Yu et al. *Journal of Inequalities and Applications* (2016) 2016:20
DOI 10.1186/s13660-015-0957-0

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

An alternative approach to partial regularity of quasilinear elliptic systems with VMO coefficients

Haiyan Yu[1,2], Shenzhou Zheng[1*] and Yuxia Tong[1,3]

*Correspondence:
shzhzheng@bjtu.edu.cn
[1]Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, China
Full list of author information is available at the end of the article

Abstract

In this paper, we provide an alternative approach to partially Hölder continuity of some quasilinear elliptic systems with discontinuous coefficients under natural growth. Here, we do it by way of the modified A-harmonic approximation and Caccippoli's inequality.

MSC: Primary 35J60; 35B65; secondary 35D30

Keywords: VMO coefficients; natural growth; A-harmonic approximation; partial regularity

1 Introduction

The aim of this paper is to consider the following quasilinear elliptic systems with VMO discontinuous coefficients under natural growth:

$$-\sum_{\alpha,\beta=1}^{n}\sum_{j=1}^{N} D_\alpha\big(A_{ij}^{\alpha\beta}(x,u)D_\beta u^j\big) = B_i(x,u,Du), \quad \text{a.e. } x\in\Omega, i=1,\ldots,N, \tag{1.1}$$

with Ω a bounded domain of $\mathbb{R}^n$ for $n\geq 2$. Here, we assume that $B(x,u,Du) = (B_i(x,u,Du))$ takes a value in $\mathbb{R}^N$ with $N\geq 1$ and $A(x,u) = (A_{ij}^{\alpha\beta}(x,u))$ takes a value in $\mathbb{R}^{n^2N^2}$. In the context, we adopt Einstein's convention by summing over repeated indices with $\alpha,\beta = 1,2,\ldots,n$ and $i,j=1,2,\ldots,N$, then one can briefly rewrite (1.1) by

$$-\operatorname{div}\big(A(x,u)Du\big) = B(x,u,Du), \quad \text{a.e. } x\in\Omega. \tag{1.2}$$

Consequently, we rewrite a vectorial-valued function $u\in W_{loc}^{1,2}(\Omega,\mathbb{R}^N)$ to be a weak solution of the systems (1.1) in the sense of distribution as follows.

$$\int_\Omega A(x,u)Du\cdot D\varphi\,dx = \int_\Omega B(x,u,Du)\varphi\,dx, \quad \forall\varphi\in C_0^\infty\big(\Omega,\mathbb{R}^N\big). \tag{1.3}$$

As we know, it is very necessary that some structural and regular assumptions are imposed on the tensorial-valued operator $A(x,u)$ and the inhomogeneity $B(x,u,Du)$. Here, we suppose that the uniformly elliptic operator $A(x,u)$ satisfies a VMO-condition in a.e.

$x \in \Omega$ uniformly with respect to $u \in \mathbb{R}^N$ and is continuous in u uniformly with respect to $x \in \Omega$, and $B(x,u,Du)$ satisfies the natural growth. More precisely, let us first recall some related notations.

Definition 1.1 *A locally integrable function u is said to belong to* BMO(Ω) (*bounded mean oscillation in* Ω), *if $u \in L^1_{loc}(\Omega)$ and for any $0 < s < \infty$, we have*

$$M_s(u,\Omega) = \sup_{x\in\Omega, 0<\rho<s} \left|\Omega(x,\rho)\right|^{-1} \int_{\Omega(x,\rho)} \left|u(y) - u_{x,\rho}\right| dy < +\infty,$$

where $\Omega(x,\rho) = \Omega \cap B(x,\rho)$ with any open ball $B(x,\rho)$ in $\mathbb{R}^n$ centered at x of radius ρ, and $u_{x,\rho} := \fint_{\Omega(x,\rho)} u(y)\,dy = \frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} u(y)\,dy$.

Definition 1.2 *A function $u \in L^1_{loc}(\Omega)$ is said to be in* VMO(Ω) (*vanishing mean oscillation in* Ω), *if*

$$M_0(u) = \lim_{s\to 0} M_s(u,\Omega) = 0.$$

Definition 1.3 *Let $p \geq 1$ and $\lambda \geq 0$. The Morrey space is defined by*

$$L^{p,\lambda}(\Omega) = \left\{ u \in L^p(\Omega) : \sup_{\substack{x\in\Omega \\ 0<\rho\leq R_0}} \left(\rho^{-\lambda} \int_{\Omega(x,\rho)} |u|^p\,dy\right)^{\frac{1}{p}} < +\infty \right\}$$

with its norm

$$\|u\|_{L^{p,\lambda}(\Omega)} = \sup_{\substack{x\in\Omega \\ 0<\rho\leq R_0}} \left(\rho^{-\lambda} \int_{\Omega(x,\rho)} |u|^p\,dy\right)^{\frac{1}{p}},$$

where R_0 = diam(Ω).

We are now in a position to state the assumptions which are imposed on $A(x,u)$ and $B(x,u,Du)$ as follows:

- H1 (Uniform ellipticity) There exist two constants $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq A^{\alpha\beta}_{ij}(x,u)\xi^i_\alpha \xi^j_\beta \leq \Lambda|\xi|^2, \quad \forall x \in \Omega, u \in \mathbb{R}^N, \xi \in \mathbb{R}^{nN}. \tag{1.4}$$

- H2 (Minimal regularity on $A(x,u)$) Suppose that $A(\cdot,u)$ is VMO in x with uniformly respect to $u \in \mathbb{R}^N$ and is continuous in u with uniformly respect to $x \in \Omega$. That is, $\lim_{s\to 0} M_s(A(\cdot,u_0)) = 0$, and there exist a constant C and a continuous concave function[a] $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) = 0$, $0 \leq \omega \leq 1$, such that

$$\left|A^{\alpha\beta}_{ij}(x,u) - A^{\alpha\beta}_{ij}(x,v)\right| \leq C\omega\left(|u-v|^2\right), \quad \forall u,v \in \mathbb{R}^N, x \in \Omega. \tag{1.5}$$

- H3 (Natural growth) The lower order term $B(x,u,Du)$ satisfies the following natural growth: for $u \in W^{1,2} \cap L^\infty(\Omega,\mathbb{R}^N)$ with $M = \|u\|_{L^\infty(\Omega)}$, we have

$$\left|B_i(x,u,Du)\right| \leq \mu(M)\left(|Du|^2 + f^i(x)\right), \quad i = 1,2,\ldots,N, \tag{1.6}$$

with

$$2\mu(M)M < \lambda, \quad \text{and} \quad f^i \in L^q(\Omega) \quad \text{with } q > \frac{n}{2}.$$

Before stating the main conclusion, let us first briefly review some recent studies involving related problems. Note that the discontinuous coefficient is not so crucial for Hölder continuity of the weak solutions to the scalar partial differential equations, which is due to the famous De Giorgi-Moser-Nash iterative technique; see [1]. However, in the case of $N > 1$ one cannot in general expect that the weak solutions will be classical C^α solutions with some $0 < \alpha \leq 1$, which was first shown by a counterexample from De Giorgi's work [2], also see Giaquinta's monograph [3]. For the systems (1.1), Giaquinta and Modica [3, 4] first studied partial regularity of their weak solutions in the Morrey space and in the Campanato space [3, 5] when each entry of the leading coefficients $A(x,u)$ is assumed to be continuous.

However, it is an important observation that many stochastic processes with discontinuous coefficients reappeared in connected with the diffusion approximation [6], which reminds us of the significance to treat particular cases of discontinuity. Sarason [7] in 1975 introduced the function spaces of the vanishing mean oscillations (briefly called VMO), which not only contains discontinuous functions but also owns a good property similar to the class of continuous functions that is not shared by general bounded measurable functions. In recent years, Calderón-Zygmund's theory of linear and nonlinear PDEs with VMO coefficients was immensely developed, which naturally originated from the singular integral operators and the estimates of commutators with a VMO function [8, 9]. In the meantime, Morrey's regularity of the weak solutions of elliptic and parabolic PDEs with the discontinuous leading coefficients was also investigated in a similar approach by Di Fazio and Ragusa [10] and Fan *et al.* [11]. Among them there are some main different arguments to deal with elliptic and parabolic PDEs with VMO leading coefficients, for example a few celebrated approaches of Chiarenza *et al.* [8], Syun and Wang [12] and Krylov *et al.* [13, 14].

For the elliptic systems (1.1), Zheng [15] and Zheng and Feng [16] derived an optimal partial regularity in Morrey spaces by using a reverse Hölder inequality and perturbation argument when $A(x,u)$ is VMO in $x \in \Omega$ and continuous in $u \in \mathbb{R}^N$ with controllable growth and natural growth, respectively. On the other hand, Chen and Tan [17] got an interior partial regularity for the nonlinear elliptic systems with controllable growth by the A-harmonic approximation, but their principal coefficients $A(x,u,Du)$ are essentially Hölder continuous in (x,u). Inspired by those achievements, in this paper we will provide an alternative approach to get Hölder continuity with an optimal Hölder exponent for the quasilinear elliptic systems (1.1) with VMO coefficients under the natural growth (*cf.* [15]). Here, we simply employ a modification of the A-harmonic approximation argument based on Duzaar and Grotowski's technique, which can avoid the use of the reverse Hölder inequality and perturbation approach. As pointed out by Duzaar and Grotowski [18], the reformulation of the A-harmonic approximation could simplify the proofs of the regularity for PDEs, we here apply the A-harmonic approximation approach to prove the regularity of our previous problem from [15]. More precisely, we have

Theorem 1.4 *Suppose that $u \in W^{1,2}_{loc} \cap L^\infty(\Omega, \mathbb{R}^N)$ is a locally weak solution of the systems* (1.1), *and $A(x,u)$, $B(x,u,Du)$ satisfy the basic assumptions* (H1)-(H3). *Then there exists an* *open subset $\Omega_0 \subset \Omega$ with $\dim_H(\Omega \setminus \Omega_0) \le n-2$ such that $u \in C^{0,\alpha}_{loc}(\Omega_0, \mathbb{R}^N)$ with $\alpha = 1 - \frac{n}{2q}$, where*

$$\Omega_0 = \left\{ x_0 \in \Omega : \limsup_{\rho \to 0} \left(\fint_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 \, dx \right) = 0 \right\},$$

and $\dim_H$ is the Hausdorff dimension. Moreover, we have $Du \in L^{2,\gamma}(\Omega_0, \mathbb{R}^{nN})$ with $\gamma = n - \frac{n}{q}$.

We close the introduction with briefly describing the strategy of the proof. At first, we establish the so-called second Caccippoli's inequality, then we give an estimate to a certain energy functional which measures the oscillations of the solution u except a small excess quantity by the modified A-harmonic approximation. On the basis of the iteration lemma we get the boundedness of the functional, which leads to the desired regularity in points while the excess quantity is small. We believe that this new approach should also be particularly helpful to understand some elliptic and parabolic systems including degenerate settings.

This paper is organized as follows. In Section 2, we recall some notations and basic facts, and we give the so-called Caccioppoli's inequality. In Section 3 we are devoted to a proof of the main conclusion.

2 Preliminaries

Throughout the paper we adopt the usual convention of denoting by C a general constant, which may vary from expression to expression in the same chain of inequalities. Let us first establish the so-called second Caccioppoli's inequality. As we know, Caccioppoli's inequality is always a standard preliminary tool to obtain the partial regularity to elliptic and parabolic PDEs; see [3].

Lemma 2.1 *Let $u \in W^{1,2}_{loc} \cap L^\infty(\Omega, \mathbb{R}^N)$ be any weak solution of the systems* (1.1) *under the assumption* (H1)-(H3) *satisfying $\|u\|_{L^\infty(\Omega)} = M < \infty$. If $2\mu(M)M < \lambda$, then for any $B_\rho(x_0) \subset \Omega$, we have*

$$\int_{B_{\frac{\rho}{2}}(x_0)} |Du|^2 \, dx \le \frac{C_1}{\rho^2} \int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 \, dx + C_2 \rho^{n+2-\frac{2n}{q}} \|f\|^2_{L^q}, \tag{2.1}$$

where $u_{x,\rho} := \fint_{B_\rho(x_0)} u(y)\, dy$ is an integral average over $B_\rho(x_0)$.

Proof For any $x_0 \in \Omega$ and $0 < \rho < \operatorname{dist}(x_0, \partial\Omega)$, denoting $B_\rho := B_\rho(x_0)$, we take $\eta \in C_0^\infty(B_\rho(x_0))$ as a cut-off function with $0 \le \eta \le 1$, $|D\eta| \le \frac{4}{\rho}$ and $\eta \equiv 1$ on $B_{\frac{\rho}{2}}(x_0)$. As usual, we choose the test function $\varphi = \eta^2(u - u_{x_0,\rho})$. Note that $D\varphi = 2\eta D\eta(u - u_{x_0,\rho}) + \eta^2 Du$, and we substitute it in (1.3) finding

$$\begin{aligned} &\int_{B_\rho} A(x,u)Du \cdot \left[2\eta D\eta(u - u_{x_0,\rho}) + \eta^2 Du\right] dx \\ &\quad = \int_{B_\rho} B(x,u,Du)\eta^2(u - u_{x_0,\rho})\, dx, \end{aligned}$$

which implies

$$\begin{aligned}
&\int_{B_\rho} \eta^2 A(x,u)Du\cdot Du\,dx \\
&\quad = -2\int_{B_\rho} A(x,u)Du\cdot\bigl(\eta(u-u_{x_0,\rho})D\eta\bigr) + \int_{B_\rho} B(x,u,Du)\eta^2(u-u_{x_0,\rho})\,dx.
\end{aligned}$$

By the ellipticity (H1) and the natural growth (H3) we get

$$\begin{aligned}
&\lambda\int_{B_\rho} |\eta Du|^2\,dx \\
&\quad\le 2\Lambda\int_{B_\rho} |\eta Du|\cdot\bigl|(u-u_{x_0,\rho})D\eta\bigr|\,dx + \mu(M)\int_{B_\rho}\eta^2\bigl(|Du|^2+\bigl|f(x)\bigr|\bigr)|u-u_{x_0,\rho}|\,dx \\
&\quad\le \mu(M)\int_{B_\rho}|u-u_{x_0,\rho}|\cdot|\eta Du|^2\,dx + \mu(M)\int_{B_\rho}|u-u_{x_0,\rho}|\cdot\bigl|f(x)\bigr|\,dx \\
&\qquad + 2\Lambda\biggl(\int_{B_\rho}|\eta Du|^2\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{B_\rho}|D\eta|^2|u-u_{x_0,\rho}|^2\,dx\biggr)^{\frac{1}{2}},
\end{aligned}\tag{2.2}$$

where the last term of the right-hand side is obtained by the Hölder inequality. Thanks to the Hölder inequality, the Young inequality with $\varepsilon > 0$, and the boundedness of u it follows that

$$\begin{aligned}
\lambda\int_{B_\rho}|\eta Du|^2\,dx \le{}& 2\mu(M)M\int_{B_\rho}|\eta Du|^2\,dx \\
&+ \mu(M)\biggl(\int_{B_\rho}\biggl|\frac{u-u_{x_0,\rho}}{\rho}\biggr|^2\,dx\biggr)^{\frac{1}{2}}\biggl(\int_{B_\rho}\bigl|\rho f(x)\bigr|^2\,dx\biggr)^{\frac{1}{2}} \\
&+ 2\Lambda\varepsilon\int_{B_\rho}|\eta Du|^2\,dx + C(\varepsilon)\cdot\frac{1}{\rho^2}\int_{B_\rho}|u-u_{x_0,\rho}|^2\,dx,
\end{aligned}$$

that is,

$$\begin{aligned}
\lambda\int_{B_\rho}|\eta Du|^2\,dx \le{}& \bigl(2\mu(M)M+2\Lambda\varepsilon\bigr)\int_{B_\rho}|\eta Du|^2\,dx + \frac{C}{\rho^2}\int_{B_\rho}|u-u_{x_0,\rho}|^2\,dx \\
&+ C\rho^2\int_{B_\rho}\bigl|f(x)\bigr|^2\,dx.
\end{aligned}$$

If we choose ε small enough such that $2\mu(M)M + 2\Lambda\varepsilon < \lambda$, it follows from the Hölder inequality that

$$\int_{B_{\frac{\rho}{2}}}|Du|^2\,dx \le \frac{C_1}{\rho^2}\int_{B_\rho(x_0)}|u-u_{x_0,\rho}|^2\,dx + C_2\rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2,$$

which yields the desired result. □

We will make use of a modification of the so-called A-harmonic approximation lemma [19, 20]. Let us first recall the so-called locally A-harmonic.

Definition 2.2 (A-harmonic) *Let* $A \in \operatorname{Bil}(B_R(x_0) \times \mathbb{R}^N, \mathbb{R}^{n^2 \times N^2})$ *be a bilinear form with constant coefficients, which satisfies the assumptions of* (1.4). *We call a map* $h \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ *A-harmonic in* $B_R(x_0)$ *if it satisfies*

$$\int_{B_R(x_0)} A(Dh, D\varphi)\,dx = 0, \quad \forall \varphi \in C_0^1\big(B_R(x_0), \mathbb{R}^N\big).$$

We should emphasize that $A \in \operatorname{Bil}(B_R(x_0) \times \mathbb{R}^N, \mathbb{R}^{n^2 \times N^2})$ is a bilinear form with constant tensorial coefficients, therefore it is well known that for any A-harmonic h we have the following inequality, which is similar to harmonic functions (*cf.* [1, 3]).

Lemma 2.3 *Let* $h(x) \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ *be a weak solution of the following system*:

$$D_\alpha\big(A_{ij}^{\alpha\beta} D_\beta h_j\big) = 0, \quad i = 1, \ldots, N,$$

where $A_{ij}^{\alpha\beta}$ *is a constant tensor. Then there exists a constant* $C = C(n, \lambda, \Lambda)$ *such that, for any* $0 < \rho < R \leq \operatorname{dist}(x_0, \partial\Omega)$ *with* $x_0 \in \Omega$, *we have*

$$\int_{B_\rho(x_0)} \big|Dh - (Dh)_{x_0,\rho}\big|^2\,dx \leq C\left(\frac{\rho}{R}\right)^{n+2} \int_{B_R(x_0)} \big|Dh - (Dh)_{x_0,R}\big|^2\,dx. \tag{2.3}$$

Next, we recall the notation of the A-harmonic approximation introduced by Duzaar and Grotowski [18, 21]. In the sequel, suppose that there exist two constants $0 < \lambda \leq \Lambda < \infty$ such that the bilinear form $A \in \operatorname{Bil}(B_R(x_0) \times \mathbb{R}^N, \mathbb{R}^{n^2 \times N^2})$ satisfies

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \xi_\beta^j \geq \lambda|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^{nN}, \tag{2.4}$$

$$A_{ij}^{\alpha\beta}(x,u)\xi_\alpha^i \bar{\xi}_\beta^j \leq \Lambda|\xi||\bar{\xi}|, \quad \text{for all } \xi, \bar{\xi} \in \mathbb{R}^{nN}. \tag{2.5}$$

In the main proof, as a main approach we will employ the following modification of the A-harmonic approximation [19, 20].

Lemma 2.4 *Let* $0 < \lambda \leq \Lambda < \infty$ *and* $n \geq 2$. *Then, for any given* $\varepsilon > 0$ *there exists a positive constant* $k = k(n, N, \lambda, \Lambda, \varepsilon) > 0$ *with the following property*: *for any* $A \in \operatorname{Bil}(B_R(x_0) \times \mathbb{R}^N, \mathbb{R}^{n^2 \times N^2})$ *satisfying* (2.4), (2.5) *and any* $u \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$, *there exists an A-harmonic function* $h \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ *such that*

$$\int_{B_R(x_0)} |Dh|^2\,dx \leq \int_{B_R(x_0)} |Du|^2\,dx; \tag{2.6}$$

moreover, there exists $\varphi \in C_0^\infty(B_R(x_0), \mathbb{R}^N)$ *with*

$$\|D\varphi\|_{L^\infty(B_R(x_0), \mathbb{R}^N)} \leq \frac{1}{R}, \tag{2.7}$$

such that

$$\int_{B_R(x_0)} |u - h|^2\,dx \leq \varepsilon R^2 \int_{B_R(x_0)} |Du|^2\,dx + k(\varepsilon)\left[R^{4-n}\left(\int_{B_R(x_0)} ADu \cdot D\varphi\,dx\right)^2\right]. \tag{2.8}$$

To give a Hölder estimate to the solution by using Morrey's lemma we employ the following iterating lemma; see [3].

Lemma 2.5 *Let $\Phi(\rho)$ be a non-negative and non-decreasing function on $(0,R)$. Suppose that*

$$\Phi(\rho) \le A\left[\left(\frac{\rho}{R}\right)^{\alpha} + \epsilon\right]\Phi(R) + BR^{\beta}, \quad \forall 0 < \rho < R < R_0 = \operatorname{dist}(x_0, \partial\Omega),$$

with non-negative constants A, B, α and β, and $\alpha > \beta$. Then there exist two constants $\epsilon_0 = \epsilon_0(A,\alpha,\beta)$ and $C = C(A,\alpha,\beta)$ such that for any $0 < \epsilon < \epsilon_0$ we have

$$\Phi(\rho) \le C\left[\left(\frac{\rho}{R}\right)^{\beta}\Phi(R) + B\rho^{\beta}\right],$$

for any $0 < \rho \le R \le R_0 = \operatorname{dist}(x_0, \partial\Omega)$.

In the process of proving our main results, we need to describe the decay of integral on the ball with respect to the radius σ. Let us recall the concept of Campanato space; see [22].

Definition 2.6 *Let $p \ge 1$ and $\lambda \ge 0$. The Campanato space is defined as*

$$\mathcal{L}^{p,\lambda}(\Omega) = \left\{u \in L^p(\Omega), [u]^p_{p,\lambda;\Omega} = \sup_{\substack{x\in\Omega \\ 0<\rho\le R_0}} \rho^{-\lambda}\int_{\Omega(x,\rho)} |u - u_{x,\rho}|^p\,dy < +\infty\right\}$$

with its norm

$$\|u\|_{\mathcal{L}^{p,\lambda}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{p,\lambda;\Omega},$$

where $u_{x,\rho}$ is the average of $u(x)$ over $B_\rho(x)$.

Lemma 2.7 (Theorem 3.1 in [22]) *For $n < \lambda \le n+2$, we have*

$$\mathcal{L}^{2,\lambda}_{loc}(\Omega) \cong C^{0,\alpha}_{loc}(\Omega)$$

with $\alpha = \frac{\lambda-n}{2}$.

Finally, we need the following estimate lemma of the Hausdorff dimensional measure while estimating the singular set to the weak solution.

Lemma 2.8 *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in L^1_{loc}(\Omega, \mathbb{R}^N)$. Then for $0 \le s < n$ and setting*

$$E_s := \left\{x \in \Omega : \liminf_{\rho\to 0} \rho^{-s}\int_{B_\rho(x)} |u|\,dy > 0\right\}, \tag{2.9}$$

we have the following estimate:

$$H^s(E_s) = 0,$$

see page 77 *in Chapter* 2 *of* [23].

3 Proof of main result

In the section, we prove our main result on the basis of the modification of the A-harmonic approximation argument.

Proof of Theorem 1.4 Given $x_0 \in \Omega$ and $R := \operatorname{dist}(x_0, \partial\Omega)$, for fixed $\rho : 0 < \rho < R$ let $\bar{A} = (A(\cdot, u_{x_0,\rho}))_{x_0,\frac{\rho}{2}}$ be defined by

$$\bar{A} := \big(A(x, u_{x_0,\rho})\big)_{x_0,\frac{\rho}{2}} = \fint_{B_{\frac{\rho}{2}}(x_0)} A(x, u_{x_0,\rho})\,dx.$$

For simplicity, we denote $B_\rho(x_0) = B_\rho$, $u_{x_0,\rho} = u_\rho$ in the following. For any $\varphi \in C_0^\infty(B_{\frac{\rho}{2}}, \mathbb{R}^N)$, we may write

$$\begin{aligned}\int_{B_{\frac{\rho}{2}}} \bar{A}Du \cdot D\varphi\,dx &= \int_{B_{\frac{\rho}{2}}} \big(\bar{A} - A(\cdot, u_\rho)\big)Du \cdot D\varphi\,dx + \int_{B_{\frac{\rho}{2}}} \big(A(\cdot, u_\rho) - A(x,u)\big)Du \cdot D\varphi\,dx \\ &\quad + \int_{B_{\frac{\rho}{2}}} A(x,u)Du \cdot D\varphi\,dx. \end{aligned} \tag{3.1}$$

Note that u is a weak solution of the systems (1.1) with test function $\varphi \in C_0^\infty(B_{\frac{\rho}{2}}, \mathbb{R}^N)$,

$$\begin{aligned}&\left|\int_{B_{\frac{\rho}{2}}} \bar{A}Du \cdot D\varphi\,dx\right| \\ &\quad\le \int_{B_{\frac{\rho}{2}}} \big|\bar{A} - A(\cdot, u_\rho)\big| \cdot |Du| \cdot |D\varphi|\,dx \\ &\qquad + \int_{B_{\frac{\rho}{2}}} \big|A(\cdot, u_\rho) - A(x,u)\big| \cdot |Du| \cdot |D\varphi|\,dx + \int_{B_{\frac{\rho}{2}}} \big|B(x,u,Du)\big| \cdot |\varphi|\,dx.\end{aligned}$$

Now we take the test function φ with $\sup_{B_{\frac{\rho}{2}}} |D\varphi| \le \frac{2}{\rho}$ and $\sup_{B_{\frac{\rho}{2}}} |\varphi| \le 2$. Based on natural growth (H3) and Hölder inequality, then we obtain

$$\begin{aligned}&\left|\int_{B_{\frac{\rho}{2}}} \bar{A}Du \cdot D\varphi\,dx\right| \\ &\quad\le \frac{2}{\rho}\int_{B_{\frac{\rho}{2}}} \big|\bar{A} - A(\cdot, u_\rho)\big| \cdot |Du|\,dx \\ &\qquad + \frac{2}{\rho}\int_{B_{\frac{\rho}{2}}} \big|A(\cdot, u_\rho) - A(x,u)\big| \cdot |Du|\,dx + 2\mu(M)\int_{B_{\frac{\rho}{2}}} |Du|^2 + |f|\,dx \\ &\quad\le \frac{2}{\rho}\left(\int_{B_{\frac{\rho}{2}}} \big|\bar{A} - A(\cdot, u_\rho)\big|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_{\frac{\rho}{2}}} |Du|^2\,dx\right)^{\frac{1}{2}} \\ &\qquad + \frac{2}{\rho}\left(\int_{B_{\frac{\rho}{2}}} \big|A(\cdot, u_\rho) - A(x,u)\big|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_{\frac{\rho}{2}}} |Du|^2\,dx\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&+ 2\mu(M)\int_{B_{\frac{\rho}{2}}} |Du|^2\,dx + 2\mu(M)\rho^{n(1-\frac{1}{q})}\|f\|_{L^q} \\
&:= I + II + III. \qquad (3.2)
\end{aligned}
$$

In the following, we estimate the items I, II, and III, respectively. For the estimate of I, thanks to the VMO regular assumption (H2) on $A(x,u)$ and the Caccioppoli inequality of Lemma 2.1, if we take $m = u_\rho$, then one finds

$$
\begin{aligned}
I &= \frac{2}{\rho}\left(\int_{B_{\frac{\rho}{2}}} \left|\bar{A} - A(\cdot,u_\rho)\right|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_{\frac{\rho}{2}}} |Du|^2\,dx\right)^{\frac{1}{2}} \\
&\le \frac{C}{\rho}\left(\rho^n M_\rho\big(A(\cdot,u_\rho)\big)\right)^{\frac{1}{2}}\left(\frac{C_1}{\rho^2}\int_{B_\rho}|u-u_\rho|^2\,dx + C_2\rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2\right)^{\frac{1}{2}}. \qquad (3.3)
\end{aligned}
$$

To estimate II, by using the continuity hypothesis (H2) in u and the Caccioppoli inequality in Lemma 2.1,

$$
\begin{aligned}
II &= \frac{2}{\rho}\left(\int_{B_{\frac{\rho}{2}}} \left|A(\cdot,u_\rho) - A(x,u)\right|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_{\frac{\rho}{2}}} |Du|^2\,dx\right)^{\frac{1}{2}} \\
&\le \frac{C}{\rho}\left(\rho^n \fint_{B_\rho}\omega\big(|u-u_\rho|^2\big)\,dx\right)^{\frac{1}{2}}\left(\frac{C_1}{\rho^2}\int_{B_\rho}|u-u_\rho|^2\,dx + C_2\rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2\right)^{\frac{1}{2}} \\
&\le \frac{C}{\rho}\left(\rho^n\omega\left(\fint_{B_\rho}|u-u_\rho|^2\,dx\right)\right)^{\frac{1}{2}}\left(\frac{C_1}{\rho^2}\int_{B_\rho}|u-u_\rho|^2\,dx + C_2\rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2\right)^{\frac{1}{2}}, \qquad (3.4)
\end{aligned}
$$

where we use the concavity of the function $\omega(\cdot)$ in the last inequality.

For the estimate of III, by the Caccioppoli inequality as above we deduce

$$
\begin{aligned}
III &= 2\mu(M)\int_{B_{\frac{\rho}{2}}} |Du|^2\,dx + 2\mu(M)\|f\|_{L^q}\big(\alpha_n\rho^n\big)^{1-\frac{1}{q}} \\
&\le \frac{C}{\rho^2}\int_{B_\rho}|u-u_\rho|^2\,dx + C_2\rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2 + C_2\rho^{n(1-\frac{1}{q})}\|f\|_{L^q}. \qquad (3.5)
\end{aligned}
$$

Now we are in a position to substitute (3.3), (3.4), and (3.5) into (3.2), then it follows that

$$
\begin{aligned}
&\left|\int_{B_{\frac{\rho}{2}}} \bar{A}Du\cdot D\varphi\,dx\right| \\
&\le C\left(\rho^{n-2}M_\rho\big(A(\cdot,u_\rho)\big)\right)^{\frac{1}{2}}\left(\rho^{n-2}\fint_{B_\rho}|u-u_\rho|^2\,dx + \rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2\right)^{\frac{1}{2}} \\
&\quad + C\left(\rho^{n-2}\omega\left(\fint_{B_\rho}|u-u_\rho|^2\,dx\right)\right)^{\frac{1}{2}}\left(\rho^{n-2}\fint_{B_\rho}|u-u_\rho|^2\,dx + \rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2\right)^{\frac{1}{2}} \\
&\quad + C\left(\rho^{n-2}\fint_{B_\rho}|u-u_\rho|^2\,dx + \rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2\right) + C\rho^{n(1-\frac{1}{q})}\|f\|_{L^q} \\
&\le C\rho^{n-2}\left(M_\rho^{\frac{1}{2}}\big(A(\cdot,u_\rho)\big) + \omega^{\frac{1}{2}}\left(\fint_{B_\rho}|u-u_\rho|^2\,dx\right)\right)
\end{aligned}
$$

$$\times \left(\fint_{B_\rho} |u-u_\rho|^2\,dx + \rho^{4-\frac{2n}{q}} \|f\|_{L^q}^2 \right)^{\frac{1}{2}}$$
$$+ C\rho^{n-2} \left(\fint_{B_\rho} |u-u_\rho|^2\,dx + \rho^{4-\frac{2n}{q}} \|f\|_{L^q}^2 + \rho^{2-\frac{n}{q}} \|f\|_{L^q} \right). \tag{3.6}$$

Note that $2-\frac{n}{q} > 0$ due to the assumptions $q > \frac{n}{2}$, then we can take $\rho > 0$ such that $\rho^{2-\frac{n}{q}} \|f\|_{L^q} \le 1$, and we rewrite (3.6):

$$\begin{aligned} &\left| \int_{B_{\frac{\rho}{2}}} \bar{A} Du \cdot D\varphi\,dx \right| \\ &\quad \le C\rho^{n-2} \left(M_\rho^{\frac{1}{2}}\big(A(\cdot,u_\rho)\big) + \omega^{\frac{1}{2}} \left(\fint_{B_\rho} |u-u_\rho|^2\,dx \right) \right) \\ &\qquad \times \left(\fint_{B_\rho} |u-u_\rho|^2\,dx + \rho^{2-\frac{n}{q}} \|f\|_{L^q} \right)^{\frac{1}{2}} \\ &\qquad + C\rho^{n-2} \left(\fint_{B_\rho} |u-u_\rho|^2\,dx + \rho^{2-\frac{n}{q}} \|f\|_{L^q} \right) \\ &\quad \le C\rho^{n-2} \sqrt{J} \big(M_\rho^{\frac{1}{2}}\big(A(\cdot,u_{x_0,\rho})\big) + \omega^{\frac{1}{2}}(J) + \sqrt{J} \big), \end{aligned} \tag{3.7}$$

where $J := \fint_{B_\rho(x_0)} |u-u_\rho|^2\,dx + \rho^{2-\frac{n}{q}} \|f\|_{L^q}$.

According to Lemma 2.4 on the modification of the A-harmonic approximation, Caccioppoli's inequality (2.1) and the estimate (3.7) above, for a given $\varepsilon > 0$ there exists an A-harmonic $h \in W^{1,2}(B_{\frac{\rho}{2}}, \mathbb{R}^N)$ such that

$$\begin{aligned} &\int_{B_{\frac{\rho}{2}}} |u-h|^2\,dx \\ &\quad \le \varepsilon\rho^2 \int_{B_{\frac{\rho}{2}}} |Du|^2\,dx + k(\varepsilon) \left[\rho^{4-n} \left(\int_{B_{\frac{\rho}{2}}} \bar{A} Du \cdot D\varphi\,dx \right)^2 \right] \\ &\quad \le \varepsilon\rho^2 \left(\frac{C_1}{\rho^2} \int_{B_\rho} |u-u_\rho|^2\,dx + C\rho^{n+2-\frac{2n}{q}} \|f\|_{L^q}^2 \right) \\ &\qquad + Ck(\varepsilon)\rho^{4-n} \big[\rho^{n-2} \sqrt{J} \big(M_\rho^{\frac{1}{2}}\big(A(\cdot,u_\rho)\big) + \omega^{\frac{1}{2}}(J) + \sqrt{J} \big) \big]^2 \\ &\quad \le \varepsilon C\rho^n \fint_{B_\rho} |u-u_\rho|^2\,dx + \varepsilon C\rho^{n+4-\frac{2n}{q}} \|f\|_{L^q}^2 \\ &\qquad + C\rho^n J \big(M_\rho\big(A(\cdot,u_\rho)\big) + \omega(J) + J \big) \\ &\quad \le \varepsilon C\rho^n J + C\rho^n J \big(M_\rho\big(A(\cdot,u_\rho)\big) + \omega(J) + J \big). \end{aligned} \tag{3.8}$$

In order to estimate the integral $\int_{B_\sigma(x_0)} |u-u_{x_0,\sigma}|^2\,dx$ for any $0 < \sigma < \frac{\rho}{2}$, by the A-harmonic property in Lemma 2.3 we have

$$\begin{aligned} \int_{B_\sigma} |u-u_\sigma|^2\,dx &\le \int_{B_\sigma} |u-h_\sigma|^2\,dx \le 2\int_{B_\sigma} |u-h|^2\,dx + 2\int_{B_\sigma} |h-h_\sigma|^2\,dx \\ &\le 2\int_{B_{\frac{\rho}{2}}} |u-h|^2\,dx + C\left(\frac{\sigma}{\rho}\right)^{n+2} \int_{B_{\frac{\rho}{2}}} |h-h_{\frac{\rho}{2}}|^2\,dx. \end{aligned} \tag{3.9}$$

To estimate the last item in the right-hand side above, in terms of the Poincaré inequality, the inequality (2.6), and Caccioppoli's inequality it follows that

$$\begin{aligned}\int_{B_{\frac{\rho}{2}}} |h-h_{\frac{\rho}{2}}|^2\,dx &\le C\rho^2\int_{B_{\frac{\rho}{2}}}|Dh|^2\,dx\\ &\le C\rho^2\int_{B_{\frac{\rho}{2}}}|Du|^2\,dx\\ &\le C\rho^2\biggl(\frac{1}{\rho^2}\int_{B_\rho}|u-u_\rho|^2\,dx+\rho^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2\biggr)\\ &\le C\rho^n\fint_{B_\rho}|u-u_\rho|^2\,dx+C\rho^{n+4-\frac{2n}{q}}\|f\|_{L^q}^2\\ &\le C\rho^n\fint_{B_\rho}|u-u_\rho|^2\,dx+C\rho^{n+2-\frac{n}{q}}\|f\|_{L^q}=C\rho^nJ,\end{aligned}\tag{3.10}$$

where we use the inequality $\rho^{2-\frac{n}{q}}\|f\|_{L^q}\le 1$ and $J=\fint_{B_\rho(x_0)}|u-u_\rho|^2\,dx+\rho^{2-\frac{n}{q}}\|f\|_{L^q}$ in the last step.

Now let us put the estimates (3.8) and (3.10) into the inequality (3.9),

$$\begin{aligned}&\int_{B_\sigma(x_0)}|u-u_{x_0,\sigma}|^2\,dx\\ &\quad\le C\biggl(\biggl(\frac{\sigma}{\rho}\biggr)^{n+2}+\varepsilon+M_\rho\bigl(A(\cdot,u_\rho)\bigr)+\omega(J)+J\biggr)\\ &\qquad\times\biggl(\int_{B_\rho(x_0)}|u-u_\rho|^2+\rho^{n+2-\frac{n}{q}}\|f\|_{L^q}\biggr).\end{aligned}$$

Note that if we assume $\fint_{B_\rho(x_0)}|u-u_\rho|^2\,dx\to 0$ on any $x\in\Omega_0$ with some $\Omega_0\subset\Omega$ as $\rho\to 0$, we have $\varepsilon+M_\rho(A(\cdot,u_\rho))+\omega(J)+J<\epsilon_0$ with ϵ_0 as the same in Lemma 2.5 with a small $\rho>0$, which is due to $M_\rho(A(\cdot,u_\rho))\to 0$ by the VMO property of $A(x,u)$ in $x\in\Omega$ and the continuous modulus of $\omega(J)$. Therefore, by the iteration Lemma 2.5 we have

$$\int_{B_\sigma(x_0)}|u-u_{x_0,\sigma}|^2\,dx\le C\sigma^{n+2-\frac{n}{q}},\tag{3.11}$$

which implies

$$u\in\mathcal{L}^{2,n+2-\frac{n}{q}}\bigl(\Omega_0,\mathbb{R}^N\bigr).$$

In virtue of Lemma 2.7 on the equivalence between Campanato's spaces and Hölder continuous spaces, $u\in C^{0,\alpha}_{loc}(\Omega_0,\mathbb{R}^N)$ with $\alpha=1-\frac{n}{2q}$ due to $n<n+2-\frac{n}{q}<n+2$.

Further, by the Caccippoli inequality from Lemma 2.1, we conclude that

$$\begin{aligned}&\sup_{x\in\Omega_0,\sigma\in(0,\frac{\rho}{2})}\sigma^{\frac{n}{q}-n}\int_{B_{\frac{\sigma}{2}}(x_0)}|Du|^2\,dx\\ &\quad\le\sup_{x\in\Omega_0,\sigma\in(0,\frac{\rho}{2})}\sigma^{\frac{n}{q}-n}\biggl[\frac{C_1}{\sigma^2}\int_{B_\sigma(x_0)}|u-u_{x_0,\sigma}|^2\,dx+C_2\sigma^{n+2-\frac{2n}{q}}\|f\|_{L^q}^2\biggr]\end{aligned}$$

$$\begin{aligned}&\leq \sup_{x\in\Omega_0,\sigma\in(0,\frac{\rho}{2})} \sigma^{\frac{n}{q}-n}\left[\frac{C}{\sigma^2}\cdot\sigma^{n+2-\frac{n}{q}} + C_2\sigma^{n+2-\frac{2n}{q}}\|f\|^2_{L^q}\right]\\ &\leq C\big(1+\rho^{2-\frac{n}{q}}\big) < +\infty,\end{aligned}$$

which implies that $Du \in L^{2,\gamma}$ with $\gamma = n - \frac{n}{q}$.

Finally, we are in a position to estimate the Hausdorff dimension of the singular set $\Omega \setminus \Omega_0$. Recall that the singular set is written by

$$\Omega\setminus\Omega_0 := \left\{x\in\Omega : \liminf_{\rho\to 0} \fint_{B_\rho(x_0)} |u-u_\rho|^2\,dx > 0\right\}.$$

Notice that by the Poincaré's inequality one deduces

$$\fint_{B_\rho(x_0)} |u-u_\rho|^2\,dx \leq C\rho^{2-n}\int_{B_\rho(x_0)} |Du|^2\,dx := E(\rho),$$

that is,

$$\Omega\setminus\Omega_0 \subset \Omega\setminus\Sigma_0 = \left\{x\in\Omega : \liminf_{\rho\to 0} E(\rho) > 0\right\}. \tag{3.12}$$

On the basis of Lemma 2.8, it yields

$$\mathcal{H}^{n-2}(\Omega\setminus\Sigma_0) = 0,$$

which implies $\mathcal{H}^{n-2}(\Omega\setminus\Omega_0) = 0$. This completes the proof of Theorem 1.4. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed to each part of this work equally, and they all read and approved the final manuscript.

Author details
[1]Department of Mathematics, Beijing Jiaotong University, Beijing, 100044, China. [2]*Present address:* College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, 028043, China. [3]*Present address:* College of Science, North China University of Science and Technology, Tangshan, Hebei 063009, China.

Acknowledgements
This work is supported by the National Science Foundation of China grant 11371050. The authors would like to thank the referees for their valuable comments and suggestions.

Endnote
[a] The modulus of continuity may take a continuous concave function by
$\omega(t) = \inf\{\lambda(t) : \lambda(t)$ concave and continuous with $\lambda(t) \geq \alpha(t)$ for any modulus of continuity $\alpha(t)\}$.

Received: 18 September 2015 Accepted: 23 December 2015 Published online: 19 January 2016

References
1. Gilbarg, D, Trudinger, NS: Elliptic Partial Differential Equations of Second Order, 3th edn. Spinger, Berlin (2001)
2. De Giorgi, E: Un esempio di estremali discontinue per un problema variazionale di tipo ellitico. Boll. UMI **4**, 135-137 (1968)
3. Giaquinta, M: Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Ann. Math. Stud., vol. 105. Princeton University Press, Princeton (1983)
4. Giaquinta, M, Modica, G: Regularity results for some classes of higher order nonlinear elliptic systems. J. Reine Angew. Math. **311**, 145-169 (1979)
5. Nečas, J: Introduction to the Theory of Nonlinear Elliptic Equations. Teubner, Leipzig (1983)
6. Krylov, NV: On weak uniqueness for some diffusions with discontinuous coefficients. Stoch. Process. Appl. **113**(1), 37-64 (2004)

7. Sarason, D: Functions of vanishing mean oscillation. Trans. Am. Math. Soc. **207**, 391-405 (1975)
8. Chiarenza, F, Frasca, M, Longo, P: $W^{2,p}$ solvability of the Dirichlet problem for nonlinear elliptic equations with VMO coefficients. Trans. Am. Math. Soc. **336**(2), 841-853 (1993)
9. Bramanti, M, Cerutti, MC: $W_p^{1,2}$ solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients. Commun. Partial Differ. Equ. **18**(9-10), 1735-1763 (1993)
10. Di Fazio, G, Ragusa, MA: Interior estimate in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients. J. Funct. Anal. **112**(2), 241-256 (1993)
11. Fan, DS, Lu, SZ, Yang, DC: Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients. Georgian Math. J. **5**(5), 425-440 (1998)
12. Byun, S, Wang, L: Elliptic equations with BMO coefficients in Reifenberg domains. Commun. Pure Appl. Math. **57**(10), 1283-1310 (2004)
13. Krylov, NV: Parabolic and elliptic equations with VMO coefficients. Commun. Partial Differ. Equ. **32**(1-3), 453-475 (2007)
14. Dong, HJ, Kim, D: Parabolic and elliptic systems with VMO coefficients. Methods Appl. Anal. **16**(3), 365-388 (2010)
15. Zheng, SZ: Partial regularity for quasi-linear elliptic systems with VMO coefficients under natural growth condition. Chin. Ann. Math. **29**, 49-58 (2008)
16. Zheng, SZ, Feng, ZS: Regularity for quasi-linear elliptic systems with discontinuous coefficients. Dyn. Partial Differ. Equ. **5**(1), 87-99 (2008)
17. Chen, SH, Tan, Z: The method of A-harmonic approximation and optimal interior partial regularity for nonlinear elliptic systems under the controllable growth condition. J. Math. Anal. Appl. **335**, 20-42 (2007)
18. Duzaar, F, Grotowsli, J: Optimal interior partial regularity for nonlinear elliptic systems: the method of A-harmonic approximation. Manuscr. Math. **103**, 267-298 (2000)
19. Daněček, J, John, O, Stará, J: Morrey space regularity for weak solutions of Stokes systems with VMO coefficients. Ann. Mat. Pura Appl. **190**, 681-701 (2011)
20. Yu, HY, Zheng, SZ: Optimal partial regularity for quasilinear elliptic systems with VMO coefficients based on A-harmonic approximations. Electron. J. Differ. Equ. **2015**, 16 (2015)
21. Duzaar, F, Mingione, G: Harmonic type approximation lemmas. J. Math. Anal. Appl. **352**(1), 301-335 (2009)
22. Giaquinta, M: Introduction to Regularity Theory for Nonlinear Elliptic Systems. Birkhäuser, Basel (1993)
23. Evans, LC, Gariepy, RF: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics. CRC Press, New York (1992)