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A short note on C^*-valued contraction mappings

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Abstract
In this short note we point out that the recently announced notion, the C^*-valued metric, does not bring about a real extension in metric fixed point theory. Besides, fixed point results in the C^*-valued metric can be derived from the desired Banach mapping principle and its famous consecutive theorems.

1 Introduction and preliminaries

Very recently, Ma *et al.* [1] reported a generalization of the Banach contraction principle for self mappings on C^*-valued metric spaces by defining the notion of a C^*-valued metric space. Following this initial article, some further extension of the Banach contraction principle has been reported (see *e.g.* [2, 3]). In this note, we shall show that the announced fixed point results in [1–5] in the context of C^*-valued metric spaces can be derived from the corresponding existing fixed point results in the literature.

First of all, we recall some basic definitions, which will be used later.

Suppose that A is a unital algebra with the unit e. An involution on A is a conjugate linear map a $*: A \to A$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. The pair $(A, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra A together with a complete submultiplicative norm such that $\|a\| = \|a^*\|$ for all $a \in A$. A C^*-algebra is a Banach $*$-algebra such that $\|a\| = \|aa^*\|$.

Throughout this paper, A will denote an unital C^*-algebra with a unit e. Set $A_h = \{x \in A : x = x^*\}$. We call an element $x \in A$ a positive element, denote it by $x \in A$, a positive element if $x \in A_h$ and $\sigma(x) \subset R^+ = [0, +\infty)$, where $\sigma(x)$ is the spectrum of x. Using positive elements, one can define a partial ordering $\preceq$ on A_h as follows: $x \preceq y$ if and only if $y - x \succeq \theta$, where θ means the zero element in A. From now on, by A^+ we denote the set $\{x \in A : x \succeq \theta\}$ and $|x| = (x.x^*)^{\frac{1}{2}}$. We say a is normal if $a^*a = aa^*$.

A character on an abelian algebra A is a non-zero homomorphism $\tau : A \to \mathbb{C}$. We denote by $\Omega(A)$ the set of characters on A.

Suppose that A is an abelian Banach algebra for which the space $\Omega(A)$ is nonempty. If $a \in A$, we define the function $\hat{a}$ by

$$\begin{cases} \hat{a} : \Omega(A) \to \mathbb{C}, \\ \tau \mapsto \tau(a). \end{cases}$$

Clearly, the topology on $\Omega(A)$ is the smallest one making all of the functions a continuous.

The set $\{\tau \in \Omega(A) : |\tau(a)| \geq \epsilon\}$ is weak* closed in the closed unit ball of A^* for each $\epsilon > 0$, and weak* compact by the Banach-Alaoglu theorem. Hence, we deduce that $a \in C(\Omega(A))$.

We call $\hat{a}$ the Gelfand transform of a.

Theorem 1.1 ([6], Gelfand representation) *Suppose that A is an abelian Banach algebra and that $\Omega(A)$ is nonempty. Then the map*

$$\begin{cases} \hat{a} : A \to C(\Omega(A)), \\ a \mapsto \hat{a}, \end{cases}$$

is a norm-decreasing homomorphism, and

$$r(a) = \|\hat{a}\|_\infty \quad (a \in A).$$

If A is unital, $\sigma(a) = \sigma(\hat{a}(\Omega(A)))$, and if A is non-unital, $\sigma(a) = \sigma(\hat{a}(\Omega(A))) \cup \{0\}$, for each $a \in A$.

Theorem 1.2 ([6]) *Let A be a unital Banach algebra generated by 1 and an element a. Then A is abelian and the map*

$$\begin{cases} \hat{a} : \Omega(A) \to \sigma(a), \\ \tau \mapsto \tau(a), \end{cases}$$

is a homeomorphism.

Theorem 1.3 ([6], Theorem 2.2.5) *Let A be a C^*-algebra and $a \in A^+$. Then*

(1) *There exists a unique element $b \in A^+$ such that $b^2 = a$.*
(2) *The set A^+ is equal to $\{a^*a : a \in A\}$.*
(3) *If $a, b \in A$ and $0 \leq a \leq b$, then $\|a\| \leq \|b\|$.*

We recall the definition of C^*-algebra-valued metric.

Definition 1.1 Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{A}$ satisfies:

(d1) $\theta \leq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta \iff x = y$;
(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(d3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a C^*-algebra-valued metric on X and $(X, \mathbb{A}, d)$ is called a C^*-algebra-valued metric space.

2 Main result

Theorem 2.1 *Let $(X, \mathbb{A}, d)$ be a C^*-algebra-valued complete metric space and $T : X \to X$ be a mapping such that there exists $a \in A$ with $\|a\| < 1$ such that*

$$d(Tx, Ty) \preceq a^* d(x,y) a \quad \textit{for all } x, y \in X.$$

Then T has a unique fixed point in X.

Proof Since $d(x,y)$ and $d(Tx,Ty)$ are positive and we have

$$0_A \preceq d(Tx,Ty) \preceq a^* d(x,y)a.$$

Also, by (2) of Theorem 1.3 there exists $u_{x,y} \in A$ such that $d(x,y) = u_{x,y}^* u_{x,y}$. Thus $\|d(x,y)\| = \|u_{x,y}^* u_{x,y}\| = \|u_{x,y}\|^2$ and

$$\begin{aligned} 0_A \preceq d(Tx,Ty) &\preceq a^* d(x,y)a \\ &= a^* u_{x,y}^* u_{x,y} a \\ &= (u_{x,y}a)^*(u_{x,y}a). \end{aligned}$$

Applying (3) of Theorem 1.3 we have

$$\begin{aligned} \|d(Tx,Ty)\| &\leq \|(u_{x,y}a)^*(u_{x,y}a)\| \\ &= \|u_{x,y}a\|^2 \\ &\leq \|a\|^2 \|u_{x,y}\|^2 \\ &= \|a\|^2 \|d(x,y)\|. \end{aligned}$$

Taking $D(x,y) = \|d(x,y)\|$ and $k = \|a\|^2 < 1$ and applying the Banach contraction principle we deduce the desired results. □

As a result, the main result of Ma *et al.* [1] follows from the Banach contraction mapping principle. The other results in [1] and the fixed point theorems in [2, 3] can be derived from the existing corresponding fixed point theorems in the setting of the standard metric space in the literature.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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