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On approximating the modified Bessel function of the first kind and Toader-Qi mean

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Abstract

In the article, we present several sharp bounds for the modified Bessel function of the first kind $I_0(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{2^{2n}(n!)^2}$ and the Toader-Qi mean $TQ(a,b)=\frac{2}{\pi}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}\,d\theta$ for all $t>0$ and $a,b>0$ with $a\neq b$.

MSC: 33C10; 26E60

Keywords: modified Bessel function; Toader-Qi mean; logarithmic mean; identric mean

1 Introduction

Let $a,b>0$, $p:(0,\infty)\to\mathbb{R}^+$ be a strictly monotone real function, $\theta\in(0,2\pi)$ and

$$r_n(\theta)=\begin{cases}(a^n\cos^2\theta+b^n\sin^2\theta)^{1/n}, & n\neq 0,\\ a^{\cos^2\theta}b^{\sin^2\theta}, & n=0.\end{cases} \tag{1.1}$$

Then the mean $M_{p,n}(a,b)$ was first introduced by Toader in [1] as follows:

$$M_{p,n}(a,b)=p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi}p\bigl(r_n(\theta)\bigr)\,d\theta\right)=p^{-1}\left(\frac{2}{\pi}\int_0^{\pi/2}p\bigl(r_n(\theta)\bigr)\,d\theta\right), \tag{1.2}$$

where p^{-1} is the inverse function of p.

From (1.1) and (1.2) we clearly see that

$$M_{1/x,2}(a,b)=\frac{\pi}{2\int_0^{\pi/2}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}}=\mathrm{AGM}(a,b)$$

is the classical arithmetic-geometric mean, which is related to the complete elliptic integral of the first kind $\mathcal{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}\,d\theta$. The Toader mean

$$M_{x,2}(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta=T(a,b)$$

is related to the complete elliptic integral of the second kind $\mathcal{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}\,d\theta$. We have

$$M_{x^q,0}(a,b)=\left(\frac{2}{\pi}\int_0^{\pi/2}a^{q\cos^2\theta}b^{q\sin^2\theta}\,d\theta\right)^{1/q}\quad(q\neq 0).$$

In particular,

$$M_{x,0}(a,b) = \frac{2}{\pi}\int_0^{\pi/2} a^{\cos^2\theta} b^{\sin^2\theta}\, d\theta = TQ(a,b) \tag{1.3}$$

is the Toader-Qi mean.

Recently, the arithmetic-geometric mean $\mathrm{AGM}(a,b)$ and the Toader mean $T(a,b)$ have attracted the attention of many researchers. In particular, many remarkable inequalities for $\mathrm{AGM}(a,b)$ and $T(a,b)$ can be found in the literature [2–20].

For $q \neq 0$, the mean $M_{x^q,0}(a,b)$ seems to be mysterious, Toader [1] said that he did not know how to determine any sense for this mean.

Let $z \in \mathbb{C}$, $\nu \in \mathbb{R}\backslash\{-1,-2,-3,\ldots\}$ and $\Gamma(z) = \lim_{n\to\infty} n!n^z/[\prod_{k=0}^{\infty}(z+k)]$ be the classical gamma function. Then the modified Bessel function of the first kind $I_\nu(z)$ [21] is given by

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{z^{2n+\nu}}{n!2^{2n+\nu}\Gamma(\nu+n+1)}. \tag{1.4}$$

Very recently, Qi *et al.* [22] proved the identity

$$M_{x^q,0}(a,b) = \left(\frac{2}{\pi}\int_0^{\pi/2} a^{q\cos^2\theta} b^{q\sin^2\theta}\, d\theta\right)^{1/q} = \sqrt{ab}\,I_0^{1/q}\left(\frac{q}{2}\log\frac{a}{b}\right) \tag{1.5}$$

and inequalities

$$L(a,b) < TQ(a,b) < \frac{A(a,b)+G(a,b)}{2} < \frac{2A(a,b)+G(a,b)}{3} < I(a,b) \tag{1.6}$$

for all $q \neq 0$ and $a,b > 0$ with $a \neq b$, where $L(a,b) = (b-a)/(\log b - \log a)$, $A(a,b) = (a+b)/2$, $G(a,b) = \sqrt{ab}$, and $I(a,b) = (b^b/a^a)^{1/(b-a)}/e$ are, respectively, the logarithmic, arithmetic, geometric, and identric means of a and b.

Let $b > a > 0$, $p \in \mathbb{R}$, $t = (\log b - \log a)/2 > 0$, and the pth power mean $A_p(a,b)$ be defined by

$$A_p(a,b) = \left(\frac{a^p+b^p}{2}\right)^{1/p} \quad (p \neq 0), \qquad A_0(a,b) = \sqrt{ab} = G(a,b).$$

Then the logarithmic mean $L(a,b)$, the identric mean $I(a,b)$, and the pth power mean $A_p(a,b)$ can be expressed as

$$\begin{aligned} &L(a,b) = \sqrt{ab}\frac{\sinh t}{t}, \qquad I(a,b) = \sqrt{ab}e^{t/\tanh t - 1}, \\ &A_p(a,b) = \sqrt{ab}\cosh^{1/p}(pt) \quad (p \neq 0) \end{aligned} \tag{1.7}$$

and (1.3)-(1.5) lead to

$$\begin{aligned} \frac{TQ(a,b)}{\sqrt{ab}} &= \frac{M_{x,0}(a,b)}{\sqrt{ab}} = \frac{2}{\pi}\int_0^{\pi/2} e^{t\cos(2\theta)}\, d\theta = I_0(t) \\ &= \frac{2}{\pi}\int_0^{\pi/2}\cosh(t\cos\theta)\, d\theta = \frac{2}{\pi}\int_0^{\pi/2}\cosh(t\sin\theta)\, d\theta. \end{aligned} \tag{1.8}$$

The main purpose of this paper is to present several sharp bounds for the modified Bessel function of the first kind $I_0(t)$ and the Toader-Qi mean $TQ(a,b)$.

2 Lemmas

In order to establish our main results we need several lemmas, which we present in this section.

Lemma 2.1 (See [23]) *Let $\binom{n}{k}$ be the number of combinations of n objects taken k at a time, that is,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Then

$$\sum_{k=0}^{\infty} \binom{n}{k}^2 = \binom{2n}{n}.$$

Lemma 2.2 (See [23]) *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real sequences with $b_n > 0$ and $\lim_{n\to\infty} a_n/b_n = s$. Then the power series $\sum_{n=0}^{\infty} a_n t^n$ is convergent for all $t \in \mathbb{R}$ and*

$$\lim_{t\to\infty} \frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s$$

if the power series $\sum_{n=0}^{\infty} b_n t^n$ is convergent for all $t \in \mathbb{R}$.

Lemma 2.3 *The Wallis ratio*

$$W_n = \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1)} \tag{2.1}$$

is strictly decreasing with respect to all integers $n \geq 0$ and strictly log-convex with respect to all real numbers $n \geq 0$.

Proof It follows from (2.1) that

$$\frac{W_{n+1}}{W_n} = 1 - \frac{1}{2(n+1)} < 1 \tag{2.2}$$

for all integers $n \geq 0$.

Therefore, W_n is strictly decreasing with respect to all integers $n \geq 0$ follows from (2.2).

Let $f(x) = \Gamma(x+1/2)/\Gamma(x+1)$ and $\psi(x) = \Gamma'(x)/\Gamma(x)$ be the psi function. Then it follows from the monotonicity of $\psi'(x)$ that

$$\left[\log f(x)\right]'' = \psi'\left(x+\frac{1}{2}\right) - \psi'(x+1) > 0 \tag{2.3}$$

for all $x \geq 0$.

Therefore, W_n is strictly log-convex with respect to all real numbers $n \geq 0$ follows from (2.1) and (2.3). □

Lemma 2.4 (See [24]) *The double inequality*

$$\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}}$$

holds for all $x > 0$ *and* $a \in (0,1)$.

Lemma 2.5 *Let* $s_n = (2n)!(2n+1)!/[2^{4n}(n!)^4]$. *Then the sequence* $\{s_n\}_{n=0}^{\infty}$ *is strictly decreasing and*

$$\lim_{n\to\infty} s_n = \frac{2}{\pi}. \tag{2.4}$$

Proof The monotonicity of the sequence $\{s_n\}_{n=0}^{\infty}$ follows from

$$\frac{s_{n+1}}{s_n} = \frac{(2n+1)(2n+3)}{4(n+1)^2} < 1.$$

To prove (2.4), we rewrite s_n as

$$\begin{aligned} s_n &= (2n+1)\left[\frac{(2n-1)!!}{2^n n!}\right]^2 = \frac{2n+1}{\Gamma^2(\frac{1}{2})}\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2 \\ &= \frac{2(n+\frac{1}{2})}{\pi}\left[\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\right]^2. \end{aligned} \tag{2.5}$$

It follows from Lemma 2.4 and (2.5) that

$$\frac{2}{\pi} = \frac{2}{\pi}\frac{n+\frac{1}{2}}{n+\frac{1}{2}} < s_n < \frac{2}{\pi}\frac{n+\frac{1}{2}}{n}. \tag{2.6}$$

Therefore, equation (2.4) follows from (2.6). □

Lemma 2.6 (See [25]) *Let* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *and* $B(t) = \sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r, r)$ $(r > 0)$ *with* $b_k > 0$ *for all* k. *If the non-constant sequence* $\{a_k/b_k\}$ *is increasing* (*decreasing*) *for all* k, *then the function* $A(t)/B(t)$ *is strictly increasing* (*decreasing*) *on* $(0, r)$.

Lemma 2.7 (See [26]) *Let* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *and* $B(t) = \sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $\mathbb{R}$ *with* $b_k > 0$ *for all* k. *If there exists* $m \in \mathbb{N}$ *such that the non-constant sequence* $\{a_k/b_k\}$ *is increasing* (*decreasing*) *for* $0 \le k \le m$ *and decreasing* (*increasing*) *for* $k \ge m$, *then there exists* $t_0 \in (0,\infty)$ *such that the function* $A(t)/B(t)$ *is strictly increasing* (*decreasing*) *on* $(0, t_0)$ *and strictly decreasing* (*increasing*) *on* (t_0, ∞).

Lemma 2.8 *The identity*

$$I_0^2(t) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^4} t^{2n}$$

holds for all $t \in \mathbb{R}$.

Proof From (1.4) and Lemma 2.1 together with the Cauchy product we have

$$I_0^2(t) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{1}{2^{2k}(k!)^2}\frac{1}{2^{2(n-k)}[(n-k)!]^2}\right)t^{2n}$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{2n}(n!)^2}\sum_{k=0}^{n}\frac{(n!)^2}{(k!)^2[(n-k)!]^2}\right)t^{2n} = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^4}t^{2n}. \qquad \square$$

Lemma 2.9 (See [27]) *Let* $-\infty < a < b < \infty$ *and* $f, g : [a,b] \to \mathbb{R}$. *Then*

$$\int_a^b f(x)g(x)\,dx \ge \frac{1}{b-a}\int_a^b f(x)\,dx\int_a^b g(x)\,dx$$

if both f *and* g *are increasing or decreasing on* (a,b).

Lemma 2.10 (See [28]) *Let* $-\infty < a < b < \infty$ *and* $f, g : (a,b) \to \mathbb{R}$. *Then*

$$\int_a^b f(x)g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\int_a^b g(x)\,dx$$
$$\ge \frac{12}{(b-a)^3}\int_a^b\left(x-\frac{a+b}{2}\right)f(x)\,dx\int_a^b\left(x-\frac{a+b}{2}\right)g(x)\,dx \tag{2.7}$$

if both f *and* g *are convex on the interval* (a,b), *and inequality* (2.7) *becomes an equality if and only if* f *or* g *is a linear function on* (a,b).

3 Main results

Theorem 3.1 *The double inequalities*

$$\frac{e^t}{1+2t} < I_0(t) < \frac{e^t}{\sqrt{1+2t}} \tag{3.1}$$

and

$$\frac{b}{1+\log(b/a)} < TQ(a,b) < \frac{b}{\sqrt{1+\log(b/a)}} \tag{3.2}$$

hold for all $t > 0$ *and* $b > a > 0$.

Proof From (1.8) we have

$$I_0(t) = \frac{2}{\pi}\int_0^{\pi/2}\cosh(t\sin\theta)\,d\theta = \frac{2}{\pi}\int_0^1\frac{\cosh(tx)}{\sqrt{1-x^2}}\,dx \tag{3.3}$$

and

$$e^{-t}I_0(t) = \frac{2}{\pi}\int_0^{\pi/2}e^{t[\cos(2\theta)-1]}\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{e^{2t\sin^2\theta}}$$
$$< \frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{1+2t\sin^2\theta} = \frac{1}{\sqrt{1+2t}}. \tag{3.4}$$

We clearly see that both $\cosh(tx)$ and $1/\sqrt{1-x^2}$ are increasing with respect to x on $(0,1)$. Then Lemma 2.9 and (3.3) lead to

$$\begin{aligned} I_0(t) &\geq \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^2}}\int_0^1 \cosh(tx)\,dx = \frac{\sinh t}{t} \\ &= \frac{e^t}{2t}\left(1-\frac{1}{e^{2t}}\right) > \frac{e^t}{2t}\left(1-\frac{1}{1+2t}\right) = \frac{e^t}{1+2t}. \end{aligned} \tag{3.5}$$

Therefore, inequality (3.1) follows from (3.4) and (3.5).

Let $t = \log(b/a)/2$. Then it follows from (1.8) and (3.1) that

$$\frac{\sqrt{b/a}}{1+\log(b/a)} < \frac{TQ(a,b)}{\sqrt{ab}} < \frac{\sqrt{b/a}}{\sqrt{1+\log(b/a)}}. \tag{3.6}$$

Therefore, inequality (3.2) follows from (3.6). □

Remark 3.1 From Theorem 3.1 we clearly see that

$$\lim_{t\to\infty} e^{-t}I_0(t) = \lim_{x\to 0^+} TQ(x,1) = 0.$$

Theorem 3.2 *The double inequalities*

$$\alpha_1\sqrt{\frac{\sinh(2t)}{t}} < I_0(t) < \beta_1\sqrt{\frac{\sinh(2t)}{t}} \tag{3.7}$$

and

$$\alpha_2\sqrt{L(a,b)A(a,b)} < TQ(a,b) < \beta_2\sqrt{L(a,b)A(a,b)} \tag{3.8}$$

hold for all $t > 0$ *and* $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha_1 \leq 1/\sqrt{\pi}$, $\beta_1 \geq \sqrt{2}/2$, $\alpha_2 \leq \sqrt{2/\pi}$ *and* $\beta_2 \geq 1$.

Proof Let

$$R_0(t) = \frac{I_0^2(t)}{\sinh(2t)/(2t)}, \tag{3.9}$$

$$a_n = \frac{(2n)!}{2^{2n}(n!)^4}, \qquad b_n = \frac{2^{2n}}{(2n+1)!}. \tag{3.10}$$

Then simple computation leads to

$$\frac{a_n}{b_n} = \frac{(2n)!(2n+1)!}{2^{4n}(n!)^4}. \tag{3.11}$$

It follows from Lemma 2.5 and (3.11) that the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is strictly decreasing and

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{2}{\pi}. \tag{3.12}$$

From Lemma 2.8 we have

$$R_0(t)=\frac{\sum_{n=0}^{\infty}a_nt^{2n}}{\sum_{n=0}^{\infty}b_nt^{2n}}. \tag{3.13}$$

Lemma 2.6 and (3.13) together with the monotonicity of the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ lead to the conclusion that $R_0(t)$ is strictly decreasing on the interval $(0,\infty)$. Therefore, we have

$$\lim_{t\to\infty}R_0(t)<R_0(t)<\lim_{t\to 0^+}R_0(t)=\frac{a_0}{b_0}=1. \tag{3.14}$$

From Lemma 2.2, (3.12), and (3.13) we know that

$$\lim_{t\to\infty}R_0(t)=\frac{2}{\pi}. \tag{3.15}$$

Therefore, inequality (3.7) holds for all $t>0$ if and only if $\alpha_1\le 1/\sqrt{\pi}$ and $\beta_1\ge\sqrt{2}/2$ follows easily from (3.9), (3.14), and (3.15) together with the monotonicity of $R_0(t)$.

Let $b>a>0$ and $t=\log(b/a)/2$. Then inequality (3.8) holds for $a,b>0$ with $a\ne b$ if and only if $\alpha_2\le\sqrt{2/\pi}$, and $\beta_2\ge 1$ follows from (1.7) and (1.8) together with inequality (3.7) for all $t>0$ if and only if $\alpha_1\le 1/\sqrt{\pi}$ and $\beta_1\ge\sqrt{2}/2$. □

Remark 3.2 Equations (3.9) and (3.15) imply that

$$\lim_{t\to\infty}e^{-t}\sqrt{t}I_0(t)=\frac{1}{\sqrt{2\pi}}$$

or we have the asymptotic formula

$$I_0(t)\sim\frac{e^t}{\sqrt{2\pi t}}\quad(t\to\infty).$$

Theorem 3.3 *Let $\lambda_1,\lambda_2>0$, $t_0=2.7113\ldots$ be the unique solution of the equation*

$$\frac{d}{dt}\left[\frac{tI_0^2(t)-\sinh t}{(\cosh t-1)\sinh t}\right]=0 \tag{3.16}$$

on $(0,\infty)$ and

$$\lambda_0=\frac{t_0I_0^2(t_0)-\sinh t_0}{(\cosh t_0-1)\sinh t_0}=0.6766\ldots. \tag{3.17}$$

Then the double inequality

$$\sqrt{(\lambda_1\cosh t+1-\lambda_1)\frac{\sinh t}{t}}<I_0(t)<\sqrt{(\lambda_2\cosh t+1-\lambda_2)\frac{\sinh t}{t}} \tag{3.18}$$

or

$$\sqrt{\big[\lambda_1A(a,b)+(1-\lambda_1)G(a,b)\big]L(a,b)}<TQ(a,b)<\sqrt{\big[\lambda_2A(a,b)+(1-\lambda_2)G(a,b)\big]L(a,b)}$$

holds for all $t>0$ or $a,b>0$ with $a\ne b$ if and only if $\lambda_1\le 2/\pi$, $\lambda_2>\lambda_0$.

Proof Let

$$R_1(t) = \frac{I_0^2(t) - \frac{\sinh t}{t}}{\frac{(\cosh t-1)\sinh t}{t}}, \tag{3.19}$$

$$c_n = \frac{(2n)!}{2^{2n}(n!)^4} - \frac{1}{(2n+1)!}, \qquad d_n = \frac{2^{2n}-1}{(2n+1)!}, \qquad s_n = \frac{(2n)!(2n+1)!}{2^{4n}(n!)^4}, \tag{3.20}$$

and

$$s_n' = \left(2^{2n} + 3n^2 + 6n + 2\right)s_n - \left(3n^2 + 6n + 3\right). \tag{3.21}$$

Then it follows from Lemma 2.2, Lemma 2.5, Lemma 2.8, and (3.19)-(3.21) that

$$R_1(t) = \frac{\sum_{n=1}^{\infty} c_n t^{2n}}{\sum_{n=1}^{\infty} d_n t^{2n}}, \tag{3.22}$$

$$\lim_{t\to\infty} R_1(t) = \lim_{n\to\infty} \frac{c_n}{d_n} = \lim_{n\to\infty} \frac{2^{2n}s_n - 1}{2^{2n}-1} = \frac{2}{\pi}, \tag{3.23}$$

$$\frac{c_1}{d_1} = \frac{2}{3} < \frac{c_2}{d_2} = \frac{41}{60} > \frac{c_3}{d_3} = \frac{19}{28}, \tag{3.24}$$

$$\frac{c_{n+1}}{d_{n+1}} - \frac{c_n}{d_n} = -\frac{2^{2n}s_n'}{(n+1)^2(2^{2n+2}-1)(2^{2n}-1)}, \tag{3.25}$$

and we have the inequality

$$\begin{aligned} s_n' &> \frac{2}{\pi}\left(2^{2n} + 3n^2 + 6n + 2\right) - \left(3n^2 + 6n + 3\right) \\ &> \frac{3}{5}\left(2^{2n} + 3n^2 + 6n + 2\right) - \left(3n^2 + 6n + 3\right) = \frac{3}{5}\left[2^{2n} - \left(2n^2 + 4n + 3\right)\right] > 0 \end{aligned} \tag{3.26}$$

for all $n \geq 3$.

From (3.24)-(3.26) we know that the sequence $\{c_n/d_n\}_{n=1}^{\infty}$ is strictly increasing for $1 \leq n \leq 2$ and strictly decreasing for $n \geq 2$. Then Lemma 2.7 and (3.22) lead to the conclusion that there exists $t_0 \in (0,\infty)$ such that $R_1(t)$ is strictly increasing on $(0,t_0)$ and decreasing on (t_0,∞). Therefore, we have

$$\min\left\{R_1\left(0^+\right), \lim_{t\to\infty} R_1(t)\right\} < R_1(t) \leq R_1(t_0) \tag{3.27}$$

for all $t > 0$, and t_0 is the unique solution of equation (3.16) on $(0,\infty)$.

Note that

$$R_1\left(0^+\right) = \frac{c_1}{d_1} = \frac{2}{3}. \tag{3.28}$$

From (3.17), (3.19), (3.23), (3.27), and (3.28) we get

$$\frac{2}{\pi} < R_1(t) \leq R_1(t_0) = \lambda_0. \tag{3.29}$$

Therefore, inequality (3.18) holds for all $t > 0$ if and only if $\lambda_1 \leq 2/\pi$, $\lambda_2 \geq \lambda_0$ follows from (3.19) and (3.29) together with the piecewise monotonicity of $R_1(t)$ on $(0,\infty)$. Numerical computations show that $t_0 = 2.7113\ldots$ and $\lambda_0 = 0.6766\ldots$. □

Theorem 3.4 *Let $p, q \in \mathbb{R}$. Then the double inequality*

$$\cosh^{1-p} t\left(\frac{\sinh t}{t}\right)^p < I_0(t) < q\frac{\sinh t}{t} + (1-q)\cosh t \tag{3.30}$$

or

$$L^p(a,b)A^{1-p}(a,b) < TQ(a,b) < qL(a,b) + (1-q)A(a,b)$$

holds for all $t > 0$ or $a, b > 0$ with $a \neq b$ if and only if $p \geq 3/4$ and $q \leq 3/4$.

Proof If the first inequality of (3.30) holds for all $t > 0$, then

$$\lim_{t\to 0^+} \frac{I_0(t) - \cosh^{1-p} t(\frac{\sinh t}{t})^p}{t^2} = \frac{1}{3}\left(p - \frac{3}{4}\right) \geq 0,$$

which implies that $p \geq 3/4$.

It is not difficult to verify that the function $\cosh^{1-p} t(\sinh t/t)^p$ is strictly decreasing with respect to $p \in \mathbb{R}$ for any fixed $t > 0$, hence we only need to prove the first inequality of (3.30) for all $t > 0$ and $p = 3/4$, that is,

$$I_0^4(t) > \left(\frac{\sinh t}{t}\right)^3 \cosh t. \tag{3.31}$$

Making use of the power series and Cauchy product formulas together with Lemma 2.8 we have

$$\begin{aligned} &I_0^4(t) - \left(\frac{\sinh t}{t}\right)^3 \cosh t \\ &\quad = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\left(\frac{(2k)!}{2^{2k}(k!)^4}\frac{(2(n-k))!}{2^{2(n-k)}((n-k)!)^4}\right) - \frac{2^{4n+3} - 2^{2n+1}}{(2n+3)!}\right]t^{2n}. \end{aligned} \tag{3.32}$$

Let W_n and s_n be, respectively, defined by Lemma 2.3 and Lemma 2.5, and

$$u_n = \sum_{k=0}^{n}\left(\frac{(2k)!}{2^{2k}(k!)^4}\frac{(2(n-k))!}{2^{2(n-k)}((n-k)!)^4}\right) - \frac{2^{4n+3} - 2^{2n+1}}{(2n+3)!}. \tag{3.33}$$

Then simple computations lead to

$$u_0 = u_1 = 0, \qquad u_2 = \frac{3}{80}, \qquad u_3 = \frac{4}{189}. \tag{3.34}$$

It follows from Lemma 2.1 and Lemmas 2.3-2.5 together with (3.33) that

$$\begin{aligned} u_n &= \sum_{k=0}^{n}\left(\frac{1}{(k!)^2((n-k)!)^2}\frac{(2k)!}{2^{2k}(k!)^2}\frac{(2(n-k))!}{2^{2(n-k)}((n-k)!)^2}\right) - \frac{2^{4n+3} - 2^{2n+1}}{(2n+3)!} \\ &= \sum_{k=0}^{n}\left(\frac{1}{(k!)^2((n-k)!)^2}W_k W_{n-k}\right) - \frac{2^{4n+3} - 2^{2n+1}}{(2n+3)!} \end{aligned}$$

$$
\begin{aligned}
&> \sum_{k=0}^{n}\left(\frac{1}{(k!)^2((n-k)!)^2}W_{n/2}^2\right) - \frac{2^{4n+3}-2^{2n+1}}{(2n+3)!} \\
&= \sum_{k=0}^{n}\left[\frac{1}{(k!)^2((n-k)!)^2}\left(\frac{\Gamma(n/2+1/2)}{\Gamma(1/2)\Gamma(n/2+1)}\right)^2\right] - \frac{2^{4n+3}-2^{2n+1}}{(2n+3)!} \\
&> \sum_{k=0}^{n}\left(\frac{1}{\pi(n/2+1/2)(k!)^2((n-k)!)^2}\right) - \frac{2^{4n+3}-2^{2n+1}}{(2n+3)!} \\
&= \frac{2}{\pi(n+1)(n!)^2}\sum_{k=0}^{n}\frac{(n!)^2}{(k!)^2((n-k)!)^2} - \frac{2^{4n+3}-2^{2n+1}}{(2n+3)!} \\
&= \frac{2}{\pi(n+1)(n!)^2}\frac{(2n)!}{(n!)^2} - \frac{2^{4n+3}-2^{2n+1}}{(2n+3)!} \\
&= \frac{2^{2n+1}(2^{2n+2}-1)}{\pi(2n+3)!}\left[\frac{2^{2n+2}}{2^{2n+2}-1}\left(n+\frac{3}{2}\right)s_n - \pi\right] \\
&> \frac{2^{2n+1}(2^{2n+2}-1)}{\pi(2n+3)!}\left[\left(4+\frac{3}{2}\right)\frac{2}{\pi} - \pi\right] > 0
\end{aligned} \tag{3.35}
$$

for all $n \geq 4$.

Therefore, inequality (3.31) follows from (3.32)-(3.35).

If the second inequality of (3.30) holds for all $t > 0$, then we have

$$\lim_{t\to 0^+}\frac{I_0(t) - q\frac{\sinh t}{t} - (1-q)\cosh t}{t^2} = \frac{1}{3}\left(q - \frac{3}{4}\right) \leq 0,$$

which implies that $q \leq 3/4$.

Since $\cosh t > \sinh t/t$, we only need to prove that the second inequality of (3.3) holds for all $t > 0$ and $q = 3/4$, that is,

$$\frac{\cosh t - I_0(t)}{\cosh t - \sinh t/t} > \frac{3}{4}. \tag{3.36}$$

Let

$$\alpha_n = \frac{2^n n! - (2n-1)!!}{2^n n!(2n)!}, \qquad \beta_n = \frac{2n}{(2n+1)!}, \qquad \gamma_n = \frac{(n+2)(2n+1)}{2(n+1)}W_n,$$

and W_n be defined by (2.1).

Then simple computations lead to

$$\frac{\cosh t - I_0(t)}{\cosh t - \sinh t/t} = \frac{\sum_{n=1}^{\infty}\alpha_n t^{2n}}{\sum_{n=1}^{\infty}\beta_n t^{2n}}, \tag{3.37}$$

$$\frac{\alpha_{n+1}}{\beta_{n+1}} - \frac{\alpha_n}{\beta_n} = \frac{2n+3}{2n+2}(1-W_{n+1}) - \frac{2n+1}{2n}(1-W_n) = \frac{\gamma_n - 1}{2n(n+1)}, \tag{3.38}$$

$$\frac{\gamma_{n+1}}{\gamma_n} = 1 + \frac{n+1}{2(n+2)^2} > 1, \qquad \gamma_1 = \frac{9}{8} > 1. \tag{3.39}$$

From (3.38) and (3.39) we clearly see that the sequence $\{\alpha_n/\beta_n\}_{n=1}^{\infty}$ is strictly increasing, then Lemma 2.6 and (3.37) lead to the conclusion that the function $(\cosh t - I_0(t))/[\cosh t -$

$\sinh t/t]$ is strictly increasing on the interval $(0,\infty)$. Therefore, inequality (3.36) follows from the monotonicity of $(\cosh t - I_0(t))/[\cosh t - \sinh t/t]$ and the fact that

$$\lim_{t\to 0^+}\frac{\cosh t - I_0(t)}{\cosh t - \sinh t/t} = \frac{\alpha_1}{\beta_1} = \frac{3}{4}. \qquad \square$$

Theorem 3.5 *Let $p,q>0$, t_0 be the unique solution of the equation*

$$\frac{d[\frac{p^2(I_0(t)-1)}{\cosh(pt)-1}]}{dt} = 0 \tag{3.40}$$

and

$$\mu_0 = \frac{p^2(I_0(t_0)-1)}{\cosh(pt_0)-1}. \tag{3.41}$$

Then the following statements are true:

(i) *The double inequality*

$$1 - \frac{1}{2p^2} + \frac{1}{2p^2}\cosh(pt) < I_0(t) < 1 - \frac{1}{2q^2} + \frac{1}{2q^2}\cosh(qt) \tag{3.42}$$

or

$$\begin{aligned}&\left(1-\frac{1}{2p^2}\right)G(a,b) + \frac{1}{2p^2}A_p^p(a,b)G^{1-p}(a,b)\\ &\quad < TQ(a,b) < \left(1-\frac{1}{2q^2}\right)G(a,b) + \frac{1}{2q^2}A_q^q(a,b)G^{1-q}(a,b)\end{aligned}$$

holds for all $t>0$ or $a,b>0$ with $a\neq b$ if and only if $p\leq\sqrt{3}/2$ and $q\geq 1$.

(ii) *The inequality*

$$I_0(t) \geq 1 - \frac{\mu_0}{p^2} + \frac{\mu_0}{p^2}\cosh(pt) \tag{3.43}$$

or

$$TQ(a,b) \geq \left(1-\frac{\mu_0}{p^2}\right)G(a,b) + \frac{\mu_0}{p^2}A_p^p(a,b)G^{1-p}(a,b)$$

holds for all $t>0$ or $a,b>0$ with $a\neq b$ if $p\in(\sqrt{3}/2,1)$.

Proof (i) Let

$$R_2(t) = \frac{p^2(I_0(t)-1)}{\cosh(pt)-1}, \tag{3.44}$$

$$u_n = \frac{1}{2^{2n}(n!)^2}, \qquad v_n = \frac{p^{2n-2}}{(2n)!}.$$

Then simple computations lead to

$$R_2(t) = \frac{\sum_{n=1}^{\infty}u_n t^{2n}}{\sum_{n=1}^{\infty}v_n t^{2n}}, \tag{3.45}$$

$$\frac{u_{n+1}}{v_{n+1}} - \frac{u_n}{v_n} = -\frac{(2n)!}{(2p)^{2n}(n!)^2}\left(p^2 - \frac{2n+1}{2n+2}\right). \tag{3.46}$$

From (3.46) we clearly see that the sequence $\{u_n/v_n\}_{n=1}^{\infty}$ is strictly decreasing if $p \geq 1$ and strictly increasing if $p \leq \sqrt{3}/2$. Then Lemma 2.6 and (3.45) lead to the conclusion that the function $R_2(t)$ is strictly decreasing if $p \geq 1$ and strictly increasing if $p \leq \sqrt{3}/2$. Hence, we have

$$R_2(t) < \lim_{t\to 0^+} R_2(t) = \frac{u_1}{v_1} = \frac{1}{2} \tag{3.47}$$

for all $t > 0$ if $p \geq 1$ and

$$R_2(t) > \lim_{t\to 0^+} R_2(t) = \frac{u_1}{v_1} = \frac{1}{2} \tag{3.48}$$

for all $t > 0$ if $p \leq \sqrt{3}/2$.

Therefore, inequality (3.42) holds for all $t > 0$ if $p \leq \sqrt{3}/2$ and $q \geq 1$ follows easily from (3.44) and (3.47) together with (3.48).

If the first inequality (3.42) holds for all $t > 0$, then we have

$$\lim_{t\to 0^+} \frac{I_0(t) - (1 - \frac{1}{2p^2} + \frac{1}{2p^2}\cosh(pt))}{t^4} = \frac{1}{48}\left(\frac{3}{4} - p^2\right) \geq 0,$$

which implies that $p \leq \sqrt{3}/2$.

If there exists $q_0 \in (\sqrt{3}/2, 1)$ such that the second inequality of (3.42) holds for all $t > 0$, then we have

$$\lim_{t\to\infty} \frac{I_0(t) - (1 - \frac{1}{2q_0^2} + \frac{1}{2q_0^2}\cosh(q_0 t))}{e^{q_0 t}} \leq 0. \tag{3.49}$$

But the first inequality of (3.1) leads to

$$\begin{aligned} &\frac{I_0(t) - (1 - \frac{1}{2q_0^2} + \frac{1}{2q_0^2}\cosh(q_0 t))}{e^{q_0 t}} \\ &\quad > \frac{e^{(1-q_0)t}}{1+2t} - \left(1 - \frac{1}{2q_0^2}\right)e^{-q_0 t} - \frac{1+e^{-2q_0 t}}{4q_0^2} \to \infty \quad (t \to \infty), \end{aligned}$$

which contradicts inequality (3.49).

(ii) If $p \in (\sqrt{3}/2, 1)$, then from (3.46) we know that there exists $n_0 \in \mathbb{N}$ such that the sequence $\{u_n/v_n\}_{n=1}^{\infty}$ is strictly decreasing for $n \leq n_0$ and strictly increasing for $n \geq n_0$. Then (3.45) and Lemma 2.7 lead to the conclusion that there exists $t_0 \in (0, \infty)$ such that the function $R_2(t)$ is strictly decreasing on $(0, t_0]$ and strictly increasing on $[t_0, \infty)$. We clearly see that t_0 satisfies equation (3.40). It follows from (3.41) and (3.44) together with the piecewise monotonicity of $R_2(t)$ that

$$R_2(t) \geq R_2(t_0) = \mu_0. \tag{3.50}$$

Therefore, inequality (3.43) holds for all $t > 0$ follows from (3.44) and (3.50). □

It is not difficult to verify that the function

$$1-\frac{1}{2p^2}+\frac{1}{2p^2}\cosh(pt)$$

is strictly increasing with respect to p on the interval $(0,\infty)$ and

$$2\cosh\left(\frac{t}{2}\right)-1>\cosh^{1/2}t$$

for $t>0$.

Letting $p=\sqrt{3}/2, 3/4, \sqrt{2}/2, 2/3, 1/2$ and $q=1$ in Theorem 3.5(i), then we get Corollary 3.1 immediately.

Corollary 3.1 *The inequalities*

$$\begin{aligned}\cosh^{1/2}(t)&<2\cosh\left(\frac{t}{2}\right)-1<\frac{9}{8}\cosh\left(\frac{2t}{3}\right)-\frac{1}{8}<\cosh\left(\frac{\sqrt{2}t}{2}\right)\\&<\frac{8}{9}\cosh\left(\frac{3t}{4}\right)+\frac{1}{9}<\frac{2}{3}\cosh\left(\frac{\sqrt{3}t}{2}\right)+\frac{1}{3}<I_0(t)<\frac{1+\cosh t}{2}\end{aligned}$$

or

$$\begin{aligned}G^{1/2}(a,b)A^{1/2}(a,b)&<2A_{1/2}^{1/2}(a,b)G^{1/2}(a,b)-G(a,b)\\&<\frac{9}{8}A_{2/3}^{2/3}(a,b)G^{1/3}(a,b)-\frac{1}{8}G(a,b)\\&<A_{\sqrt{2}/2}^{\sqrt{2}/2}(a,b)G^{1-\sqrt{2}/2}(a,b)\\&<\frac{8}{9}A_{3/4}^{3/4}(a,b)G^{1/4}(a,b)+\frac{1}{9}G(a,b)\\&<\frac{2}{3}A_{\sqrt{3}/2}^{\sqrt{3}/2}(a,b)G^{1-\sqrt{3}/2}(a,b)+\frac{1}{3}G(a,b)\\&<TQ(a,b)<\frac{A(a,b)+G(a,b)}{2}\end{aligned}$$

hold for all $t>0$ or all $a,b>0$ with $a\neq b$.

Theorem 3.6 *Let $p>0$. Then the following statements are true*:

(i) *The inequality*

$$I_0(t)>\left[\cosh(pt)\right]^{\frac{1}{2p^2}} \tag{3.51}$$

or

$$TQ(a,b)>G^{1-\frac{1}{2p}}(a,b)A_p^{\frac{1}{2p}}(a,b) \tag{3.52}$$

holds for all $t>0$ or $a,b>0$ with $a\neq b$ if and only if $p\geq\sqrt{6}/4$.

(ii) *The inequality* (3.51) *or* (3.52) *is reversed if and only if $p\leq 1/2$.*

(iii) *The inequalities*

$$\cosh^{1/2} t < \cosh\left(\frac{\sqrt{2}t}{2}\right) < \left[\cosh\left(\frac{\sqrt{6}t}{4}\right)\right]^{4/3} < I_0(t) < \cosh^2\left(\frac{t}{2}\right) < e^{t^2/4} \tag{3.53}$$

or

$$\begin{aligned} G^{1/2}(a,b)A^{1/2}(a,b) &< A^{\sqrt{2}/2}_{\sqrt{2}/2}(a,b)G^{1-\sqrt{2}/2}(a,b) \\ &< A^{\sqrt{6}/3}_{\sqrt{6}/4}(a,b)G^{1-\sqrt{6}/3}(a,b) \\ &< TQ(a,b) < \frac{A(a,b)+G(a,b)}{2} \\ &< G(a,b)e^{(A^2(a,b)-G^2(a,b))/(4L^2(a,b))} \end{aligned}$$

hold for all $t > 0$ or all $a, b > 0$ with $a \neq b$.

Proof (i) If inequality (3.51) holds for all $t > 0$, then we have

$$\lim_{t\to 0^+} \frac{I_0(t) - [\cosh(pt)]^{\frac{1}{2p^2}}}{t^4} = \frac{1}{24}\left(p^2 - \frac{3}{8}\right) \geq 0,$$

which implies that $p \geq \sqrt{6}/4$.

It follows from Lemma 2 of [29] that the function $[\cosh(pt)]^{1/(2p^2)}$ is strictly decreasing with respect to $p \in (0, \infty)$ for any fixed $t > 0$, hence we only need to prove that inequality (3.51) holds for all $t > 0$ and $p = \sqrt{6}/4$. From the sixth inequality of Corollary 3.1 we clearly see that it suffices to prove that

$$\frac{2}{3}\cosh\left(\frac{\sqrt{3}t}{2}\right) + \frac{1}{3} > \left[\cosh\left(\frac{\sqrt{6}t}{4}\right)\right]^{4/3}$$

for all $t > 0$, which is equivalent to

$$\log\left[\frac{2}{3}\cosh(\sqrt{2}x) + \frac{1}{3}\right] > \frac{4}{3}\log(\cosh x) \tag{3.54}$$

for all $x > 0$, where $x = \sqrt{6}t/4$.

Let

$$f_1(x) = \log\left[\frac{2}{3}\cosh(\sqrt{2}x) + \frac{1}{3}\right] - \frac{4}{3}\log(\cosh x), \tag{3.55}$$

$$f_2(x) = 6\sqrt{2}\sinh(\sqrt{2}x)\cosh x - 8\cosh(\sqrt{2}x)\sinh x - 4\sinh x,$$

$$\xi_n = (3\sqrt{2}-4)(\sqrt{2}+1)^{2n-1} + (3\sqrt{2}+4)(\sqrt{2}-1)^{2n-1} - 4,$$

$$\eta_n = (\sqrt{2}+1)^{2n-1}.$$

Then simple computations lead to

$$f_1(0) = 0, \tag{3.56}$$

$$f_1'(x) = \frac{f_2(x)}{3\cosh x[2\cosh(\sqrt{2}x)+1]}, \tag{3.57}$$

$$f_2(x) = \sum_{n=1}^{\infty} \frac{\xi_n}{(2n-1)!} x^{2n-1}, \tag{3.58}$$

$$\xi_1 = \xi_2 = 0, \tag{3.59}$$

$$\eta_n \xi_n = (3\sqrt{2}-4)(\eta_n - \eta_1)(\eta_n - \eta_2). \tag{3.60}$$

From (3.58)-(3.60) and $\eta_n > \eta_2 > \eta_1 > 0$ for $n \geq 3$ we know that

$$f_2(x) > 0 \tag{3.61}$$

for all $x > 0$.

Therefore, inequality (3.54) follows easily from (3.55)-(3.57) and (3.61).

(ii) The sufficiency follows easily from the monotonicity of the function $p \to [\cosh(pt)]^{1/(2p^2)}$ and the last inequality in Corollary 3.1 together with the identity $(1 + \cosh t)/2 = \cosh^2(t/2)$.

Next, we prove the necessity. If there exists $p_0 \in (1/2, \sqrt{6}/4)$ such that $I_0(t) < [\cosh(p_0 t)]^{1/(2p_0^2)}$ for all $t > 0$, then we have

$$\lim_{t\to\infty} \frac{I_0(t) - [\cosh(p_0 t)]^{1/(2p_0^2)}}{e^{t/(2p_0)}} \leq 0. \tag{3.62}$$

But the first inequality of (3.1) leads to

$$\frac{I_0(t) - [\cosh(p_0 t)]^{1/(2p_0^2)}}{e^{t/(2p_0)}} > \frac{1}{1+2t}\frac{e^t}{e^{t/(2p_0)}} - \left(\frac{1+e^{-2p_0 t}}{2}\right)^{1/(2p_0^2)} \to \infty \quad (t \to \infty),$$

which contradicts (3.62).

(iii) Let $p = 1, \sqrt{2}/2, \sqrt{6}/4, 1/2, 0^+$. Then parts (i) and (ii) together with the monotonicity of the function $p \to [\cosh(pt)]^{1/(2p^2)}$ lead to (3.53). □

Theorem 3.7 *Let $\theta \in [0, \pi/2]$. Then the inequality*

$$I_0(t) > \frac{\cosh(t\cos\theta) + \cosh(t\sin\theta)}{2} \tag{3.63}$$

or

$$TQ(a,b) > \frac{A_{\cos\theta}^{\cos\theta}(a,b)G^{1-\cos\theta}(a,b) + A_{\sin\theta}^{\sin\theta}(a,b)G^{1-\sin\theta}(a,b)}{2}$$

holds for all $t > 0$ or all $a, b > 0$ with $a \neq b$ if and only if $\theta \in [\pi/8, 3\pi/8]$. In particular, the inequalities

$$\begin{aligned} I_0(t) &> \frac{1}{2}\left[\cosh\left(\frac{\sqrt{2-\sqrt{2}}}{2}t\right) + \cosh\left(\frac{\sqrt{2+\sqrt{2}}}{2}t\right)\right] \\ &> \frac{1}{2}\left[\cosh\left(\frac{\sqrt{3}}{2}t\right) + \cosh\left(\frac{1}{2}t\right)\right] > \cosh\left(\frac{\sqrt{2}}{2}t\right) \end{aligned} \tag{3.64}$$

or

$$TQ(a,b) > \frac{A_{\sqrt{2-\sqrt{2}}/2}^{\sqrt{2-\sqrt{2}}/2}(a,b)G^{1-\sqrt{2-\sqrt{2}}/2}(a,b) + A_{\sqrt{2+\sqrt{2}}/2}^{\sqrt{2+\sqrt{2}}/2}(a,b)G^{1-\sqrt{2+\sqrt{2}}/2}(a,b)}{2}$$
$$> \frac{A_{\sqrt{3}/2}^{\sqrt{3}/2}(a,b)G^{1-\sqrt{3}/2} + A_{1/2}^{1/2}(a,b)G^{1/2}(a,b)}{2} > A_{\sqrt{2}/2}^{\sqrt{2}/2}(a,b)G^{1-\sqrt{2}/2}(a,b)$$

hold for all $t>0$ *or all* $a,b>0$ *with* $a\neq b$.

Proof If inequality (3.63) holds for all $t>0$, then we have

$$\lim_{t\to 0^+}\frac{I_0(t)-\frac{\cosh(t\cos\theta)+\cosh(t\sin\theta)}{2}}{t^4} = -\frac{1}{192}\cos(4\theta)\geq 0,$$

which implies that $\theta\in[\pi/8,3\pi/8]$.

Next, we prove the sufficiency of inequality (3.63). Simple computations lead to

$$\frac{\partial[\cosh(t\cos\theta)+\cosh(t\sin\theta)]}{\partial\theta} = \frac{t^2\sin(2\theta)}{2}\left[\frac{\sinh(t\sin\theta)}{t\sin\theta}-\frac{\sinh(t\cos\theta)}{t\cos\theta}\right], \tag{3.65}$$

$$\left(\frac{\sinh x}{x}\right)' = \frac{1}{x}\left(\cosh x-\frac{\sinh x}{x}\right) > 0 \tag{3.66}$$

for $x>0$.

Equation (3.65) and inequality (3.66) imply that the function $\theta\to[\cosh(t\cos\theta)+\cosh(t\sin\theta)]$ is decreasing on $[0,\pi/4]$ and increasing on $[\pi/4,\pi/2]$ for any fixed $t>0$. Hence, it suffices to prove that inequality (3.63) holds for all $t>0$ and $\theta=\theta_0=\pi/8$.

Let

$$\rho_n = \frac{(\frac{2-\sqrt{2}}{4})^n+(\frac{2+\sqrt{2}}{4})^n}{(2n)!}, \qquad \sigma_n = \frac{2}{2^{2n}(n!)^2},$$
$$R_3(t) = \frac{\cosh(t\cos\theta_0)+\cosh(t\sin\theta_0)}{2I_0(t)}. \tag{3.67}$$

Then simple computations lead to

$$R_3(t) = \frac{\sum_{n=0}^{\infty}\rho_n t^{2n}}{\sum_{n=0}^{\infty}\sigma_n t^{2n}}, \tag{3.68}$$

$$\frac{\rho_0}{\sigma_0} = \frac{\rho_1}{\sigma_1} = \frac{\rho_2}{\sigma_2} = \frac{\rho_3}{\sigma_3} = 1, \tag{3.69}$$

$$\frac{\frac{\rho_{n+1}}{\sigma_{n+1}}}{\frac{\rho_n}{\sigma_n}}-1 = -\frac{\sqrt{2}[(n+\sqrt{2}-1)(\sqrt{2}-1)^{n-1}+(n-\sqrt{2}-1)(\sqrt{2}+1)^{n-1}]}{2(2n+1)[(\sqrt{2}-1)^n+(\sqrt{2}+1)^n]} < 0 \tag{3.70}$$

for $n\geq 3$.

It follows from Lemma 2.6 and (3.68)-(3.70) that $R_3(t)$ is strictly decreasing on $(0,\infty)$. Therefore,

$$I_0(t) > \frac{\cosh(t\cos\theta_0)+\cosh(t\sin\theta_0)}{2} \tag{3.71}$$

follows from (3.67) and the monotonicity of $R_3(t)$ together with $R_3(0^+)=\rho_0/\sigma_0=1$.

Let $\theta = \pi/8, \pi/6, \pi/4$. Then inequality (3.64) follows easily from (3.63) and the monotonicity of the function $\theta \to [\cosh(t\cos\theta) + \cosh(t\sin\theta)]$. $\square$

Theorem 3.8 *The inequality*

$$I_0(t) > \frac{\sinh t}{t} + \frac{3(4-\pi)(t\sinh t - 2\cosh t + 2)}{\pi t^2} \tag{3.72}$$

holds for all $t > 0$.

Proof It is easy to verify that

$$\frac{d^2}{dx^2}\left(\frac{1}{\sqrt{1-x^2}}\right) = \frac{1+2x^2}{(1-x^2)^{5/2}} > 0, \qquad \frac{\partial^2 \cosh(tx)}{\partial x^2} = t^2\cosh(tx) > 0$$

for all $t > 0$ and $x \in (0,1)$, which implies that the two functions $1/\sqrt{1-x^2}$ and $\cosh(tx)$ are convex with respect to x on the interval $(0,1)$. Then from Lemma 2.10 and (3.3) we have

$$\begin{aligned}
&\frac{\pi}{2}I_0(t) - \frac{\pi}{2}\frac{\sinh t}{t} \\
&\quad = \int_0^1 \frac{\cosh(tx)}{\sqrt{1-x^2}}\,dx - \int_0^1 \frac{dx}{\sqrt{1-x^2}}\int_0^1 \cosh(tx)\,dx \\
&\quad > 12\int_0^1 \frac{x-\frac{1}{2}}{\sqrt{1-x^2}}\,dx \int_0^1 \left(x - \frac{1}{2}\right)\cosh(tx)\,dx \\
&\quad = \frac{3(4-\pi)(t\sinh t - 2\cosh t + 2)}{2t^2}.
\end{aligned} \tag{3.73}$$

Therefore, inequality (3.72) follows from (3.73). $\square$

Remark 3.3 The inequality $I_0(t) > \sinh(t)/t$ in (3.5) is equivalent to the first inequality $TQ(a,b) > L(a,b)$ in (1.6). Therefore, Theorem 3.8 is an improvement of the first inequality in (1.6).

Let $p \in \mathbb{R}$ and $M(a,b)$ be a bivariate mean of two positive a and b. Then the pth power-type mean $M_p(a,b)$ is defined by

$$M_p(a,b) = M^{1/p}\left(a^p, b^p\right) \quad (p \neq 0), \qquad M_0(a,b) = \sqrt{ab}.$$

We clearly see that

$$M_{\lambda p}(a,b) = M_p^{1/\lambda}\left(a^\lambda, b^\lambda\right)$$

for all $\lambda, p \in \mathbb{R}$ and $a, b > 0$ if M is a bivariate mean.

Theorem 3.9 *The inequality*

$$TQ(a,b) < I_p(a,b)$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $p \geq 3/4$.

Proof The second inequality (1.6) can be rewritten as

$$TQ(a,b) < A_{1/2}(a,b). \tag{3.74}$$

In [30, 31], the authors proved that the inequality

$$I(a,b) > A_{2/3}(a,b) \tag{3.75}$$

holds for all distinct positive real numbers a and b with the best possible constant 2/3.

Inequalities (3.74) and (3.75) lead to

$$TQ(a,b) < A_{1/2}(a,b) = A_{2/3}^{4/3}\left(a^{3/4}, b^{3/4}\right) < I^{4/3}\left(a^{3/4}, b^{3/4}\right) = I_{3/4}(a,b) \tag{3.76}$$

for all $a, b > 0$ with $a \neq b$.

If $p \geq 3/4$, then $TQ(a,b) < I_{3/4}(a,b) \leq I_p(a,b)$ follows from (3.76) and the function $p \to I_p(a,b)$ is strictly increasing [32].

If $TQ(a,b) < I_p(a,b)$ for all $a, b > 0$ with $a \neq b$. Then

$$I_0(t) - e^{t/\tanh(pt)-1/p} < 0 \tag{3.77}$$

for all $t > 0$.

Inequality (3.77) leads to

$$\lim_{t\to 0^+} \frac{I_0(t) - e^{t/\tanh(pt)-1/p}}{t^2} = \frac{1}{3}\left(\frac{3}{4} - p\right) \leq 0,$$

which implies that $p \geq 3/4$. □

Remark 3.4 For all $a, b > 0$ with $a \neq b$, the Toader mean $T(a,b)$ satisfies the double inequality [5, 7]

$$A_{3/2}(a,b) < T(a,b) < A_{\log 2/(\log \pi - \log 2)}(a,b) \tag{3.78}$$

with the best possible constants 3/2 and $\log 2/(\log \pi - \log 2)$, and the one-sided inequality [33]

$$T(a,b) < I_{9/4}(a,b). \tag{3.79}$$

It follows from (3.78) and (3.79) that

$$\begin{aligned} A_{1/2}^{1/3}(a,b) &= A_{3/2}\left(a^{1/3}, b^{1/3}\right) < T\left(a^{1/3}, b^{1/3}\right) \\ &= T_{1/3}^{1/3}(a,b) < I_{9/4}\left(a^{1/3}, b^{1/3}\right) = I_{3/4}^{1/3}(a,b), \end{aligned}$$

which can be rewritten as

$$A_{1/2}(a,b) < T_{1/3}(a,b) < I_{3/4}(a,b). \tag{3.80}$$

Inequalities (3.74) and (3.80) lead to the inequalities

$$TQ(a,b) < A_{1/2}(a,b) < T_{1/3}(a,b) < I_{3/4}(a,b) \tag{3.81}$$

for all $a, b > 0$ with $a \neq b$.

Remark 3.5 For all $a, b > 0$ with $a \neq b$, Theorem 3.4 shows that

$$L^{3/4}(a,b)A^{1/4}(a,b) < TQ(a,b) < \frac{3L(a,b) + A(a,b)}{4}. \tag{3.82}$$

It follows from $L(a,b) < A(a,b)/3 + 2G(a,b)/3$, given by Carlson in [34], and $A(a,b) > L(a,b)$ that

$$L(a,b) < L^{3/4}(a,b)A^{1/4}(a,b), \qquad \frac{A(a,b) + G(a,b)}{2} > \frac{3L(a,b) + A(a,b)}{4}.$$

Therefore, inequality (3.82) is an improvement of the first and second inequalities of (1.6).

Remark 3.6 In [2, 20, 35], the authors proved that the inequalities

$$L(a,b) < \mathrm{AGM}(a,b) < L^{3/4}(a,b)A^{1/4}(a,b) < L_{3/2}(a,b) \tag{3.83}$$

hold for all $a, b > 0$ with $a \neq b$.

Inequalities (3.81)-(3.83) lead to the chain of inequalities

$$\begin{aligned} L(a,b) &< \mathrm{AGM}(a,b) < L^{3/4}(a,b)A^{1/4}(a,b) \\ &< TQ(a,b) < A_{1/2}(a,b) < T_{1/3}(a,b) < I_{3/4}(a,b) \end{aligned} \tag{3.84}$$

for all $a, b > 0$ with $a \neq b$.

Motivated by the first inequality in (3.82) and the third inequality in (3.83), we propose Conjecture 3.1.

Conjecture 3.1 *The inequality*

$$TQ(a,b) > L_{3/2}(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

For all $a, b > 0$ with $a \neq b$, inspired by the double inequality

$$\sqrt{A(a,b)G(a,b)} < TQ(a,b) < \frac{A(a,b) + G(a,b)}{2}$$

given in Corollary 3.1 and the inequalities

$$\sqrt{A(a,b)G(a,b)} < \sqrt{I(a,b)L(a,b)} < \frac{I(a,b) + L(a,b)}{2} < \frac{A(a,b) + G(a,b)}{2}$$

proved by Alzer in [36] we propose Conjecture 3.2.

Conjecture 3.2 *The inequality*

$$TQ(a,b) < \sqrt{I(a,b)L(a,b)}$$

holds for all $a, b > 0$ *with* $a \neq b$.

Remark 3.7 Let W_n be the Wallis ratio defined by (2.1), and c_n, d_n, and s_n be defined by (3.20). Then it follows from Lemma 2.5 and the proof of Theorem 3.3 that the sequence $\{s_n\}_{n=1}^{\infty}$ is strictly decreasing and $\lim_{n\to\infty} s_n = 2/\pi$, and the sequence $\{c_n/d_n\}_{n=1}^{\infty}$ is strictly increasing for $n = 1, 2$ and strictly decreasing for $n \geq 2$. Hence, we have

$$\frac{2}{\pi} < s_n = (2n+1)W_n^2 \leq s_1 = \frac{3}{4} \tag{3.85}$$

and

$$\frac{2}{\pi} = \min\left\{\frac{c_1}{d_1}, \lim_{n\to\infty}\frac{c_n}{d_n}\right\} < \frac{c_n}{d_n} = \frac{2^{2n}s_n - 1}{2^{2n} - 1} \leq \frac{c_2}{d_2} = \frac{41}{60} \tag{3.86}$$

for all $n \in \mathbb{N}$.

Inequalities (3.85) and (3.86) lead to the Wallis ratio inequalities

$$\frac{1}{\sqrt{\pi(n+\frac{1}{2})}} < W_n \leq \frac{\sqrt{6}}{4\sqrt{n+\frac{1}{2}}}$$

and

$$\sqrt{\frac{2^{-2n}(\pi-2)+2}{\pi(2n+1)}} < W_n \leq \sqrt{\frac{41+19\times 2^{-2n}}{60(2n+1)}}$$

for all $n \in \mathbb{N}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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