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# Almost sure central limit theorem for products of sums of partial sums

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**Abstract**
Considering a sequence of i.i.d. positive random variables, for products of sums of partial sums we establish an almost sure central limit theorem, which holds for some class of unbounded measurable functions.

**MSC:** 60F15

**Keywords:** almost sure central limit theorem; products of sums of partial sums; unbounded measurable functions

## 1 Introduction and main results

Let $\{X_n; n \geq 1\}$ be a sequence of random variables and define $S_n = \sum_{i=1}^{n} X_i$. Some results as regards the limit theorem of products $\prod_{j=1}^{n} S_j$ were obtained in recent years. Rempala and Wesolowski [1] obtained the following asymptotics for products of sums for a sequence of i.i.d. random variables.

**Theorem A** *Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. positive square integrable random variables with $\mathbb{E}X_1 = \mu$, the coefficient of variation $\gamma = \sigma/\mu$, where $\sigma^2 = \operatorname{Var}(X_1)$. Then*

$$\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \xrightarrow{d} e^{\sqrt{2}\mathcal{N}} \quad \text{as } n \to \infty. \tag{1.1}$$

*Here and in the sequel, $\mathcal{N}$ is a standard normal random variable and $\xrightarrow{d}$ denotes the convergence in distribution.*

Gonchigdanzan and Rempala [2] discussed the almost sure central limit theorem (ASCLT) for the products of partial sums and obtained the following result.

**Theorem B** *Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. positive random variables with $\mathbb{E}X_1 = \mu$, $\operatorname{Var}(X_1) = \sigma^2$ the coefficient of variation $\gamma = \sigma/\mu$. Then*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} I\left\{\left(\frac{\prod_{k=1}^{n} S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}} \leq x\right\} = F(x) \quad \text{a.s. for any } x \in \mathbb{R}, \tag{1.2}$$

*where $F$ is the distribution function of the random variable $e^{\sqrt{2}\mathcal{N}}$. Here and in the sequel, $I\{\cdot\}$ denotes the indicator function.*

Tan and Peng [3] proved the result of Theorem B still holds for some class of unbounded measurable functions and obtained the following result.

**Theorem C** *Let* $\{X_n; n \geq 1\}$ *be a sequence of i.i.d. positive random variables with* $\mathbb{E}X_1 = \mu$, $\operatorname{Var}(X_1) = \sigma^2$, $\mathbb{E}|X_1|^3 < \infty$, *the coefficient of variation* $\gamma = \sigma/\mu$. *Let* $g(x)$ *be a real valued almost everywhere continuous function on* $\mathbb{R}$ *such that* $|g(e^x)\phi(x)| \leq c(1+|x|)^{-\alpha}$ *with some* $c > 0$ *and* $\alpha > 5$. *Then*

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} g\left\{ \left( \frac{\prod_{k=1}^{n} S_k}{n! \mu^n} \right)^{\frac{1}{\gamma\sqrt{n}}} \right\} = \int_0^\infty g(x)\,\mathrm{d}F(x) \quad \text{a.s. for any } x \in \mathbb{R}, \tag{1.3}$$

*where* $F(\cdot)$ *is the distribution function of the random variable* $e^{\sqrt{2}\mathcal{N}}$ *and* $\phi(x)$ *is the density function of the standard normal random variable.*

Zhang *et al.* [4] discussed the almost sure central limit theory for products of sums of partial sums and obtained the following result.

**Theorem D** *Let* $\{X, X_n; n \geq 1\}$ *be a sequence of i.i.d. positive square integrable random variables with* $\mathbb{E}X = \mu$, $\operatorname{Var}(X) = \sigma^2 < \infty$, *the coefficient of variation* $\gamma = \sigma/\mu$. *Denote* $S_n = \sum_{i=1}^{n} X_i$, $T_k = \sum_{i=1}^{k} S_i$. *Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} I\left\{ \left( \frac{2^k \prod_{j=1}^{k} T_j}{k!(k+1)! \mu^k} \right)^{\frac{1}{\gamma\sqrt{k}}} \leq x \right\} = F(x) \quad \text{a.s. for any } x \in \mathbb{R}, \tag{1.4}$$

*where* $F(\cdot)$ *is the distribution function of the random variable* $e^{\sqrt{10/3}\mathcal{N}}$.

The purpose of this article is to establish that Theorem D holds for some class of unbounded measurable functions.

Our main result is the following theorem.

**Theorem 1.1** *Let* $\{X_n; n \geq 1\}$ *be a sequence of i.i.d. positive random variables with* $\mathbb{E}X_1 = \mu$, $\operatorname{Var}(X_1) = \sigma^2$, $\mathbb{E}|X_1|^3 < \infty$, *the coefficient of variation* $\gamma = \sigma/\mu$. *Let* $g(x)$ *be a real valued almost everywhere continuous function on* $\mathbb{R}$ *such that* $|g(e^{\sqrt{10/3}x})\phi(x)| \leq c(1+|x|)^{-\alpha}$ *with some* $c > 0$ *and* $\alpha > 5$. *Denote* $S_n = \sum_{i=1}^{n} X_i$, $T_k = \sum_{i=1}^{k} S_i$. *Then*

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} g\left( \left( \frac{2^n \prod_{k=1}^{n} T_k}{n!(n+1)! \mu^n} \right)^{\frac{1}{\gamma\sqrt{n}}} \right) \\ &\quad = \int_0^\infty g(x)\,\mathrm{d}F(x) \quad \text{a.s. for any } x \in \mathbb{R}, \end{aligned} \tag{1.5}$$

*where* $F(\cdot)$ *is the distribution function of the random variable* $e^{\sqrt{10/3}\mathcal{N}}$. *Here and in the sequel,* $\phi(x)$ *is the density function of the standard normal random variable.*

**Remark 1** Let $f(x) = g(e^{\sqrt{10/3}x})$, $t = e^{\sqrt{10/3}x}$. Then

$$x = \sqrt{\frac{3}{10}} \log t, \qquad g(t) = f\left( \sqrt{\frac{3}{10}} \log t \right),$$

$$g\left(\left(\frac{2^n\prod_{k=1}^n T_k}{n!(n+1)!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}}\right) = f\left(\sqrt{\frac{3}{10}}\log\left(\prod_{k=1}^n \frac{2T_k}{k(k+1)\mu}\right)^{\frac{1}{\gamma\sqrt{n}}}\right)$$
$$= f\left(\frac{1}{\gamma\sqrt{10n/3}}\sum_{k=1}^n \log\frac{T_k}{k(k+1)\mu/2}\right).$$

Since $F(x)$ is the distribution function of the random variable $e^{\sqrt{10/3}\mathcal{N}}$, we can get $F(x) = \Phi(\sqrt{\frac{3}{10}}\log x)$, where $\Phi(x)$ is the distribution function of the standard normal random variable. Hence we have the following: Let $f(x) = g(e^{\sqrt{10/3}x})$ and $f(x)$ be a real valued almost everywhere continuous function on $\mathbb{R}$ such that $|f(x)\phi(x)| \le c(1+|x|)^{-\alpha}$ with some $c > 0$ and $\alpha > 5$, then (1.5) is equivalent to

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^N \frac{1}{n} f\left(\frac{1}{\gamma\sqrt{10n/3}}\sum_{k=1}^n \log\frac{T_k}{k(k+1)\mu/2}\right)$$
$$= \int_{-\infty}^{\infty} f(x)\phi(x)\,\mathrm{d}x \quad \text{a.s. for any } x\in\mathbb{R}. \tag{1.6}$$

**Remark 2** By the proof of Theorem 2 of Berkes *et al.* [5], in order to prove (1.5), it suffices to show (1.6) holds true for $f(x)\phi(x) = (1+|x|)^{-\alpha}$ with $\alpha > 5$. Here and in the sequel, $f(x)$ satisfies $f(x)\phi(x) = (1+|x|)^{-\alpha}$ with $\alpha > 5$.

## 2 Preliminaries

In the following, the notation $a_n \sim b_n$ means that $\lim_{n\to\infty} a_n/b_n = 1$ and $a_n \ll b_n$ means that $\limsup_{n\to\infty}|a_n/b_n| < +\infty$. We denote $b_{k,n} = \sum_{j=k}^n \frac{1}{j}$, $c_{k,n} = 2\sum_{j=k}^n \frac{j+1-k}{j(j+1)}$, $d_{k,n} = \frac{n+1-k}{n+1}$, $\widetilde{X}_i = \frac{X_i-\mu}{\sigma}$, $\widetilde{S}_k = \sum_{i=1}^k \widetilde{X}_i$, $S_{k,n} = \sum_{i=1}^k c_{i,n}\widetilde{X}_i$. By Lemma 2.1 of Wu [6], we can get

$$c_{i,n} = 2(b_{i,n} - d_{i,n}), \qquad \sum_{i=1}^n c_{i,n}^2 \sim \frac{10n}{3}.$$

Let

$$Y_i = \frac{1}{\gamma\sqrt{10i/3}}\sum_{k=1}^i \log\frac{T_k}{k(k+1)\mu/2}.$$

Note that

$$\frac{1}{\gamma}\sum_{k=1}^i\left(\frac{T_k}{k(k+1)\mu/2} - 1\right)$$
$$= \frac{1}{\gamma}\sum_{k=1}^i\left(\frac{2\sum_{j=1}^k S_j - k(k+1)\mu}{k(k+1)\mu}\right)$$
$$= \frac{1}{\gamma}\sum_{k=1}^i \frac{2}{k(k+1)\mu}\sum_{j=1}^k\sum_{l=1}^j (X_l - \mu)$$
$$= \frac{1}{\gamma}\sum_{k=1}^i \frac{2}{k(k+1)\mu}\sum_{l=1}^k\sum_{j=l}^k (X_l - \mu)$$

$$\begin{aligned}
&= \frac{\mu}{\sigma}\sum_{k=1}^{i}\frac{2}{k(k+1)\mu}\sum_{l=1}^{k}(k+1-l)(X_l-\mu)\\
&= \sum_{k=1}^{i}\sum_{l=1}^{k}\frac{2(k+1-l)}{k(k+1)}\frac{X_l-\mu}{\sigma}\\
&= \sum_{l=1}^{i}\sum_{k=l}^{i}\frac{2(k+1-l)}{k(k+1)}\widetilde{X}_l\\
&= \sum_{l=1}^{i}c_{l,i}\widetilde{X}_l = S_{i,i}.
\end{aligned}$$

By the fact that $\log(1+x) = x + \frac{\delta}{2}x^2$, where $|x| < 1$, $\delta \in (-1, 0)$, thus we have

$$\begin{aligned}
Y_i &= \frac{1}{\gamma\sqrt{10i/3}}\sum_{k=1}^{i}\log\frac{T_k}{k(k+1)\mu/2}\\
&= \frac{1}{\gamma\sqrt{10i/3}}\sum_{k=1}^{i}\left(\frac{T_k}{k(k+1)\mu/2}-1\right) + \frac{1}{\gamma\sqrt{10i/3}}\sum_{k=1}^{i}\frac{\delta_k}{2}\left(\frac{T_k}{k(k+1)\mu/2}-1\right)^2\\
&= \frac{1}{\sqrt{10i/3}}S_{i,i} + \frac{1}{\gamma\sqrt{10i/3}}\sum_{k=1}^{i}\frac{\delta_k}{2}\left(\frac{T_k}{k(k+1)\mu/2}-1\right)^2\\
&=: \frac{1}{\sqrt{10i/3}}S_{i,i} + R_i.
\end{aligned}$$

By the fact that $\mathbb{E}|X_1|^2 < \infty$, using the Marcinkiewicz-Zygmund strong large number law, we have

$$\begin{aligned}
&S_k - k\mu = o\bigl(k^{1/2}\bigr) \quad \text{a.s.},\\
&\left|\frac{T_k}{k(k+1)\mu/2}-1\right| = \left|\frac{2\sum_{j=1}^{k}S_j - k(k+1)\mu}{k(k+1)\mu}\right|\\
&\qquad \le \frac{2|\sum_{j=1}^{k}(S_j - j\mu)|}{k(k+1)\mu}\\
&\qquad \le \frac{2\sum_{j=1}^{k}j^{1/2}}{k(k+1)\mu} \ll \frac{k^{3/2}}{k^2} = \frac{1}{k^{1/2}}.
\end{aligned}$$

Thus

$$|R_i| \ll \frac{1}{\sqrt{i}}\sum_{k=1}^{i}\frac{1}{k} \ll \frac{\log i}{\sqrt{i}} \quad \text{a.s.} \tag{2.1}$$

In order to prove Theorem 1.1, we introduce the following lemmas.

**Lemma 2.1** *Let $X$ and $Y$ be random variables. Set $F(x) = P(X < x)$, $G(x) = P(X + Y < x)$, then for any $\varepsilon > 0$ and $x \in \mathbb{R}$,*

$$F(x-\varepsilon) - P\bigl(|Y| \ge \varepsilon\bigr) \le G(x) \le F(x+\varepsilon) + P\bigl(|Y| \ge \varepsilon\bigr).$$

*Proof* See Lemma 3 on p.16 of Petrov [7]. □

**Lemma 2.2** *Let $\{X_n; n \geq 1\}$ be a sequence of i.i.d. positive random variables. Denote $S_n = \sum_{i=1}^{n} X_i$, $F^s$ denotes the distribution function obtained from $F$ by symmetrization and choose $L > 0$ so large that $\int_{|x|\leq L} x^2 \,\mathrm{d}F^s(x) \geq 1$. Then, for any $n \geq 1$, $\lambda > 0$, there exists a $c > 0$ such that*

$$\sup_a P\left(a \leq \frac{S_n}{\sqrt{n}} \leq a + \lambda\right) \leq c\lambda$$

*holds for $\lambda\sqrt{n} \geq L$.*

*Proof* See (20) on p.73 of Berkes *et al.* [5]. □

Let

$$Z_k = \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} f(Y_i),$$

$$Z_k^* = \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} f(Y_i) I\left\{f(Y_i) \leq \frac{k}{(\log k)^\beta}\right\},$$

where $1 < \beta < (\alpha - 3)/2$.

**Lemma 2.3** *Under the conditions of Theorem* 1.1, *we get*

$$\mathbb{P}(Z_k \neq Z_k^*, \text{i.o.}) = 0.$$

*Proof* It is easy to get

$$\begin{aligned}\{Z_k \neq Z_k^*\} &\subseteq \{|Y_i| \geq f^{-1}(k/(\log k)^\beta) \text{ for some } 2^k < i \leq 2^{k+1}\} \\ &= \left\{\left|\frac{1}{\sqrt{10i/3}} S_{i,i} + R_i\right| \geq f^{-1}(k/(\log k)^\beta) \geq \left(2\log k + (\alpha - 2\beta)\log\log k\right)^{1/2} \right. \\ &\qquad \left. \text{for some } 2^k < i \leq 2^{k+1}\right\}.\end{aligned}$$

Since $|R_i| \ll \frac{\log i}{\sqrt{i}}$ a.s.; see (2.1). By the law of iterated logarithm (Feller [8], Theorem 2), we get

$$\begin{aligned}\mathbb{P}(Z_k \neq Z_k^*, \text{i.o.}) &\leq \mathbb{P}\left(\left|\frac{1}{\sqrt{10i/3}} S_{i,i}\right| \geq \left(2\log\log i + (\alpha - 2\beta)\log\log\log i - O(1)\right)^{1/2}, \text{i.o.}\right) \\ &= 0.\end{aligned}$$

We complete the proof of Lemma 2.3. □

Let $G_i$, $F_i$, $F$ denote the distribution functions of $Y_i$, $\frac{\widetilde{S}_i}{\sqrt{i}}$, $\widetilde{X}_1$, respectively. $\Phi$ denotes the distribution function of the standard normal distribution function. Set

$$\sigma_i^2 = \int_{-\sqrt{i}}^{\sqrt{i}} x^2 \,\mathrm{d}F(x) - \left(\int_{-\sqrt{i}}^{\sqrt{i}} x \,\mathrm{d}F(x)\right)^2,$$

$$\varepsilon_i = \sup_x \left| F_i(x) - \Phi\left(\frac{x}{\sigma_i}\right) \right|, \qquad \theta_i = \sup_x \left| G_i(x) - \Phi\left(\frac{x}{\sigma_i}\right) \right|.$$

Obviously $\sigma_i \leq 1$, $\lim_{i\to\infty} \sigma_i = 1$.

**Lemma 2.4** *Under the conditions of Theorem* 1.1, *we have*

$$\sum_{k=1}^{N} \mathbb{E}\left(Z_k^*\right)^2 \ll \frac{N^2}{(\log N)^{2\beta}}.$$

*Proof* Note that the estimation

$$\left| \int_{-a}^{a} \Psi(x)\,\mathrm{d}\big(H_1(x) - H_2(x)\big) \right| \leq \sup_{-a\leq x\leq a} \left|\Psi(x)\right| \cdot \sup_{-a\leq x\leq a} \left|H_1(x) - H_2(x)\right| \tag{2.2}$$

holds for any bounded, measurable function $\Psi(x)$ and the distribution functions $H_1(x)$, $H_2(x)$. Thus for $2^k < i \leq 2^{k+1}$, we get

$$\begin{aligned}
&\mathbb{E}f^2(Y_i)I\left\{f(Y_i) \leq \frac{k}{(\log k)^\beta}\right\} \\
&\quad = \int_{|x|\leq a_k} f^2(x)\,\mathrm{d}G_i(x) \\
&\quad \leq \int_{|x|\leq a_k} f^2(x)\,\mathrm{d}\Phi\left(\frac{x}{\sigma_i}\right) + \theta_i \frac{k^2}{(\log k)^{2\beta}} \\
&\quad \ll \int_{|x|\leq a_k} f^2(x)\,\mathrm{d}\Phi(x) + \theta_i \frac{k^2}{(\log k)^{2\beta}};
\end{aligned}$$

here and in the sequel $a_k = f^{-1}(\frac{k}{(\log k)^\beta})$. Hence, by the Cauchy-Schwarz inequality and the fact that $f(x)\phi(x) = (1 + |x|)^{-\alpha}$, we obtain

$$\begin{aligned}
\mathbb{E}\left(Z_k^*\right)^2 &\ll \mathbb{E}\left(\left(\sum_{i=2^k+1}^{2^{k+1}} \left(\frac{1}{i}\right)^2\right)^{1/2} \left(\sum_{i=2^k+1}^{2^{k+1}} f^2(Y_i)I\left\{f(Y_i) \leq \frac{k}{(\log k)^\beta}\right\}\right)^{1/2}\right)^2 \\
&\ll \left(\sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i^2}\right)\left(\sum_{i=2^k+1}^{2^{k+1}} \left(\int_{|x|\leq a_k} f^2(x)\,\mathrm{d}\Phi(x) + \theta_i \frac{k^2}{(\log k)^{2\beta}}\right)\right) \\
&\ll \frac{1}{2^k}\left(2^k \int_{|x|\leq a_k} f^2(x)\,\mathrm{d}\Phi(x) + \frac{k^2}{(\log k)^{2\beta}} \sum_{i=2^k+1}^{2^{k+1}} \theta_i\right) \\
&\ll \int_{|x|\leq a_k} \frac{e^{x^2/2}}{(1 + |x|)^{2\alpha}}\,\mathrm{d}x + \frac{k^2}{(\log k)^{2\beta}} \sum_{i=2^k+1}^{2^{k+1}} \frac{\theta_i}{i}.
\end{aligned}$$

By the same methods as that on p.72 of Berkes *et al.* [5], we get

$$\int_{|x|\leq a_k} \frac{e^{x^2/2}}{(1 + |x|)^{2\alpha}}\,\mathrm{d}x \ll \frac{k}{(\log k)^{\beta+(\alpha+1)/2}}.$$

Now we estimate $\theta_i$. By Lemma 2.1, for any $\varepsilon > 0$, we have

$$\begin{aligned}\theta_i &= \sup_x \left| G_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right| \\ &\le \sup_x \left| G_i(x) - F_i(x)\right| + \sup_x \left| F_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right| \\ &= \sup_x \left| P(Y_i \le x) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right)\right| + \varepsilon_i \\ &\le \sup_x \left| P(Y_i \le x) - P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right)\right| + \sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right)\right| + \varepsilon_i \\ &\le \sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} + R_i \le x\right) - P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x + \varepsilon\right)\right| \\ &\quad + \sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x + \varepsilon\right) - P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right)\right| \\ &\quad + \sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right)\right| + \varepsilon_i \\ &\le P\big(|R_i| \ge \varepsilon\big) + \sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x + \varepsilon\right) - P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right)\right| \\ &\quad + \sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right)\right| + \varepsilon_i.\end{aligned}$$

By the Markov inequality and (2.1), we have

$$P\big(|R_i| \ge \varepsilon\big) \le \frac{\mathbb{E}|R_i|}{\varepsilon} \ll \frac{\log i}{\sqrt{i}\varepsilon}.$$

By Lemma 2.2, we have

$$\sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x + \varepsilon\right) - P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right)\right| \ll \varepsilon.$$

By the Berry-Esseen inequality, we have

$$\begin{aligned}&\sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right) - P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right)\right| \\ &\quad\le \sup_x \left| P\left(\frac{S_{i,i}}{\sqrt{10i/3}} \le x\right) - \Phi(x)\right| + \sup_x \left| P\left(\frac{\widetilde{S}_i}{\sqrt{i}} \le x\right) - \Phi(x)\right| \\ &\quad\ll \frac{1}{i^{1/2}} + \frac{1}{i^{1/2}}.\end{aligned}$$

Let $\varepsilon = i^{-1/3}$, then

$$\theta_i \ll \frac{\log i}{i^{1/6}} + \frac{1}{i^{1/3}} + \frac{1}{i^{1/2}} + \varepsilon_i.$$

Therefore, there exists $\varepsilon_0 > 0$ such that

$$\theta_i \ll \frac{1}{i^{\varepsilon_0}} + \varepsilon_i.$$

By Theorem 1 of Friedman *et al.* [9], we have

$$\sum_{i=1}^{\infty} \frac{\varepsilon_i}{i} < \infty.$$

Hence

$$\sum_{i=1}^{\infty} \frac{\theta_i}{i} \ll \sum_{i=1}^{\infty} \frac{\frac{1}{i^{\varepsilon_0}} + \varepsilon_i}{i} < \infty.$$

By the fact that $(\alpha + 1)/2 > \beta$, we have

$$\sum_{k=1}^{N} \mathbb{E}\big(Z_k^*\big)^2 \ll \sum_{k=1}^{N} \frac{k}{(\log k)^{\beta+(\alpha+1)/2}} + \sum_{k=1}^{N} \frac{k^2}{(\log k)^{2\beta}} \sum_{i=2^k+1}^{2^{k+1}} \frac{\theta_i}{i} \ll \frac{N^2}{(\log N)^{2\beta}}.$$

We complete the proof of Lemma 2.4. □

**Lemma 2.5** *Under the conditions of Theorem* 1.1, *for* $l \geq l_0$, *we have*

$$\big|\mathrm{Cov}\big(Z_k^*, Z_l^*\big)\big| \ll \frac{kl}{(\log k)^{\beta}(\log l)^{\beta}} 2^{-(l-k)\tau},$$

*where* $\tau$ *is a constant* $0 < \tau \leq 1/8$.

*Proof* For $1 \leq i \leq j/2$, $j \geq j_0$ and any $x$, $y$, we first prove

$$\big|P(Y_i \leq x, Y_j \leq y) - P(Y_i \leq x)P(Y_j \leq y)\big| \ll \left(\frac{i}{j}\right)^{\tau}. \tag{2.3}$$

Let $\rho = \frac{i}{j}$. By the Chebyshev inequality, we have

$$P\left(\left|\frac{S_{i,i}}{\sqrt{10j/3}}\right| \geq \rho^{1/8}\right) = P\left(\left|\frac{S_{i,i}}{\sqrt{10i/3}}\right| \geq \sqrt{\frac{j}{i}}\rho^{1/8}\right) \leq \frac{i}{j}\rho^{-1/4} \leq \rho^{1/8} \leq \rho^{\tau_1},$$

where $\tau_1$ is a constant $0 < \tau_1 \leq 1/8$.

By the Markov inequality and (2.1), for $j \geq j_0$, we have

$$P\big(|R_j| \geq \rho^{1/8}\big) \leq \frac{\mathbb{E}|R_j|}{\rho^{1/8}} \ll \frac{\log j}{j^{1/2}\rho^{1/8}} = \rho^{1/8}\frac{\log j}{j^{1/4} i^{1/4}} \ll \rho^{\tau_2},$$

where $\tau_2$ is a constant, $0 < \tau_2 \leq 1/8$.

By the Markov inequality, we have

$$\begin{aligned} &P\left(\left|\sqrt{1-\rho}\frac{c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}}\right| \geq \rho^{1/8}\right) \\ &\quad = P\left(\left|\frac{\widetilde{S}_i}{\sqrt{i}}\right| \geq \sqrt{\frac{10/3j}{i}}\frac{1}{c_{i+1,j}}\rho^{1/8}\right) \\ &\quad \leq \frac{3}{10}\rho^{3/4}(c_{i+1,j})^2 = \frac{3}{10}\rho^{3/4}(2b_{i+1,j} - 2d_{i+1,j})^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{10}\rho^{3/4}\left[\left(2\sum_{k=i+1}^{j}\frac{1}{k}\right)^2 + 4\left(\frac{j+1-i-1}{j+1}\right)^2 - 8\left(\sum_{k=i+1}^{j}\frac{1}{k}\right)\frac{j+1-i-1}{j+1}\right] \\
&\ll \rho^{3/4}\left[\left(\log\frac{j}{i}\right)^2 + \left(\frac{j-i}{j+1}\right)^2 - \frac{j-i}{j+1}\log\frac{j}{i}\right] \\
&\ll \rho^{\tau_3},
\end{aligned}
$$

where $\tau_3$ is a constant that satisfies $0 < \tau_3 \leq 1/8$.

By Lemma 2.2 and the fact that $\rho = \frac{i}{j}$, $1 \leq i \leq j/2$, we have

$$
P\left(y - 3\rho^{1/8} \leq \sqrt{1-\rho}\frac{S_{j,j} - S_{i,i} - c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} \leq y\right) \ll \frac{\rho^{1/8}}{\sqrt{1-\rho}} \ll \rho^{1/8}.
$$

Set $\tau = \min\{\tau_1, \tau_2, \tau_3, 1/8\}$, we get

$$
\begin{aligned}
&P(Y_i \leq x, Y_j \leq y) \\
&\quad = P\left(Y_i \leq x, \frac{S_{j,j}}{\sqrt{10j/3}} + R_j \leq y\right) \\
&\quad = P\left(Y_i \leq x, \frac{S_{i,i}}{\sqrt{10j/3}} + \sqrt{1-\rho}\frac{S_{j,j} - S_{i,i} - c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} + \sqrt{1-\rho}\frac{c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} + R_j \leq y\right) \\
&\quad \geq P\left(Y_i \leq x, \sqrt{1-\rho}\frac{S_{j,j} - S_{i,i} - c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} \leq y\right) \\
&\qquad - P\left(y - 3\rho^{1/8} \leq \sqrt{1-\rho}\frac{S_{j,j} - S_{i,i} - c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} \leq y\right) - P\left(\left|\frac{S_{i,i}}{\sqrt{10j/3}}\right| \geq \rho^{1/8}\right) \\
&\qquad - P\left(\left|\sqrt{1-\rho}\frac{c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}}\right| \geq \rho^{1/8}\right) - P\left(|R_j| \geq \rho^{1/8}\right) \\
&\quad \geq P\left(Y_i \leq x, \sqrt{1-\rho}\frac{S_{j,j} - S_{i,i} - c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} \leq y\right) - \rho^{\tau} \\
&\quad = P(Y_i \leq x)P\left(\sqrt{1-\rho}\frac{S_{j,j} - S_{i,i} - c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} \leq y\right) - \rho^{\tau}.
\end{aligned}
$$

We can get a similar upper estimate for $P(Y_i \leq x, Y_j \leq y)$ in the same way. Thus there exists some constant $M$ such that

$$
P(Y_i \leq x, Y_j \leq y) = P(Y_i \leq x)P\left(\sqrt{1-\rho}\frac{S_{j,j} - S_{i,i} - c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} \leq y\right) + M\rho^{\tau}.
$$

A similar argument,

$$
P(Y_i \leq x)P(Y_j \leq y) = p(Y_i \leq x)P\left(\sqrt{1-\rho}\frac{S_{j,j} - S_{i,i} - c_{i+1,j}\widetilde{S}_i}{\sqrt{10(j-i)/3}} \leq y\right) + M'\rho^{\tau},
$$

holds for some constant $M'$. Thus we prove that (2.3) holds.

Let $G_{i,j}(x,y)$ be the joint distribution function of $Y_i$ and $Y_j$. By (2.2) and (2.3), for $2^k < i \le 2^{k+1}$, $2^l < j \le 2^{l+1}$, $l-k \ge 3$, $l \ge l_0$, we can get

$$\begin{aligned}
&\left|\mathrm{Cov}\left(f(Y_i)I\left\{f(Y_i) \le \frac{k}{(\log k)^\beta}\right\}, f(Y_j)I\left\{f(Y_j) \le \frac{l}{(\log l)^\beta}\right\}\right)\right| \\
&\quad = \left|\int_{|x|\le a_k}\int_{|y|\le a_l} f(x)f(y)\,\mathrm{d}\bigl(G_{i,j}(x,y) - G_i(x)G_j(y)\bigr)\right| \\
&\quad \ll \frac{kl}{(\log k)^\beta(\log l)^\beta}\left(\frac{i}{j}\right)^\tau \ll \frac{kl}{(\log k)^\beta(\log l)^\beta}2^{-(l-k-1)\tau}.
\end{aligned}$$

Thus we have

$$\bigl|\mathrm{Cov}\bigl(Z_k^*, Z_l^*\bigr)\bigr| \ll \frac{kl}{(\log k)^\beta(\log l)^\beta}2^{-(l-k)\tau}.$$

We complete the proof of Lemma 2.5. □

**Lemma 2.6** *Under the conditions of Theorem* 1.1, *denoting* $\eta_k = Z_k^* - \mathbb{E}Z_k^*$, *we have*

$$\mathbb{E}\left(\sum_{k=1}^N \eta_k\right)^2 = O\left(\frac{N^2}{(\log N)^{2\beta-1}}\right).$$

*Proof* It follows from Lemma 2.4 and Lemma 2.5 that Lemma 2.6 also holds true. The proof is similar to that of Lemma 4 of Berkes *et al.* [5]. So we omit it here. □

## 3 Proof of theorem

By Lemma 2.6, we have

$$\mathbb{E}\left(\frac{1}{N}\sum_{k=1}^N \eta_k\right)^2 = O\bigl((\log N)^{1-2\beta}\bigr).$$

Letting $N_k = [e^{k\lambda}]$, $(2\beta-1)^{-1} < \lambda < 1$, we get

$$\mathbb{E}\left(\frac{1}{N_k}\sum_{k=1}^{N_k} \eta_k\right)^2 < \infty,$$

which implies

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{k=1}^{N_k}\eta_k = 0 \quad \text{a.s.} \tag{3.1}$$

Note that for $2^k < i \le 2^{k+1}$,

$$\begin{aligned}
&\mathbb{E}f(Y_i)I\left\{f(Y_i) \le \frac{k}{(\log k)^\beta}\right\} \\
&\quad = \int_{|x|\le a_k} f(x)\,\mathrm{d}G_i(x) = \int_{|x|\le a_k} f(x)\,\mathrm{d}\Phi\left(\frac{x}{\sigma_i}\right) + \int_{|x|\le a_k} f(x)\,\mathrm{d}\left(G_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right).
\end{aligned} \tag{3.2}$$

Set $a = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}\Phi(x)$. Noting that $\sigma_i \le 1$, $\lim_{i\to\infty}\sigma_i = 1$, we have

$$\lim_{k\to\infty}\sup_{2^k<i\le 2^{k+1}}\left|\int_{|x|\le a_k} f(x)\,\mathrm{d}\Phi\left(\frac{x}{\sigma_i}\right) - a\right| = 0. \tag{3.3}$$

Then by (3.2), (3.3), and (2.2) we get

$$\begin{aligned}
&\left|\mathbb{E}f(Y_i)I\left\{f(Y_i)\le \frac{k}{(\log k)^\beta}\right\} - a\right| \\
&\quad\le \left|\int_{|x|\le a_k} f(x)\,\mathrm{d}\Phi\left(\frac{x}{\sigma_i}\right) - a\right| + \left|\int_{|x|\le a_k} f(x)\,\mathrm{d}\left(G_i(x) - \Phi\left(\frac{x}{\sigma_i}\right)\right)\right| \\
&\quad\le o_k(1) + \frac{k\theta_i}{(\log k)^\beta}.
\end{aligned}$$

Thus

$$\mathbb{E}Z_k^* = a\sum_{i=2^k+1}^{2^{k+1}}\frac{1}{i} + \zeta_k\frac{k}{(\log k)^\beta}\sum_{i=2^k+1}^{2^{k+1}}\frac{\theta_i}{i} + o_k(1), \quad |\zeta_k|\le 1.$$

Using $\sum_{i=1}^{L} 1/i = \log L + O(1)$ and $\sum_{i=1}^{\infty}\frac{\theta_i}{i} < \infty$, we get

$$\begin{aligned}
\left|\frac{\mathbb{E}(\sum_{k=1}^{N} Z_k^*)}{\log 2^{N+1}} - a\right| &\ll \frac{1}{N}\sum_{k=1}^{N}\frac{k}{(\log k)^\beta}\sum_{i=2^k+1}^{2^{k+1}}\frac{\theta_i}{i} + o_N(1) \\
&= O\big((\log N)^{-\beta}\big) + o_N(1) \\
&= o_N(1).
\end{aligned}$$

Thus by (3.1), we get

$$\lim_{k\to\infty}\frac{\sum_{k=1}^{N_k} Z_k^*}{\log 2^{N_k+1}} = a \quad \text{a.s.}$$

Then by Lemma 2.3, we have

$$\lim_{k\to\infty}\frac{\sum_{k=1}^{N_k} Z_k}{\log 2^{N_k+1}} = a \quad \text{a.s.} \tag{3.4}$$

The relation $\lambda < 1$ implies $\lim_{k\to\infty} N_{k+1}/N_k = 1$, thus (3.4) and the positivity of the $Z_k$ yield

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N} Z_k}{\log 2^{N+1}} = a \quad \text{a.s.,}$$

*i.e.* (1.6) holds for the subsequence $N = 2^k$. Using again the positivity of the terms, we get (1.6). We complete the proof of Theorem 1.1.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**

FF conceived of the study and drafted and completed the manuscript. DW participated in the discussion of the manuscript. FF and DW read and approved the final manuscript.

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**References**

1. Rempala, G, Wesolowski, J: Asymptotics for products of sums and *U*-statistics. Electron. Commun. Probab. **7**, 47-54 (2002)
2. Gonchigdanzan, K, Rempala, G: A note on the almost sure limit theorem for the product of partial sums. Appl. Math. Lett. **19**, 191-196 (2006)
3. Tan, ZQ, Peng, ZX: Almost sure central limit theorem for the product of partial sums. Acta Math. Sci. **29**, 1689-1698 (2009)
4. Zhang, Y, Yang, XY, Dong, ZS: An almost sure central limit theorem for products of sums of partial sums under association. J. Math. Anal. Appl. **355**, 708-716 (2009)
5. Berkes, I, Csáki, E, Horváth, L: Almost sure central limit theorems under minimal conditions. Stat. Probab. Lett. **37**, 67-76 (1998)
6. Wu, QY: Almost sure central limit theory for products of sums of partial sums. Appl. Math. J. Chin. Univ. Ser. B **27**, 169-180 (2012)
7. Petrov, V: Sums of Independent Random Variables. Springer, New York (1975)
8. Feller, W: The law of iterated logarithm for identically distributed random variables. Ann. Math. **47**, 631-638 (1946)
9. Friedman, N, Katz, M, Koopmans, LH: Convergence rates for the central limit theorem. Proc. Natl. Acad. Sci. USA **56**, 1062-1065 (1966)