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A refinement of the left-hand side of Hermite-Hadamard inequality for simplices

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Abstract

In this paper, we establish a new refinement of the left-hand side of Hermite-Hadamard inequality for convex functions of several variables defined on simplices.

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1 Introduction, definitions, and notations

The classical Hermite-Hadamard inequality [1] states that if a function $f\colon [a,b] \to \mathbb{R}$ is convex, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \le \frac{f(a)+f(b)}{2}.$$

This inequality has been discussed by many mathematicians. We refer to [2–4] and the references therein. In the last few decades, several generalizations of the Hermite-Hadamard inequality have been established and studied. One of them [5–7] says that if $\Delta \subset \mathbb{R}^n$ is a simplex with barycenter $\mathbf{b}$ and vertices $\mathbf{x}_0, \dots, \mathbf{x}_n$ and $f\colon \Delta \to \mathbb{R}$ is convex, then

$$f(\mathbf{b}) \le \frac{1}{\mathrm{Vol}(\Delta)}\int_\Delta f(\mathbf{x})\,\mathrm{d}\mathbf{x} \le \frac{f(\mathbf{x}_0)+\cdots+f(\mathbf{x}_n)}{n+1}. \tag{1.1}$$

Wąsowicz and Witkowski in [8] and Mitroi and Spiridon in [9] investigated the relationship between the left- and right-hand sides of (1.1).

An interesting refinement of both inequalities in (1.1) was obtained by Raïssouli and Dragomir in [10]. In this paper we use their method to obtain another refinement of the left-hand side of Hermite-Hadamard inequality on simplices.

Before we formulate the main theorem of this paper, we first give some definitions and notations. For a fixed natural number $n \ge 1$ let $N = \{0, 1, \dots, n\}$. Suppose $\mathbf{x}_0, \dots, \mathbf{x}_n \in \mathbb{R}^n$ are such that the vectors $\overrightarrow{\mathbf{x}_0\mathbf{x}_i}$, $i = 1, \dots, n$ are linearly independent. The set $\Delta = \mathrm{conv}\{\mathbf{x}_i\colon i \in N\}$ is called a *simplex*. Such a simplex is an n-dimensional object and we shall call it sometimes an n-simplex if we would like to emphasize its dimension. The point

$$\mathbf{b} = \frac{1}{n+1}(\mathbf{x}_0 + \cdots + \mathbf{x}_n)$$

is called the *barycenter* of Δ. For any subset K of N of cardinality $k \le n$ we define an $(n-k)$-simplex $\Delta^{[K]}$ as follows. For each $j \in N \setminus K$ let

$$\mathbf{x}_j^{[K]} = \frac{1}{n+1} \sum_{i \in K} \mathbf{x}_i + \frac{n+1-k}{n+1} \mathbf{x}_j \tag{1.2}$$

and

$$\Delta^{[K]} = \operatorname{conv}\{\mathbf{x}_j^{[K]} : j \in N \setminus K\}. \tag{1.3}$$

Obviously $\Delta^{[\emptyset]} = \Delta$ and $\Delta^{[K]} = \mathbf{b}$ if $\operatorname{card} N \setminus K = 1$.

The integration over a k-dimensional simplex will be always with respect to the k-dimensional Lebesgue measure denoted by $\mathrm{d}\mathbf{x}$ and the k-dimensional volume will be denoted by Vol. There will be no ambiguity, as the dimension will be obvious from the context.

The purpose of this paper is to prove that if $f : \Delta \to \mathbb{R}$ is convex and $K \subset L \subsetneq N$, then the average value of f on $\Delta^{[L]}$ does not exceed its average on $\Delta^{[K]}$.

By $H(\mathbf{a}, \lambda) : \mathbb{R}^n \to \mathbb{R}^n$ we denote the homothety with center $\mathbf{a}$ and scale λ, given by the formula

$$H(\mathbf{a}, \lambda)(\mathbf{x}) = \mathbf{a} + \lambda(\mathbf{x} - \mathbf{a}).$$

2 Refinement of the left-hand side

This is the main result of our paper.

Theorem 2.1 *Let $n \in \mathbb{N}$ and $\mathbf{x}_0, \dots, \mathbf{x}_n \in \mathbb{R}^n$ be such that the vectors $\overrightarrow{\mathbf{x}_0\mathbf{x}_i}$, $i = 1, \dots, n$ are linearly independent. If $f : \operatorname{conv}\{\mathbf{x}_0, \dots, \mathbf{x}_n\} \to \mathbb{R}$ is a convex function and $K \subset L \subsetneq \{0, \dots, n\}$, then*

$$\frac{1}{\operatorname{Vol} \Delta^{[L]}} \int_{\Delta^{[L]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \le \frac{1}{\operatorname{Vol} \Delta^{[K]}} \int_{\Delta^{[K]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

where $\Delta^{[K]} := \operatorname{conv}\{\frac{1}{n+1} \sum_{i \in K} \mathbf{x}_i + \frac{n+1-k}{n+1} \mathbf{x}_j : j \in \{0, \dots, n\} \setminus K\}$ and k is the cardinality of K.

Given the remark stated after equation (1.3) it is clear that Theorem 2.1 refines the left-hand side of (1.1).

Let us begin with two observations, which will make clear the nature of simplices $\Delta^{[K]}$.

The first observation follows immediately from (1.2).

Observation 2.2 *All simplices $\Delta^{[K]}$ have a common barycenter.*

Observation 2.3 *If $K \subset L \subsetneq N$ and $\operatorname{card} L = \operatorname{card} K + 1$, then $\Delta^{[L]}$ arises from $\Delta^{[K]}$ in the following way*:

let $l \in L \setminus K$ and let $\Delta_l^{[K]}$ be the face of $\Delta^{[K]}$ opposite to $\mathbf{x}_l^{[K]}$.

Then

$$\Delta^{[L]} = H\left(\mathbf{x}_l^{[K]}, \frac{n - \operatorname{card} K}{n + 1 - \operatorname{card} K}\right)(\Delta_l^{[K]}).$$

Proof Assume, without loss of generality, that $K = \{1,\dots,k\}$ and $L = \{0\} \cup K$. Let $k < s \le n$. By (1.2) the vertices of $\Delta^{[L]}$ are

$$x_s^{[L]} = \frac{1}{n+1}\sum_{i=0}^{k}\mathbf{x}_i + \frac{n-k}{n+1}\mathbf{x}_s.$$

Then

$$\begin{aligned} x_s^{[L]} &= \frac{1}{n+1}\sum_{i=1}^{k}\mathbf{x}_i + \frac{1}{n+1}\mathbf{x}_0 + \frac{n-k}{n+1}\mathbf{x}_s \\ &= \frac{1}{n+1}\sum_{i=1}^{k}\mathbf{x}_i + \frac{n+1-k}{n+1}\mathbf{x}_0 + \frac{n-k}{n+1}\mathbf{x}_s - \frac{n-k}{n+1}\mathbf{x}_0 \\ &= \frac{1}{n+1}\sum_{i=1}^{k}\mathbf{x}_i + \frac{n+1-k}{n+1}\mathbf{x}_0 + \frac{n-k}{n+1-k}\left(\frac{n+1-k}{n+1}\mathbf{x}_s - \frac{n+1-k}{n+1}\mathbf{x}_0\right) \\ &= \mathbf{x}_0^{[K]} + \frac{n-k}{n+1-k}\left(\mathbf{x}_s^{[K]} - \mathbf{x}_0^{[K]}\right) = H\left(\mathbf{x}_0^{[K]}, \frac{n-k}{n+1-k}\right)\left(\mathbf{x}_s^{[K]}\right). \end{aligned}$$ □

Let us brief on the approach proposed by Dragomir and Raïssouli in [10]. They constructed the sequence of subsimplices of Δ as follows.

Let $\mathbf{b}$ be the barycenter of Δ. One can divide Δ into $n+1$ subsimplices

$$D_i = \operatorname{conv}\{\mathbf{x}_0,\dots,\mathbf{x}_{i-1},\mathbf{b},\mathbf{x}_{i+1},\dots,\mathbf{x}_n\}, \quad i = 0,1,\dots,n.$$

It is important to note that all these simplices have the same volume.

Denote by $\mathcal{D}_1$ the set of simplices created this way. The set $\mathcal{D}_{p+1}$ is constructed by applying the above procedure to all simplices in $\mathcal{D}_p$. Dragomir and Raïssouli proved that for a convex function $f : \Delta \to \mathbb{R}$ one has

$$f(\mathbf{b}) \le \frac{1}{\operatorname{card}\mathcal{D}_p}\sum_{\delta\in\mathcal{D}_p} f(\mathbf{b}_\delta) \le \frac{1}{\operatorname{card}\mathcal{D}_{p+1}}\sum_{\delta\in\mathcal{D}_{p+1}} f(\mathbf{b}_\delta)$$

and

$$\lim_{p\to\infty}\frac{1}{\operatorname{card}\mathcal{D}_p}\sum_{\delta\in\mathcal{D}_p} f(\mathbf{b}_\delta) = \frac{1}{\operatorname{Vol}\Delta}\int_\Delta f(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{2.1}$$

where $\mathbf{b}_\delta$ denotes the barycenter of δ.

We shall use the above procedure to prove the main result of this paper.

Proof of Theorem 2.1 Obviously it is enough to prove the result in the case $\operatorname{card}K + 1 = \operatorname{card}L$. As above we may assume $K = \{1,\dots,k\}$ and $L = \{0\} \cup K$. Let $\Sigma = \Delta_0^{[K]}$ denote the face of $\Delta^{[K]}$ opposite to $\mathbf{x}_0^{[K]}$. For simplicity denote by H the homothety with center $\mathbf{x}_0^{[K]}$ and scale $\frac{n-k}{n+1-k}$. Then, by Observation 2.3 we see that $\Delta^{[L]} = H(\Sigma)$.

Let us apply the Dragomir-Raïssouli process to Σ. Thus we obtain a sequence of sets of subsimplices of Σ denoted by $\mathcal{D}_p$.

Fix $p \geq 1$. For every $\sigma \in \mathcal{D}_p$ let $\Sigma_\sigma = \operatorname{conv}(\sigma \cup \{x_0^{[K]}\})$. Clearly the simplices Σ_σ form a partition of $\Delta^{[K]}$ into simplices of the same height thus $\operatorname{Vol} \Sigma_\sigma = \operatorname{Vol} \Delta^{[K]} / \operatorname{card} \mathcal{D}_p$.

Now we apply the left-hand side of the Hermite-Hadamard inequality to all simplices Σ_σ to obtain

$$\frac{1}{\operatorname{card} \mathcal{D}_p} \sum_{\sigma \in \mathcal{D}_p} f(\mathbf{b}_{\Sigma_\sigma}) \leq \sum_{\sigma \in \mathcal{D}_p} \frac{1}{\operatorname{Vol} \Sigma_\sigma} \int_{\Sigma_\sigma} f(\mathbf{x})\, \mathrm{d}\mathbf{x} = \frac{1}{\operatorname{Vol} \Delta^{[K]}} \int_{\Delta^{[K]}} f(\mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{2.2}$$

Since $\Delta^{[L]}$ is the image of Σ by H, the sets

$$H(\mathcal{D}_p) = \bigl\{ H(\sigma) : \sigma \in \mathcal{D}_p \bigr\}$$

form the Dragomir-Raïssouli sequence for $\Delta^{[L]}$. Moreover, *comme par miracle* [11], the barycenters of Σ_σ and that of $H(\sigma)$ coincide, *i.e.*

$$\mathbf{b}_{\Sigma_\sigma} = \mathbf{b}_{H(\sigma)}. \tag{2.3}$$

From (2.2) and (2.3) we conclude

$$\frac{1}{\operatorname{card} \mathcal{D}_p} \sum_{\sigma \in \mathcal{D}_p} f(\mathbf{b}_{H(\sigma)}) \leq \frac{1}{\operatorname{Vol} \Delta^{[K]}} \int_{\Delta^{[K]}} f(\mathbf{x})\, \mathrm{d}\mathbf{x}, \tag{2.4}$$

and applying (2.1)

$$\lim_{p \to \infty} \frac{1}{\operatorname{card} \mathcal{D}_p} \sum_{\sigma \in \mathcal{D}_p} f(\mathbf{b}_{H(\sigma)}) = \frac{1}{\operatorname{Vol} \Delta^{[L]}} \int_{\Delta^{[L]}} f(\mathbf{x})\, \mathrm{d}\mathbf{x}. \tag{2.5}$$

Now the assertion follows immediately from (2.5) and (2.4). □

From Theorem 2.1 we obtain the following corollary.

Corollary 2.4 *Let $K_0, K_1, \ldots, K_n$ be a sequence of subsets of N such that*

$$K_0 \subset K_1 \subset \cdots \subset K_n \quad \text{and} \quad \operatorname{card} K_i = i, \quad i = 0, 1, \ldots, n.$$

If $f : \Delta \to \mathbb{R}$ is convex, then

$$\begin{aligned} f(\mathbf{b}) &= \frac{1}{\operatorname{Vol}(\Delta^{[K_n]})} \int_{\Delta^{[K_n]}} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq \frac{1}{\operatorname{Vol}(\Delta^{[K_{n-1}]})} \int_{\Delta^{[K_{n-1}]}} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &\leq \cdots \leq \frac{1}{\operatorname{Vol}(\Delta^{[K_1]})} \int_{\Delta^{[K_1]}} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \leq \frac{1}{\operatorname{Vol}(\Delta^{[K_0]})} \int_{\Delta^{[K_0]}} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \\ &= \frac{1}{\operatorname{Vol}(\Delta)} \int_{\Delta} f(\mathbf{x})\, \mathrm{d}\mathbf{x} \end{aligned}$$

(*note that* $\operatorname{Vol}(\Delta^{[K_i]})$ *denotes* $(n - i)$*-dimensional volume and* $\int_{\Delta^{[K_i]}} \cdots \mathrm{d}\mathbf{x}$ *denotes integration with respect to* $(n - i)$*-dimensional Lebesgue measure*).

Applying Theorem 2.1 to all possible proper subsets of N of the same cardinality and summing the obtained inequalities, we obtain the following corollary.

Corollary 2.5 *If $f: \Delta \to \mathbb{R}$ is a convex function, then*

$$\frac{1}{\operatorname{Vol}\Delta}\int_{\Delta} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \geq \frac{1}{\binom{n+1}{k}} \sum_{\substack{K\subsetneq N\\ \operatorname{card}K=k}} \frac{1}{\operatorname{Vol}\Delta^{[K]}}\int_{\Delta^{[K]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

From Theorem 2.1 we can derive the following corollary.

Corollary 2.6 *Let $f: \Delta \to \mathbb{R}$ be a convex function and let $k < l \leq n$. Then*

$$\frac{1}{\binom{n+1}{l}} \sum_{\substack{L\\ \operatorname{card}L=l}} \frac{1}{\operatorname{Vol}\Delta^{[L]}}\int_{\Delta^{[L]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq \frac{1}{\binom{n+1}{k}} \sum_{\substack{K\\ \operatorname{card}K=k}} \frac{1}{\operatorname{Vol}\Delta^{[K]}}\int_{\Delta^{[K]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Proof Clearly it is sufficient to prove the corollary only in the case $l = k + 1$. Fix $K = \{1,\ldots,k\}$. We have $n + 1 - k$ oversets of K of cardinality $k + 1$. Applying Theorem 2.1 to K and all such oversets and summing the obtained inequalities, we deduce

$$\sum_{\substack{L\supset K\\ \operatorname{card}L=k+1}} \frac{1}{\operatorname{Vol}\Delta^{[L]}}\int_{\Delta^{[L]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq (n+1-k)\frac{1}{\operatorname{Vol}\Delta^{[K]}}\int_{\Delta^{[K]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Summing this over all possible K, we obtain

$$(k+1)\sum_{\substack{L\\ \operatorname{card}L=k+1}} \frac{1}{\operatorname{Vol}\Delta^{[L]}}\int_{\Delta^{[L]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \leq (n+1-k)\sum_{\substack{K\\ \operatorname{card}K=k}} \frac{1}{\operatorname{Vol}\Delta^{[K]}}\int_{\Delta^{[K]}} f(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

since every L has $k+1$ subsets of cardinality k. We complete the proof by multiplying both sides by $\frac{k!(n-k)!}{(n+1)!}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AW came up with an idea, MN extended it and performed all necessary calculations. All authors read and approved the final manuscript.

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