} [f_1(\tau)\rho^p(K \tilde{+}_p L, u) + f_2(\tau)\rho^p(K \tilde{+}_p L, -u)]^{\frac{n-i}{p}} \, du}{\frac{1}{n} \int_{\mathcal{S}^{n-1}} [f_1(\tau)\rho^p(K \tilde{+}_p L, u) + f_2(\tau)\rho^p(K \tilde{+}_p L, -u)]^{\frac{n-j}{p}} \, du} \right)^{\frac{p}{j-i}}$$

$$= \left(\frac{\frac{1}{n} \int_{\mathcal{S}^{n-1}} [f_1(\tau)\rho^p(K, u) + f_2(\tau)\rho^p(-K, u) + f_1(\tau)\rho^p(L, u) + f_2(\tau)\rho^p(-L, u)]^{\frac{n-i}{p}} \, du}{\frac{1}{n} \int_{\mathcal{S}^{n-1}} [f_1(\tau)\rho^p(K, u) + f_2(\tau)\rho^p(-K, u) + f_1(\tau)\rho^p(L, u) + f_2(\tau)\rho^p(-L, u)]^{\frac{n-j}{p}} \, du} \right)^{\frac{p}{j-i}}$$

$$\leq \left(\frac{\frac{1}{n} \int_{\mathcal{S}^{n-1}} [f_1(\tau)\rho^p(K, u) + f_2(\tau)\rho^p(-K, u)]^{\frac{n-i}{p}} \, du}{\frac{1}{n} \int_{\mathcal{S}^{n-1}} [f_1(\tau)\rho^p(K, u) + f_2(\tau)\rho^p(-K, u)]^{\frac{n-j}{p}} \, du} \right)^{\frac{p}{j-i}}$$

$$+ \left(\frac{\frac{1}{n} \int_{\mathcal{S}^{n-1}} [f_1(\tau)\rho^p(L, u) + f_2(\tau)\rho^p(-L, u)]^{\frac{n-i}{p}} \, du}{\frac{1}{n} \int_{\mathcal{S}^{n-1}} [f_1(\tau)\rho^p(L, u) + f_2(\tau)\rho^p(-L, u)]^{\frac{n-j}{p}} \, du} \right)^{\frac{p}{j-i}}$$

$$= \left(\frac{C_{p,i}^{(\tau)}(K)}{C_{p,j}^{(\tau)}(K)} \right)^{\frac{p}{j-i}} + \left(\frac{C_{p,i}^{(\tau)}(L)}{C_{p,j}^{(\tau)}(L)} \right)^{\frac{p}{j-i}},$$

inequality (1.13) is proved. Using the same method, we can see that when $j \geq n \geq i \geq n - p$, the reversed inequality of (1.13) is obtained.

From the equality condition of Lemma 3.3 and Lemma 3.4, we know that equality in inequality (1.13) if and only if K and L have a similar general L_p-chord. □

Finally, we prove the interesting equality for the L_p-radial bodies.

Proof of Theorem 1.7 For $\forall \tau \in [-1, 1]$,

$$C_{p,i}^{(\tau)}(\bar{\triangle}_p K) = \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(\tau)}(\bar{\triangle}_p K, u)^{n-i} \, du$$

$$= \frac{1}{n} \int_{\mathcal{S}^{n-1}} \left[f_1(\tau)\rho^p(\bar{\triangle}_p K, u) + f_2(\tau)\rho^p(\bar{\triangle}_p - K, u) \right]^{\frac{n-i}{p}} \, du$$

$$= \frac{1}{n} \int_{\mathcal{S}^{n-1}} \left[\frac{1}{2} f_1(\tau)\rho^p(K, u) + \frac{1}{2} f_2(\tau)\rho^p(-K, u) \right]$$

$$+ \frac{1}{2}f_1(\tau)\rho^p(-K,u) + \frac{1}{2}f_2(\tau)\rho^p(K,u) \Bigg]^{\frac{n-i}{p}} du$$

$$= \frac{1}{n} \int_{\mathcal{S}^{n-1}} \left[\frac{1}{2}\rho^p(K,u) + \frac{1}{2}\rho^p(-K,u) \right]^{\frac{n-i}{p}} du$$

$$= \frac{1}{n} \int_{\mathcal{S}^{n-1}} c_p^{(0)}(K,u)^{n-i} du$$

$$= C_{p,i}(K),$$

that is, $C_{p,i}^{(\tau)}(\bar{\triangle}_p K) = C_{p,i}(K)$. □

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