Nishizawa *Journal of Inequalities and Applications* (2015) 2015:402
DOI 10.1186/s13660-015-0932-9

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Sharp Becker-Stark's type inequalities with power exponential functions

Yusuke Nishizawa*

*Correspondence:
yusuke@ube-k.ac.jp
General Education, Ube National College of Technology, Tokiwadai 2-14-1, Ube, Yamaguchi 755-8555, Japan

Abstract

In this paper, we give some inequalities with power exponential functions derived from the left hand side of Becker-Stark's inequality:

$$\frac{8}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2-4x^2}$$

for $0 < x < \pi/2$.

MSC: Primary 26D05

Keywords: Becker-Stark's inequality; Bernoulli's inequality; monotonically decreasing function; monotonically increasing function

1 Introduction

Becker-Stark's inequality is well known:

$$\frac{8}{\pi^2-4x^2} < \frac{\tan x}{x} < \frac{\pi^2}{\pi^2-4x^2} \tag{1.1}$$

for $0 < x < \pi/2$. The research of Becker-Stark's inequality is one of the active areas in mathematical analysis [1–8]. Recently, Zhu [6] gave the following refinement of Becker-Stark's inequality: For $0 < x < \pi/2$, the inequalities

$$\frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{\pi^2-9}{6\pi^4}\left(\pi^2-4x^2\right) < \frac{\tan x}{x} \tag{1.2}$$

and

$$\frac{\tan x}{x} < \frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{10-\pi^2}{\pi^4}\left(\pi^2-4x^2\right) \tag{1.3}$$

hold, where the constants $-(\pi^2-9)/(6\pi^4)$ and $-(10-\pi^2)/\pi^4$ are the best possible. Moreover, from the right hand side of the inequality (1.1), Chen and Cheung [2] gave the following inequality: For $0 < x < \pi/2$, the inequality

$$\left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\theta} < \frac{\tan x}{x} < \left(\frac{\pi^2}{\pi^2-4x^2}\right)^{\vartheta} \tag{1.4}$$

holds, where the constants $\theta = \pi^2/12$ and $\vartheta = 1$ are the best possible. In [5], Sun and Zhu gave a simple proof of the results. The above inequality (1.4) is created based on the right hand side of Becker-Stark's inequality (1.1). However, in this paper we establish some inequalities created based on the left hand side of the inequality (1.1).

2 Results and discussion

Motivated by (1.4), in this paper, we give some inequalities with power exponential functions derived from the left hand side of Becker-Stark's inequality (1.1). Since we note that $8/(\pi^2 - 4x^2) < 1$ for $0 < x < (\sqrt{\pi^2 - 8})/2$ and $8/(\pi^2 - 4x^2) > 1$ for $(\sqrt{\pi^2 - 8})/2 < x < \pi/2$, we obtain the two inequalities as follows.

Theorem 2.1 *For* $0 < x < (\sqrt{\pi^2 - 8})/2$, *we have*

$$\left(\frac{8}{\pi^2 - 4x^2}\right)^{\theta} < \frac{\tan x}{x} < \left(\frac{8}{\pi^2 - 4x^2}\right)^{\vartheta(x)}$$

with the best possible constant $\theta = 0$ *and the function*

$$\vartheta(x) = \frac{2}{2x - \sqrt{\pi^2 - 8}} + \frac{2}{\sqrt{\pi^2 - 8}}.$$

Theorem 2.2 *For* $(\sqrt{\pi^2 - 8})/2 < x < \pi/2$, *we have*

$$\left(\frac{8}{\pi^2 - 4x^2}\right)^{\theta} < \frac{\tan x}{x} < \left(\frac{8}{\pi^2 - 4x^2}\right)^{\vartheta(x)}$$

with the best possible constant $\theta = 1$ *and the function*

$$\vartheta(x) = \frac{2}{2x - \sqrt{\pi^2 - 8}} - \frac{\sqrt{\pi^2 - 8} - \pi + 2}{\pi - \sqrt{\pi^2 - 8}}.$$

From Theorems 2.1 and 2.2, we have the best possible constant θ such that

$$\left(\frac{8}{\pi^2 - 4x^2}\right)^{\theta} < \frac{\tan x}{x}.$$

If $0 < x < (\sqrt{\pi^2 - 8})/2$, the constant θ must be $\theta < 0$ in order to satisfy $1 \leq \tan x/x < (8/(\pi^2 - 4x^2))^{\theta}$. On the other hands, if $(\sqrt{\pi^2 - 8})/2 < x < \pi/2$, the constant θ must be $1 < \theta$ in order to satisfy $8/(\pi^2 - 4x^2) \leq \tan x/x < (8/(\pi^2 - 4x^2))^{\theta}$. Here, we obtain the two inequalities as follows.

Theorem 2.3 *For* $1/2 < x < (\sqrt{\pi^2 - 8})/2$, *we have*

$$\left(\frac{8}{\pi^2 - 4x^2}\right)^{\frac{\vartheta(x)}{8}} < \frac{\tan x}{x},$$

where the function $\vartheta(x)$ *is in Theorem* 2.1.

Corollary 2.4 *For* $0 < x < \pi/2$, *we do not have the best possible constant* ϑ *such that*

$$\frac{\tan x}{x} < \left(\frac{8}{\pi^2 - 4x^2}\right)^{\vartheta}.$$

3 Proofs of main theorems

3.1 Proof of Theorem 2.1

Proof of Theorem 2.1 We set

$$f(x) = \left(\frac{8}{\pi^2 - 4x^2}\right)^{\frac{2}{2x - \sqrt{\pi^2 - 8}} + \frac{2}{\sqrt{\pi^2 - 8}}} - \frac{\tan x}{x}.$$

From

$$\frac{2}{2x - \sqrt{\pi^2 - 8}} + \frac{2}{\sqrt{\pi^2 - 8}} < 0$$

for $0 < x < (\sqrt{\pi^2 - 8})/2$, by Bernoulli's inequality, we have

$$\left(\frac{8}{\pi^2 - 4x^2}\right)^{\frac{2}{2x - \sqrt{\pi^2 - 8}} + \frac{2}{\sqrt{\pi^2 - 8}}} > 1 + \left(\frac{8}{\pi^2 - 4x^2} - 1\right)\left(\frac{2}{2x - \sqrt{\pi^2 - 8}} + \frac{2}{\sqrt{\pi^2 - 8}}\right).$$

By the right hand side of the inequality (1.1), for $0 < x < (\sqrt{\pi^2 - 8})/2$,

$$\begin{aligned} f(x) &> 1 + \left(\frac{8}{\pi^2 - 4x^2} - 1\right)\left(\frac{2}{2x - \sqrt{\pi^2 - 8}} + \frac{2}{\sqrt{\pi^2 - 8}}\right) - \frac{\pi^2}{\pi^2 - 4x^2} \\ &= \frac{4x(2\sqrt{\pi^2 - 8}x^2 - 4x^2 - \pi^2 x + 8x + \pi^2 - 8)}{\sqrt{\pi^2 - 8}(\pi - 2x)(\sqrt{\pi^2 - 8} - 2x)(2x + \pi)} \\ &= \frac{4xg(x)}{\sqrt{\pi^2 - 8}(\pi - 2x)(\sqrt{\pi^2 - 8} - 2x)(2x + \pi)}, \end{aligned}$$

where

$$g(x) = 2\sqrt{\pi^2 - 8}x^2 - 4x^2 - \pi^2 x + 8x + \pi^2 - 8.$$

From $\sqrt{\pi^2 - 8} - 2x > 0$ for $0 < x < (\sqrt{\pi^2 - 8})/2$, it suffices to show that

$$g(x) > 0.$$

Here, the derivative of $g(x)$ is

$$g'(x) = 8 - \pi^2 + 4\left(\sqrt{\pi^2 - 8} - 2\right)x.$$

By $8 - \pi^2 < 0$ and $\sqrt{\pi^2 - 8} - 2 < 0$, we have $g'(x) < 0$ for any $0 < x < (\sqrt{\pi^2 - 8})/2$. Since $g(x)$ is strictly decreasing for $0 < x < (\sqrt{\pi^2 - 8})/2$, we have

$$g(x) > g\left(\frac{\sqrt{\pi^2 - 8}}{2}\right) = 0.$$

Therefore, we can get

$$\frac{\tan x}{x} < \left(\frac{8}{\pi^2 - 4x^2}\right)^{\vartheta(x)},$$

where

$$\vartheta(x) = \frac{2}{2x - \sqrt{\pi^2 - 8}} + \frac{2}{\sqrt{\pi^2 - 8}}.$$

Since $\tan x/x$ is strictly increasing for $0 < x < \pi/2$, we have

$$\frac{8}{\pi^2 - 4x^2} < 1 < \frac{\tan x}{x}$$

for any $0 < x < (\sqrt{\pi^2 - 8})/2$. Hence, for $0 < x < (\sqrt{\pi^2 - 8})/2$, we obtain

$$\left(\frac{8}{\pi^2 - 4x^2}\right)^{\theta} < \frac{\tan x}{x} < \left(\frac{8}{\pi^2 - 4x^2}\right)^{\vartheta(x)},$$

where the constant $\theta = 0$. Since $\vartheta(x)$ is strictly decreasing for $0 < x < (\sqrt{\pi^2 - 8})/2$ and

$$\vartheta(x) < \vartheta(0) = 0,$$

the constant $\theta = 0$ is the best possible. Therefore, the proof of Theorem 2.1 is complete. □

3.2 Proof of Theorem 2.2

Proof of Theorem 2.2 We set

$$f(x) = \left(\frac{8}{\pi^2 - 4x^2}\right)^{\frac{2}{2x - \sqrt{\pi^2 - 8}} - \frac{\sqrt{\pi^2 - 8} - \pi + 2}{\pi - \sqrt{\pi^2 - 8}}} - \frac{\tan x}{x}.$$

From

$$\frac{2}{2x - \sqrt{\pi^2 - 8}} - \frac{\sqrt{\pi^2 - 8} - \pi + 2}{\pi - \sqrt{\pi^2 - 8}} > 1$$

for $(\sqrt{\pi^2 - 8})/2 < x < \pi/2$, by Bernoulli's inequality, we have

$$\begin{aligned}
&\left(\frac{8}{\pi^2 - 4x^2}\right)^{\frac{2}{2x - \sqrt{\pi^2 - 8}} - \frac{\sqrt{\pi^2 - 8} - \pi + 2}{\pi - \sqrt{\pi^2 - 8}}} \\
&\quad > 1 + \left(\frac{8}{\pi^2 - 4x^2} - 1\right)\left(\frac{2}{2x - \sqrt{\pi^2 - 8}} - \frac{\sqrt{\pi^2 - 8} - \pi + 2}{\pi - \sqrt{\pi^2 - 8}}\right).
\end{aligned}$$

By the inequality (1.3), for $(\sqrt{\pi^2-8})/2 < x < \pi/2$,

$$\begin{aligned}
f(x) &> 1 + \left(\frac{8}{\pi^2-4x^2}-1\right)\left(\frac{2}{2x-\sqrt{\pi^2-8}} - \frac{\sqrt{\pi^2-8}-\pi+2}{\pi-\sqrt{\pi^2-8}}\right) \\
&\quad - \left(\frac{8}{\pi^2-4x^2}+\frac{2}{\pi^2}-\frac{10-\pi^2}{\pi^4}\left(\pi^2-4x^2\right)\right) \\
&= \frac{g(x)}{\pi^4(\sqrt{\pi^2-8}-\pi)(\sqrt{\pi^2-8}-2x)(2x+\pi)},
\end{aligned}$$

where

$$\begin{aligned}
g(x) &= 16\pi^3x^4 - 16\pi^2\sqrt{\pi^2-8}x^4 + 160\sqrt{\pi^2-8}x^4 - 160\pi x^4 \\
&\quad + 16\pi^4x^3 - 224\pi^2x^3 - 16\pi^3\sqrt{\pi^2-8}x^3 + 160\pi\sqrt{\pi^2-8}x^3 + 640x^3 \\
&\quad + 8\pi^4x^2 - 40\pi^3x^2 + 8\pi^2\sqrt{\pi^2-8}x^2 + 320\pi x^2 \\
&\quad - 4\pi^6x + 48\pi^4x - 128\pi^2x + 4\pi^5\sqrt{\pi^2-8}x \\
&\quad - 32\pi^3\sqrt{\pi^2-8}x - \pi^7 - 2\pi^6 + 16\pi^5 + 16\pi^4 \\
&\quad - 64\pi^3 + \pi^6\sqrt{\pi^2-8} - 8\pi^4\sqrt{\pi^2-8}.
\end{aligned}$$

From $(\sqrt{\pi^2-8}-\pi)(\sqrt{\pi^2-8}-2x) > 0$ for $(\sqrt{\pi^2-8})/2 < x < \pi/2$, it suffices to show that

$$g(x) > 0.$$

We have the derivatives

$$\begin{aligned}
g'(x) &= 4\bigl(16\pi^3x^3 - 16\pi^2\sqrt{\pi^2-8}x^3 + 160\sqrt{\pi^2-8}x^3 - 160\pi x^3 \\
&\quad + 12\pi^4x^2 - 168\pi^2x^2 - 12\pi^3\sqrt{\pi^2-8}x^2 + 120\pi\sqrt{\pi^2-8}x^2 + 480x^2 \\
&\quad + 4\pi^4x - 20\pi^3x + 4\pi^2\sqrt{\pi^2-8}x + 160\pi x \\
&\quad - \pi^6 + 12\pi^4 - 32\pi^2 + \pi^5\sqrt{\pi^2-8} - 8\pi^3\sqrt{\pi^2-8}\bigr) \\
&= 4h(x)
\end{aligned}$$

and

$$\begin{aligned}
h'(x) &= 4\bigl(12\pi^3x^2 - 12\pi^2\sqrt{\pi^2-8}x^2 + 120\sqrt{\pi^2-8}x^2 - 120\pi x^2 \\
&\quad + 6\pi^4x - 84\pi^2x - 6\pi^3\sqrt{\pi^2-8}x + 60\pi\sqrt{\pi^2-8}x + 240x \\
&\quad + \pi^4 - 5\pi^3 + \pi^2\sqrt{\pi^2-8} + 40\pi\bigr) \\
&= 4k(x).
\end{aligned}$$

From

$$-12\left(\pi^2-10\right)\left(\sqrt{\pi^2-8}-\pi\right) \cong -2.77627 < 0$$

and

$$-6(\pi^2-10)(\pi\sqrt{\pi^2-8}-\pi^2+4) \cong -1.23145 < 0,$$

we have

$$\begin{aligned} k(x) &= -12(\pi^2-10)(\sqrt{\pi^2-8}-\pi)x^2 - 6(\pi^2-10)(\pi\sqrt{\pi^2-8}-\pi^2+4)x \\ &\quad + \pi^2\sqrt{\pi^2-8}+\pi^4-5\pi^3+40\pi \\ &> -12(\pi^2-10)(\sqrt{\pi^2-8}-\pi)\left(\frac{\pi}{2}\right)^2 - 6(\pi^2-10)(\pi\sqrt{\pi^2-8}-\pi^2+4)\left(\frac{\pi}{2}\right) \\ &\quad + \pi^2\sqrt{\pi^2-8}+\pi^4-5\pi^3+40\pi \\ &\cong 72.7519. \end{aligned}$$

Since $h(x)$ is strictly increasing for $(\sqrt{\pi^2-8})/2 < x < \pi/2$, we have

$$h(x) > h\left(\frac{\sqrt{\pi^2-8}}{2}\right) \cong 191.598.$$

Thus, $g(x)$ is strictly increasing for $(\sqrt{\pi^2-8})/2 < x < \pi/2$ and we have

$$g(x) > g\left(\frac{\sqrt{\pi^2-8}}{2}\right) = 0.$$

Therefore, we can get

$$\frac{\tan x}{x} < \left(\frac{8}{\pi^2-4x^2}\right)^{\vartheta(x)},$$

where

$$\vartheta(x) = \frac{2}{2x-\sqrt{\pi^2-8}} - \frac{\sqrt{\pi^2-8}-\pi+2}{\pi-\sqrt{\pi^2-8}}.$$

Since we have

$$1 < \frac{8}{\pi^2-4x^2} < \frac{\tan x}{x}$$

for any $(\sqrt{\pi^2-8})/2 < x < \pi/2$, we obtain

$$\left(\frac{8}{\pi^2-4x^2}\right)^{\theta} < \frac{\tan x}{x} < \left(\frac{8}{\pi^2-4x^2}\right)^{\vartheta(x)},$$

where the constant $\theta = 1$. Since $\vartheta(x)$ is strictly decreasing for $(\sqrt{\pi^2-8})/2 < x < \pi/2$ and

$$\vartheta(x) > \vartheta\left(\frac{\pi}{2}\right) = 1,$$

the constant $\theta = 1$ is the best possible. Hence, the proof of Theorem 2.2 is complete. □

3.3 Proof of Theorem 2.3 and Corollary 2.4

We need two lemmas to prove Theorem 2.3.

Lemma 3.1 *For* $-1/5 < t < 0$, *we have*

$$\ln(t+1) > \frac{9}{8}t.$$

Proof We set

$$f(x) = \ln(t+1) - \frac{9}{8}t,$$

then

$$f'(t) = -\frac{9t+1}{8(t+1)}.$$

From $f'(t) > 0$ for $-1/5 < t < -1/9$ and $f'(t) < 0$ for $-1/9 < t < 0$, $f(t)$ is strictly increasing for $-1/5 < t < -1/9$ and $f(t)$ is strictly decreasing for $-1/9 < t < 0$. Since

$$f\left(-\frac{1}{5}\right) = \frac{9}{40} - \ln\left(\frac{5}{4}\right) \cong 0.00185645$$

and

$$f(0) = 0,$$

we can get $f(t) > 0$ for $-1/5 < t < 0$. □

Lemma 3.2 *For* $0 < s < 1/5$, *we have*

$$\ln(s+1) > \frac{8}{9}s.$$

Proof We set

$$f(s) = \ln(s+1) - \frac{8}{9}s,$$

then

$$f'(s) = -\frac{8s-1}{9(s+1)}.$$

From $f'(s) > 0$ for $0 < s < 1/8$ and $f'(s) < 0$ for $1/8 < s < 1/5$, $f(s)$ is strictly increasing for $0 < s < 1/8$ and $f(s)$ is strictly decreasing for $1/8 < s < 1/5$. Since

$$f\left(\frac{1}{5}\right) = \ln\left(\frac{6}{5}\right) - \frac{8}{45} \cong 0.00454378$$

and

$$f(0) = 0,$$

we can get $f(s) > 0$ for $0 < s < 1/5$. □

Proof of Theorem 2.3 We set

$$\begin{aligned} f(x) &= \ln\frac{\tan x}{x} - \left(\frac{\vartheta(x)}{8}\right)\ln\frac{8}{\pi^2-4x^2} \\ &= \ln\frac{\tan x}{x} - \left(\frac{1}{8x-4\sqrt{\pi^2-8}} + \frac{1}{4\sqrt{\pi^2-8}}\right)\ln\frac{8}{\pi^2-4x^2}. \end{aligned}$$

If

$$t = -1 + \frac{8}{\pi^2-4x^2},$$

then $-11/100 < t < 0$ for $1/2 < x < (\sqrt{\pi^2-8})/2$, by Lemma 3.1, we can get

$$\ln\frac{8}{\pi^2-4x^2} > \frac{9}{8}\left(-1 + \frac{8}{\pi^2-4x^2}\right).$$

If

$$s = -1 + \frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{\pi^2-9}{6\pi^4}\left(\pi^2-4x^2\right),$$

then $0 < s < 1/5$ for $1/2 < x < (\sqrt{\pi^2-8})/2$, by Lemma 3.2 and the inequality (1.2), we can get

$$\begin{aligned} \ln\frac{\tan x}{x} &> \ln\left(\frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{\pi^2-9}{6\pi^4}\left(\pi^2-4x^2\right)\right) \\ &> \frac{8}{9}\left(-1 + \frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{\pi^2-9}{6\pi^4}\left(\pi^2-4x^2\right)\right). \end{aligned}$$

Since

$$\frac{1}{8x-4\sqrt{\pi^2-8}} + \frac{1}{4\sqrt{\pi^2-8}} < 0$$

and

$$\frac{9}{8}\left(-1 + \frac{8}{\pi^2-4x^2}\right) < \ln\frac{8}{\pi^2-4x^2} < 0$$

for $1/2 < x < (\sqrt{\pi^2-8})/2$, we obtain

$$\begin{aligned} f(x) &> \frac{8}{9}\left(-1 + \frac{8}{\pi^2-4x^2} + \frac{2}{\pi^2} - \frac{\pi^2-9}{6\pi^4}\left(\pi^2-4x^2\right)\right) \\ &\quad - \left(\frac{1}{8x-4\sqrt{\pi^2-8}} + \frac{1}{4\sqrt{\pi^2-8}}\right) \times \frac{9}{8}\left(-1 + \frac{8}{\pi^2-4x^2}\right) \\ &= \frac{g(x)}{432\pi^4\sqrt{\pi^2-8}(\pi-2x)(\sqrt{\pi^2-8}-2x)(\pi+2x)}, \end{aligned}$$

where

$$\begin{aligned} g(x) = {} & -18{,}432\sqrt{\pi^2-8}x^5 + 2{,}048\pi^2\sqrt{\pi^2-8}x^5 \\ & - 73{,}728x^4 + 17{,}408\pi^2x^4 - 1{,}024\pi^4x^4 \\ & + 972\pi^4x^3 + 15{,}360\pi^2\sqrt{\pi^2-8}x^3 - 4{,}096\pi^4\sqrt{\pi^2-8}x^3 \\ & + 61{,}440\pi^2x^2 - 24{,}064\pi^4x^2 + 2{,}048\pi^6x^2 \\ & + 1{,}944\pi^4x - 243\pi^6x - 8{,}832\pi^4\sqrt{\pi^2-8}x + 896\pi^6\sqrt{\pi^2-8}x \\ & - 35{,}328\pi^4 + 8{,}000\pi^6 - 448\pi^8. \end{aligned}$$

It suffices to show that $g(x) > 0$ for $1/2 < x < (\sqrt{\pi^2-8})/2$. We have derivatives

$$\begin{aligned} g'(x) = {} & -92{,}160\sqrt{\pi^2-8}x^4 + 10{,}240\pi^2\sqrt{\pi^2-8}x^4 \\ & - 294{,}912x^3 + 69{,}632\pi^2x^3 - 4{,}096\pi^4x^3 \\ & + 2{,}916\pi^4x^2 + 46{,}080\pi^2\sqrt{\pi^2-8}x^2 - 12{,}288\pi^4\sqrt{\pi^2-8}x^2 \\ & + 122{,}880\pi^2x - 48{,}128\pi^4x + 4{,}096\pi^6x \\ & + 1{,}944\pi^4 - 243\pi^6 - 8{,}832\pi^4\sqrt{\pi^2-8} + 896\pi^6\sqrt{\pi^2-8}, \\ g''(x) = {} & 8\bigl(-46{,}080\sqrt{\pi^2-8}x^3 + 5{,}120\pi^2\sqrt{\pi^2-8}x^3 \\ & - 110{,}592x^2 + 26{,}112\pi^2x^2 - 1{,}536\pi^4x^2 \\ & + 729\pi^4x + 11{,}520\pi^2\sqrt{\pi^2-8}x - 3{,}072\pi^4\sqrt{\pi^2-8}x \\ & + 15{,}360\pi^2 - 6{,}016\pi^4 + 512\pi^6\bigr) \\ = {} & 8h(x), \end{aligned}$$

and

$$\begin{aligned} \frac{h'(x)}{3} = {} & -46{,}080\sqrt{\pi^2-8}x^2 + 5{,}120\pi^2\sqrt{\pi^2-8}x^2 \\ & - 73{,}728x + 17{,}408\pi^2x - 1{,}024\pi^4x \\ & + 243\pi^4 + 3{,}840\pi^2\sqrt{\pi^2-8} - 1{,}024\pi^4\sqrt{\pi^2-8} \\ < {} & -46{,}080\sqrt{\pi^2-8}\left(\frac{1}{2}\right)^2 + 5{,}120\pi^2\sqrt{\pi^2-8}\left(\frac{1}{2}\sqrt{\pi^2-8}\right)^2 \\ & - 73{,}728\left(\frac{1}{2}\right) + 17{,}408\pi^2\left(\frac{1}{2}\sqrt{\pi^2-8}\right) - 1{,}024\pi^4\left(\frac{1}{2}\right) \\ & + 243\pi^4 + 3{,}840\pi^2\sqrt{\pi^2-8} - 1{,}024\pi^4\sqrt{\pi^2-8} \\ = {} & -147{,}456 + 22{,}528\pi^2 - 768\pi^6 + \pi^4\left(5{,}632 + \frac{729}{2}\sqrt{\pi^2-8}\right) \\ \cong {} & -13{,}629.3. \end{aligned}$$

Thus, $h(x)$ is strictly decreasing for $1/2 < x < (\sqrt{\pi^2-8})/2$. From

$$h\left(\frac{1}{2}\right) \cong -33{,}392,$$

we have $g''(x) < 0$ for $1/2 < x < (\sqrt{\pi^2 - 8})/2$. Therefore, $g'(x)$ is strictly decreasing for $x_1 < x < (\sqrt{\pi^2 - 8})/2$. From

$$g'\left(\frac{1}{2}\right) \cong 5{,}734.6$$

and

$$g'\left(\frac{\sqrt{\pi^2 - 8}}{2}\right) \cong -67{,}578,$$

there exists uniquely a real number x_1 with $1/2 < x_1 < (\sqrt{\pi^2 - 8})/2$ such that $g'(x_1) = 0$. Hence, $g(x)$ is strictly increasing for $1/2 < x < x_1$ and $g(x)$ is strictly decreasing for $x_1 < x < (\sqrt{\pi^2 - 8})/2$. From

$$g\left(\frac{1}{2}\right) \cong 4{,}939$$

and

$$g\left(\frac{\sqrt{\pi^2 - 8}}{2}\right) = 0,$$

we can get $g(x) > 0$ for $1/2 < x < (\sqrt{\pi^2 - 8})/2$. Hence, the proof of Theorem 2.3 is complete. □

Proof of Corollary 2.4 By Theorem 2.3, for $1/2 < x < (\sqrt{\pi^2 - 8})/2$, we have the following:

$$\begin{aligned}\frac{\ln \frac{\tan x}{x}}{\ln \frac{8}{\pi^2 - 4x^2}} &< \frac{1}{8x - 4\sqrt{\pi^2 - 8}} + \frac{1}{4\sqrt{\pi^2 - 8}} \\ &= \left(-\frac{1}{4}\right)\left(\frac{x}{\sqrt{\pi^2 - 8}}\right)\left(\frac{1}{\frac{\sqrt{\pi^2-8}}{2} - x}\right).\end{aligned}$$

Therefore

$$\lim_{x \to (\sqrt{\pi^2-8})/2-0} \frac{\ln \frac{\tan x}{x}}{\ln \frac{8}{\pi^2 - 4x^2}} = -\infty.$$

The proof of Corollary 2.4 is complete. □

4 Conclusions

In this paper, we gave four inequalities derived from the left hand side of Becker-Stark's inequality (1.1), which are natural generalizations of the inequality (1.1). Since the value of $8/(\pi^2 - 4x^2)$ is less than 1 for $0 < x < (\sqrt{\pi^2 - 8})/2$ and the value of $8/(\pi^2 - 4x^2)$ is larger than 1 for $(\sqrt{\pi^2 - 8})/2 < x < \pi/2$, we established the inequalities in Theorems 2.1 and 2.2. By Theorem 2.3, we obtained Corollary 2.4 immediately.

Competing interests
The author declares that he has no competing interests.

Acknowledgements
The author would like to thank Professor Mitsuhiro Miyagi and the referees for their helpful suggestions and good advice.

Received: 29 June 2015 Accepted: 6 December 2015 Published online: 18 December 2015

References

1. Anderson, GD, Vamanamurthy, MK, Vuorinen, M: Inequalities for quasiconformal mappings in space. Pac. J. Math. **160**(1), 1-18 (1993)
2. Chen, C-P, Cheung, W-S: Sharp Cusa and Becker-Stark inequalities. J. Inequal. Appl. **2011**, Article ID 136 (2011)
3. Debnath, L, Mortici, C, Zhu, L: Refinements of Jordan-Steckin and Becker-Stark inequalities. Results Math. **67**(1-2), 207-215 (2015)
4. Mitrinović, DS: Analytic Inequalities. Springer, Berlin (1970)
5. Sun, Z-J, Zhu, L: Simple proofs of the Cusa-Huygens-type and Becker-Stark-type inequalities. J. Math. Inequal. **7**(4), 563-567 (2013)
6. Zhu, L: A refinement of the Becker-Stark inequalities. Math. Notes **93**(3-4), 421-425 (2013)
7. Zhu, L: Sharp Becker-Stark-type inequalities for Bessel functions. J. Inequal. Appl. **2010**, Article ID 838740 (2010)
8. Zhu, L, Hua, J: Sharpening the Becker-Stark inequalities. J. Inequal. Appl. **2010**, Article ID 931275 (2010)