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The lower bounds of the first eigenvalues for the biharmonic operator on manifolds

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Abstract
In this paper, we use the Reilly formula and the Hessian comparison theorem to estimate the lower bounds of the first eigenvalues for the biharmonic operator eigenvalue problems (buckling problem and clamped plate problem) on manifolds.

MSC: 35P15; 53C20

Keywords: biharmonic operator; buckling problem; clamped plate problem; eigenvalues; lower bounds

1 Introduction

It is well known that the eigenvalues and eigenfunctions of the Laplacian play an important role in global differential geometry since they reveal important relations between geometry of the manifold and analysis. Among the eigenvalue problems of the Laplacian, the biharmonic operator eigenvalue problems are interesting projects because these problems root in physics and geometric analysis. In this paper, we investigate the lower bounds of the first eigenvalues for the buckling problem and clamped plate problem on manifolds. There is a lot of research on these problems; see, for example, [1–6], among others.

Let M be an n-dimensional compact connected Riemannian manifold with smooth boundary ∂M. We consider the biharmonic operator eigenvalue problem for the famous *buckling problem*

$$\begin{cases} \Delta^2 f = -\Lambda \Delta f & \text{in } M, \\ f|_{\partial\Omega} = \frac{\partial f}{\partial \vec{n}}|_{\partial M} = 0, \end{cases} \tag{1.1}$$

and *clamped plate problem*

$$\begin{cases} \Delta^2 f = \Gamma f & \text{in } M, \\ f|_{\partial\Omega} = \frac{\partial f}{\partial \vec{n}}|_{\partial M} = 0, \end{cases} \tag{1.2}$$

where Δ^2 is the bi-Laplace operator, and $\vec{n}$ denotes the outer unit normal vector field of the boundary ∂M. Equation (1.1) describes the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary. It is known that the spectrum of this eigenvalue problem is real and purely discrete:

$$0 < \Lambda_1 \leq \Lambda_2 \leq \cdots \leq \Lambda_k \leq \cdots \uparrow +\infty.$$

Equation (1.2) describes the characteristic vibrations of a clamped plate. The spectrum of this eigenvalue problem is also real and purely discrete:

$$0 < \Gamma_1 \leq \Gamma_2 \leq \cdots \leq \Gamma_k \leq \cdots \uparrow +\infty.$$

By the results of Li and Yau [7] and the variational characterization for eigenvalues, we can obtain lower bounds for eigenvalues of problems (1.1) and (1.2) on $M \subset R^n$ (see Levine and Protter [8]), respectively,

$$\frac{1}{k}\sum_{i=1}^{k}\Lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n V(M))^{\frac{2}{n}}}k^{\frac{2}{n}}, \qquad \frac{1}{k}\sum_{i=1}^{k}\Gamma_i \geq \frac{n}{n+4}\frac{16\pi^4}{(\omega_n V(M))^{\frac{4}{n}}}k^{\frac{4}{n}},$$

where ω_n denotes the volume of a unit ball in $\mathbb{R}^n$. Recently, Chen *et al.* [1] have investigated the first eigenvalues of problems (1.1) and (1.2) on n-dimensional compact connected Riemannian manifolds with Ricci curvature bounded from below by $(n-1)$. They proved that

$$\Lambda_1 > n, \qquad \Gamma_1 > n\lambda_1,$$

where λ_1 denotes the first eigenvalue of the Dirichlet eigenvalue problem. For more research, we refer the readers to [4, 5, 9–13].

In the present paper, we will use the Reilly formula and the Hessian comparison theorem to deal with problems (1.1) and (1.2) and estimate the lower bounds of their first eigenvalues.

2 Preliminaries

In this section, we recall some preliminary knowledge, such as the Reilly formula, for later use.

Lemma 2.1 (Reilly's formula [14]) *Let M be an n-dimensional compact connected Riemannian manifold with smooth boundary $\partial\Omega$. For every $f \in C^\infty(\bar{\Omega})$, we have*

$$\begin{aligned}&\int_M \bigl[(\Delta f)^2 - \bigl|\nabla^2 f\bigr|^2 - \operatorname{Ric}(\nabla f, \nabla f)\bigr]\,\mathrm{d}V \\ &\quad = \int_{\partial M}\bigl[-2(\bar{\Delta} f)\langle \nabla f, \vec{n}\rangle + (n-1)H\langle \nabla f, \vec{n}\rangle^2 + \sigma(\bar{\nabla} f, \bar{\nabla} f)\bigr]\,\mathrm{d}A,\end{aligned}$$

where Δf, ∇f, and $\nabla^2 f$ are the Laplacian, gradient, and Hessian of f, Ric is the Ricci curvature of M, $\bar{\Delta} f$ and $\bar{\nabla} f$ are the Laplacian and gradient of f in ∂M, and σ and H are the second fundamental form and the mean curvature of ∂M with respect to the inner unit normal vector field $\vec{n}$ on ∂M.

Lemma 2.2 (Hessian comparison theorem) *Let M be an n-dimensional complete Riemannian manifold, and let $x_0, x_1 \in M$. Let $\gamma : [0, \rho(x_1)] \to M$ be a minimizing geodesic joining*

x_0 and x_1, where $\rho(x)$ is the distance function $\mathrm{dist}_M(x_0,x)$. *Let K be the sectional curvature of M, and let $\mu_i(\rho)$ ($i=0,1$) be the functions defined by*

$$\mu_0(\rho)=\begin{cases} k_0\cdot\coth(k_0\cdot\rho(x)) & \text{if } \inf_\gamma K=-k_0^2,\\ \frac{1}{\rho(x)} & \text{if } \inf_\gamma K=0,\\ k_0\cdot\cot(k_0\cdot\rho(x)) & \text{if } \inf_\gamma K=k_0^2 \text{ and } \rho<\frac{\pi}{2k_0}\end{cases}$$

and

$$\mu_1(\rho)=\begin{cases} k_1\cdot\coth(k_1\cdot\rho(x)) & \text{if } \inf_\gamma K=-k_1^2,\\ \frac{1}{\rho(x)} & \text{if } \inf_\gamma K=0,\\ k_1\cdot\cot(k_1\cdot\rho(x)) & \text{if } \inf_\gamma K=k_1^2 \text{ and } \rho<\frac{\pi}{2k_1}.\end{cases}$$

Then the Hessians of ρ and ρ^2 satisfy

$$\begin{aligned} &\mu_0\big(\rho(x)\big)\cdot\|X\|^2\geq\nabla^2\rho(x)(X,X)\geq\mu_1\big(\rho(x)\big)\cdot\|X\|^2,\\ &2\rho(x)\mu_0\big(\rho(x)\big)\cdot\|X\|^2\geq\nabla^2\rho^2(x)(X,X)\geq2\rho(x)\mu_1\big(\rho(x)\big)\cdot\|X\|^2,\\ &\nabla^2\rho(x)\big(\gamma',\gamma'\big)=0,\qquad\nabla^2\rho^2(x)\big(\gamma',\gamma'\big)=2,\end{aligned}$$

where X is any vector in T_xM perpendicular to $\gamma'(\rho(x))$.

Thus, the following inequalities hold (*cf.* [15]):

$$(n-1)\mu_0\big(\rho(x)\big)\geq\Delta\rho(x)\geq(n-1)\mu_1\big(\rho(x)\big),\tag{2.1}$$

$$2(n-1)\rho(x)\mu_0\big(\rho(x)\big)+2\geq\Delta\rho^2(x)\geq2(n-1)\rho(x)\mu_1\big(\rho(x)\big)+2.\tag{2.2}$$

We recall the following variational characterization for the first eigenvalues of the buckling problem and the clamped plate problem:

$$\Lambda_1(M)=\min_{\substack{f\in H_0^2(M)\\ f\not\equiv0}}\frac{\int_M(\Delta f)^2\,\mathrm{d}V}{\int_M|\nabla f|^2\,\mathrm{d}V}\tag{2.3}$$

and

$$\Gamma_1(M)=\min_{\substack{f\in H_0^2(M)\\ f\not\equiv0}}\frac{\int_M(\Delta f)^2\,\mathrm{d}V}{\int_M f^2\,\mathrm{d}V}.\tag{2.4}$$

3 Main results

In this section, we give and prove our main results, that is, we estimate the lower bounds of the first eigenvalues of problems (1.1) and (1.2).

Theorem 3.1 *Let (M,g) be an n-dimensional compact connected Riemannian manifold with smooth boundary ∂M and Ricci curvature bounded from below by a positive constant C. Let $\Lambda_1(M)$ be the first eigenvalue of the buckling problem* (1.1). *Then, for any vector* *field* $X \in \Gamma(TM)$ *such that* $\|X\|_\infty = \sup_M |X| < \infty$ (*where* $|X| = \sqrt{g(X,X)}$) *and* $\inf_M \operatorname{div}(X) > 0$, *we have*

$$\Lambda_1(M) \geq \left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C. \tag{3.1}$$

Proof For any $f \in C_0^\infty(\Omega)$, the vector field $|\nabla f|^2 X$ has compact support on M, and a simple calculation shows that

$$\begin{aligned} \operatorname{div}\bigl(X|\nabla f|^2\bigr) &= 2|\nabla f| X\bigl(|\nabla f|\bigr) + |\nabla f|^2 \operatorname{div}(X) \\ &\geq -2|\nabla f| \|X\|_\infty \bigl|\nabla|\nabla f|\bigr| + |\nabla f|^2 \inf_M \operatorname{div}(X). \end{aligned} \tag{3.2}$$

For any $\varepsilon > 0$, it follows from Young's inequality that

$$|\nabla f|\bigl|\nabla|\nabla f|\bigr| \leq \frac{|\nabla|\nabla f||^2}{2\varepsilon^2} + \frac{\varepsilon^2|\nabla f|^2}{2}.$$

Combining this inequality and (3.2) yields

$$\operatorname{div}\bigl(X|\nabla f|^2\bigr) \geq -2\|X\|_\infty \left(\frac{|\nabla|\nabla f||^2}{2\varepsilon^2} + \frac{\varepsilon^2|\nabla f|^2}{2}\right) + |\nabla f|^2 \inf_M \operatorname{div}(X). \tag{3.3}$$

It follows from $f|_{\partial M} = \frac{\partial f}{\partial \vec{n}}|_{\partial M} = 0$ that $(\nabla f)|_{\partial M} = 0$. Then by the divergence theorem we have

$$\int_M \operatorname{div}\bigl(X|\nabla f|^2\bigr)\,\mathrm{dV} = \int_{\partial M} |\nabla f|^2 \langle X, \vec{n}\rangle\,\mathrm{dV}_{\partial\mathrm{M}} = 0.$$

This equality and (3.3) imply

$$\int_M \bigl|\nabla|\nabla f|\bigr|^2\,\mathrm{dV} \geq \frac{\varepsilon^2}{\|X\|_\infty}\Bigl(\inf_M \operatorname{div} X - \varepsilon^2\|X\|_\infty\Bigr) \int_M |\nabla f|^2\,\mathrm{dV},$$

that is,

$$\frac{\int_M |\nabla|\nabla f||^2\,\mathrm{d}V}{\int_M |\nabla f|^2\,\mathrm{d}V} \geq \frac{\varepsilon^2}{\|X\|_\infty}\Bigl(\inf_M \operatorname{div} X - \varepsilon^2\|X\|_\infty\Bigr). \tag{3.4}$$

Let

$$g(\varepsilon) = \varepsilon^2 \inf_M \operatorname{div} X - \varepsilon^4\|X\|_\infty.$$

Then

$$g'(\varepsilon) = 2\varepsilon \inf_M \operatorname{div} X - 4\varepsilon^3\|X\|_\infty, \qquad g''(\varepsilon) = 2\inf_M \operatorname{div} X - 12\varepsilon^2\|X\|_\infty.$$

It is obvious that when $g'(\varepsilon) = 0$, we have $\varepsilon_0 = (\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty})^{\frac{1}{2}}$ and $g''(\varepsilon_0) = -4\inf_M \operatorname{div} X < 0$, and therefore,

$$\max g(\varepsilon) = \left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty^{\frac{1}{2}}}\right)^2.$$

From this equality and from (3.4) it follows that

$$\frac{\int_M |\nabla|\nabla f||^2 \,\mathrm{d}V}{\int_M |\nabla f|^2 \,\mathrm{d}V} \geq \left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2. \tag{3.5}$$

On the other hand, if $f \in C_0^\infty(\Omega)$, then

$$\left|\nabla|\nabla f|\right|^2 \leq \left|\nabla^2 f\right|^2. \tag{3.6}$$

Indeed, without loss of generality, we choose the normal coordinate $(U; x^1, \ldots, x^n)$ for any point $P \in M$, and a simple calculation shows that

$$\left|\nabla|\nabla f|\right|^2(P) = \sum_{q=1}^n \left[\frac{(\sum_{i=1}^n f_i f_{iq})^2}{\sum_{i=1}^n f_i^2}\right], \tag{3.7}$$

where $f_j = \frac{\partial f}{\partial x^j}$. From the Cauchy-Schwarz inequality it follows that

$$\sum_{q=1}^n \left(\sum_{i=1}^n f_i f_{iq}\right)^2 \leq \left[\sum_{q=1}^n \left(\sum_{i=1}^n f_i^2\right)\left(\sum_{i=1}^n f_{iq}^2\right)\right] = \left(\sum_{i=1}^n f_i^2\right)\left(\sum_{i,q=1}^n f_{iq}^2\right). \tag{3.8}$$

Combining (3.7) and (3.8) yields

$$\left|\nabla|\nabla f|\right|^2(P) \leq \sum_{i,q=1}^n f_{iq}^2 = \left|\nabla^2 f\right|^2(P),$$

which implies that

$$\left|\nabla|\nabla f|\right|^2 \leq \sum_{i,q=1}^n f_{iq}^2 = \left|\nabla^2 f\right|^2$$

on M for any $f \in C_0^\infty(\Omega)$. Since $C_0^\infty(\Omega)$ is dense in $H_0^2(\Omega)$, the last relation also holds for all $f \in H_0^2(\Omega)$.

From Reilly's formula and the equality $f|_{\partial M} = \frac{\partial f}{\partial \vec{n}}|_{\partial M} = 0$ we get

$$\int_M \left[(\Delta f)^2 - \left|\nabla^2 f\right|^2 - \operatorname{Ric}(\nabla f, \nabla f)\right] \mathrm{d}V = 0.$$

Since the Ricci curvature of M is bounded from below by a constant C, this gives

$$\int_M \left|\nabla^2 f\right|^2 \mathrm{d}V \leq \int_M (\Delta f)^2 \,\mathrm{d}V - C \int_M |\nabla f|^2 \,\mathrm{d}V.$$

From this inequality and from (3.5) and (3.6) it follows that

$$\frac{\int_M (\Delta f)^2 \,\mathrm{d}V}{\int_M |\nabla f|^2 \,\mathrm{d}V} \geq \left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C, \tag{3.9}$$

which, together with (2.3) and the arbitrariness of X, implies

$$\Lambda_1(M) \geq \left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C. \qquad \square$$

Theorem 3.2 *Assume that* (M,g) *and* $X \in \Gamma(TM)$ *satisfy the conditions of Theorem* 3.1 *and let* $\Gamma_1(M)$ *be the first eigenvalue of the clamped plate problem* (1.2). *Then*

$$\Gamma_1(M) \geq \left(\left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C\right)\lambda_1(M), \tag{3.10}$$

where $\lambda_1(M)$ *is the first eigenvalue of the Dirichlet eigenvalue problem.*

Proof By (3.9) we immediately get

$$\frac{\int_M (\Delta f)^2 \,\mathrm{d}V}{\int_M f^2 \,\mathrm{d}V} \geq \left(\left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C\right)\frac{\int_M |\nabla f|^2 \,\mathrm{d}V}{\int_M f^2 \,\mathrm{d}V} \tag{3.11}$$

for all $f \in H_0^2(M)$. From the Poincaré inequality and the fact that f is nonzero with $f|_{\partial M} = 0$ it follows that

$$\frac{\int_M |\nabla f|^2 \,\mathrm{d}V}{\int_M f^2 \,\mathrm{d}V} \geq \lambda_1(M), \tag{3.12}$$

where the equality holds if and only if f is the first eigenfunction of the Dirichlet eigenvalue problem. Thus, from (2.4), (3.11), and (3.12) and from the arbitrariness of X we have

$$\Gamma_1(M) \geq \left(\left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C\right)\lambda_1(M). \qquad \square$$

As an application of our theorems, we can get the following results.

Corollary 3.3 *Let* (M,g) *be an* n*-dimensional complete Riemannian manifold, and* $B_M(p,r)$ *a geodesic ball with radius* $r < \operatorname{inj}(p)$. *Let* $\kappa(p,r) = \sup\{K_M(x); x \in B_M(p,r)\}$, *where* $K_M(x)$ *is the sectional curvature of* M *at* x. *If for* $k > 0$, $\kappa(p,r) = k^2$, $r < \frac{\pi}{2k}$, *and the Ricci curvature of* $B_M(p,r)$ *is bounded from below by a positive constant* C, *then*

$$\Lambda_1\big(B_M(p,r)\big) \geq \big[(n-1)kr\cot(kr) + 1\big]^2/4r^2 + C,$$
$$\Gamma_1\big(B_M(p,r)\big) \geq \big(\big[(n-1)kr\cot(kr) + 1\big]^2/4r^2 + C\big)\lambda_1\big(B_M(p,r)\big).$$

Proof Let $X = \nabla\rho^2$. Then by (2.2), (3.1), and (3.10) we have

$$\begin{aligned}\Lambda_1\big(B_M(p,r)\big) &\geq \left[\frac{\inf_{B_M(p,r)} \Delta\rho^2}{2\|\nabla\rho^2\|_\infty}\right]^2 + C \\ &\geq \left[\frac{(n-1)k\inf_{B_M(p,r)}\cot(k\rho)}{2} + \frac{1}{2r}\right]^2 + C \\ &\geq \frac{[(n-1)kr\cot(kr)+1]^2}{4r^2} + C\end{aligned}$$

and

$$\Gamma_1(B_M(p,r)) \geq \left(\frac{[(n-1)kr\cot(kr)+1]^2}{4r^2} + C\right)\lambda_1(M). \qquad \square$$

Corollary 3.4 *Let $\varphi : M \hookrightarrow N$ have a locally bounded mean curvature, that is, the number $h(p,r) = \sup\{|H(x)|; x \in \varphi(M) \cap B_N(p,r)\}$ is finite, where H is the mean curvature, and let Ω be any connected component of $\varphi^{-1}(\overline{B_N(p,r)})$, where $p \in N \backslash \varphi(M)$ and $r > 0$.*

(1) *If $\kappa(p,r) = k^2 < \infty$ and the Ricci curvature of Ω is bounded from below by a positive constant C, choose*

$$r < \min\left\{\operatorname{inj}(p), \frac{\pi}{2k}, \cot^{-1}\left[h(p, \operatorname{inj}(p))/(m-1)k\right]/k\right\}.$$

Then

$$\Lambda_1(\Omega) \geq \left[(m-1)k\cot(kr) - h(p,r)\right]^2/4 + C,$$

$$\Gamma_1(\Omega) \geq \left(\left[(m-1)k\cot(kr) - h(p,r)\right]^2/4 + C\right)\lambda_1(\Omega).$$

(2) *If $\lim_{r\to\infty}\kappa(p,r) = \infty$ and the Ricci curvature of Ω is bounded from below by a positive constant C, let*

$$r(s) = \min\left\{\pi/2\sqrt{\kappa(p,s)}, \cot^{-1}\left[h(p,s)/(m-1)\kappa(p,s)\right]/\kappa(p,s)\right\}, \quad s > 0.$$

Choose $r = \max_{s>0} r(s)$. Then

$$\Lambda_1(\Omega) \geq \left(\left[(m-1)\sqrt{\kappa(p,r)}\cot\left(\sqrt{\kappa(p,r)}r\right) - h(p,r)\right]^2/4 + C\right),$$

$$\Gamma_1(\Omega) \geq \left(\left[(m-1)\sqrt{\kappa(p,r)}\cot\left(\sqrt{\kappa(p,r)}r\right) - h(p,r)\right]^2/4 + C\right)\lambda_1(\Omega).$$

Proof By the proof of Theorem 4.3 of [15], if $\rho(x) = \operatorname{dist}_N(p,x)$ and $f = \rho \circ \varphi : M \to \mathbb{R}$, then applying (2.1), we have

$$\operatorname{div}(\nabla f) \geq (m-1)k\cot(kr) - h(p,r) > 0 \quad \text{if } \kappa(p,r) = k^2.$$

Since $\|\nabla f\| \leq 1$, the estimates follow from Theorems 3.1 and 3.2. $\square$

Remark 3.5 If, in (3.5), $|\nabla f|$ is replaced by f, then

$$\frac{\int_M |\nabla f|^2 \, dV}{\int_M f^2 \, dV} \geq \left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2,$$

which, together with (3.11), implies

$$\Gamma_1(M) \geq \left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 \left(\left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C\right). \tag{3.13}$$

From (3.13) and the proof of Corollary 3.3 we have that

$$\Gamma_1(B_M(p,r)) \geq \left(\frac{[(n-1)kr\cot(kr)+1]^2}{4r^2}\right)\left(\frac{[(n-1)kr\cot(kr)+1]^2}{4r^2} + C\right).$$

Remark 3.6 Using a similar argument as in Theorems 3.1 and 3.2 and Corollary 3.3, we can investigate the following two biharmonic operator eigenvalue problems:

$$\begin{cases} \Delta^2 u = -p\Delta u & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial^2 u}{\partial \vec{n}^2}|_{\partial\Omega} = 0 \end{cases}$$

and

$$\begin{cases} \Delta^2 u = qu & \text{in } \Omega, \\ u|_{\partial\Omega} = \frac{\partial^2 u}{\partial \vec{n}^2}|_{\partial\Omega} = 0. \end{cases}$$

We get

$$\begin{aligned} p_1(M) &\geq \sup\left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C, \\ q_1(M) &\geq \left(\sup\left(\frac{\inf_M \operatorname{div} X}{2\|X\|_\infty}\right)^2 + C\right)\lambda_1(M) \end{aligned}$$

and

$$\begin{aligned} p_1\bigl(B_M(p,r)\bigr) &\geq \left(\frac{[(n-1)kr\cot(kr)+1]^2}{4r^2} + C\right), \\ q_1\bigl(B_M(p,r)\bigr) &\geq \left(\frac{[(n-1)kr\cot(kr)+1]^2}{4r^2} + C\right)\lambda_1(M), \end{aligned}$$

where p_1 and q_1 are the first eigenvalues of these two problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved for the final manuscript.

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