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A new Hardy-Hilbert-type inequality with multiparameters and a best possible constant factor

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Abstract

By means of weight coefficients and techniques of real analysis, a new Hardy-Hilbert-type inequality with multiparameters and the best possible constant factor is given. The equivalent forms, the operator expression with the norm, and the reverse and some particular inequalities with the best possible constant factors are also considered.

MSC: 26D15; 47A07

Keywords: Hardy-Hilbert-type inequality; weight coefficient; equivalent form; reverse; operator

1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $\|f\|_p = (\int_0^\infty f^p(x)\,dx)^{\frac{1}{p}} > 0$, $\|g\|_q > 0$, then we have the following Hardy-Hilbert integral inequality [1]:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \tag{1}$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$, $\|a\|_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0$, $\|b\|_q > 0$, we have the following Hardy-Hilbert inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ [1]:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications [1–5].

If $\mu_i, \upsilon_j > 0$ $(i, j \in \mathbf{N} = \{1, 2, \ldots\})$,

$$U_m := \sum_{i=1}^m \mu_i, \qquad V_n := \sum_{j=1}^n \upsilon_j \quad (m, n \in \mathbf{N}), \tag{3}$$

then we have the following inequality (see Theorem 321 of [1]):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu_m^{1/q}\upsilon_n^{1/p}a_m b_n}{U_m+V_n}<\frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{4}$$

Replacing $\mu_m^{1/q}a_m$ and $\upsilon_n^{1/p}b_n$ by a_m and b_n in (4), respectively, we have the following equivalent form of (4):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m+V_n}<\frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{5}$$

For $\mu_i=\upsilon_j=1$ ($i,j\in\mathbf{N}$), both (4) and (5) reduce to (2). We call (4) and (5) Hardy-Hilbert-type inequalities.

Note The authors of [1] did not prove that (4) is valid with the best possible constant factor.

In 1998, by introducing an independent parameter $\lambda\in(0,1]$ Yang [6] gave an extension of (1) with the kernel $\frac{1}{(x+y)^\lambda}$ for $p=q=2$. Following the results of [6], Yang [5] gave some best extensions of (1) and (2) as follows.

If $\lambda_1,\lambda_2\in\mathbf{R}$, $\lambda_1+\lambda_2=\lambda$, $k_\lambda(x,y)$ is a nonnegative homogeneous function of degree $-\lambda$ with $k(\lambda_1)=\int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}\,dt\in\mathbf{R}_+$, $\phi(x)=x^{p(1-\lambda_1)-1}$, $\psi(x)=x^{q(1-\lambda_2)-1}$, $f(x),g(y)\geq 0$,

$$f\in L_{p,\phi}(\mathbf{R}_+)=\left\{f;\|f\|_{p,\phi}:=\left(\int_0^\infty\phi(x)\left|f(x)\right|^p dx\right)^{\frac{1}{p}}<\infty\right\},$$

$g\in L_{q,\psi}(\mathbf{R}_+)$, $\|f\|_{p,\phi},\|g\|_{q,\psi}>0$, then

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)\,dx\,dy<k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{6}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}$ ($k_\lambda(x,y)y^{\lambda_2-1}$) is decreasing with respect to $x>0$ ($y>0$), then for $a_m,b_n\geq 0$,

$$a\in l_{p,\phi}=\left\{a;\|a\|_{p,\phi}:=\left(\sum_{n=1}^{\infty}\phi(n)|a_n|^p\right)^{\frac{1}{p}}<\infty\right\},$$

$b=\{b_n\}_{n=1}^\infty\in l_{q,\psi}$, $\|a\|_{p,\phi},\|b\|_{q,\psi}>0$, it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k_\lambda(m,n)a_m b_n<k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{7}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda=1$, $k_1(x,y)=\frac{1}{x+y}$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, inequality (6) reduces to (1), whereas (7) reduces to (2). For $0<\lambda_1,\lambda_2\leq 1$, $\lambda_1+\lambda_2=\lambda$, we set

$$k_\lambda(x,y)=\frac{1}{(x+y)^\lambda}\quad\bigl((x,y)\in\mathbf{R}_+^2\bigr).$$

Then by (7) it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{8}$$

where the constant $B(\lambda_1,\lambda_2)$ is the best possible. Some other results including multidimensional Hilbert-type inequalities are provided in [7–24].

In this paper, by means of weight coefficients and techniques of real analysis, a new Hardy-Hilbert-type inequality with multiparameters and the best possible constant factor is given, which is with the kernel

$$k_{\lambda}(x,y)=\frac{(\min\{x,y\})^{\alpha}}{(\max\{x,y\})^{\lambda+\alpha}}$$

similar to (4). The equivalent forms, the operator expression with the norm, the reverse and some particular inequalities with the best possible constant factors are also considered.

2 An example and some lemmas

In the following, we make appointment that $\mu_i,\upsilon_j>0$ $(i,j\in\mathbf{N})$, U_m and V_n are defined by (3), $p\neq 0,1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n\geq 0$ $(m,n\in\mathbf{N})$,

$$\|a\|_{p,\Phi_{\lambda}}=\left(\sum_{m=1}^{\infty}\Phi_{\lambda}(m)a_m^p\right)^{\frac{1}{p}},\qquad \|b\|_{q,\Psi_{\lambda}}=\left(\sum_{n=1}^{\infty}\Psi_{\lambda}(n)b_n^q\right)^{\frac{1}{q}},$$

where

$$\Phi_{\lambda}(m):=\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}},\qquad \Psi_{\lambda}(n):=\frac{V_n^{q(1-\lambda_2)-1}}{\upsilon_n^{q-1}}\quad (m,n\in\mathbf{N}).$$

We also set

$$\widetilde{\Phi}_{\lambda}(m):=\bigl(1-\theta(\lambda_2,m)\bigr)\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}},\qquad \widetilde{\Psi}_{\lambda}(n):=\bigl(1-\vartheta(\lambda_1,n)\bigr)\frac{V_n^{q(1-\lambda_2)-1}}{\upsilon_n^{q-1}}\quad (m,n\in\mathbf{N}).$$

Note For $0<p<1$ or $p<0$, we still use the formal symbols $\|a\|_{p,\Phi_{\lambda}}$, $\|b\|_{q,\Psi_{\lambda}}$, $\|a\|_{p,\widetilde{\Phi}_{\lambda}}$, and $\|b\|_{q,\widetilde{\Psi}_{\lambda}}$.

Example 1 For $-\alpha<\lambda_1,\lambda_2\leq 1-\alpha$, $\lambda_1+\lambda_2=\lambda$, we set

$$k_{\lambda}(x,y)=\frac{(\min\{x,y\})^{\alpha}}{(\max\{x,y\})^{\lambda+\alpha}}\quad \bigl((x,y)\in\mathbf{R}_+^2\bigr).$$

We find

$$\begin{aligned} k(\lambda_1)&=\int_0^{\infty}k_{\lambda}(t,1)t^{\lambda_1-1}\,dt=\int_0^{\infty}\frac{(\min\{t,1\})^{\alpha}}{(\max\{t,1\})^{\lambda+\alpha}}t^{\lambda_1-1}\,dt\\ &=\int_0^1 t^{\lambda_1+\alpha-1}\,dt+\int_1^{\infty}\frac{1}{t^{\lambda+\alpha}}t^{\lambda_1-1}\,dt\\ &=\frac{1}{\lambda_1+\alpha}+\frac{1}{\lambda_2+\alpha}=\frac{\lambda+2\alpha}{(\lambda_1+\alpha)(\lambda_2+\alpha)}\in\mathbf{R}_+. \end{aligned}\tag{9}$$

Since

$$k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}} = \begin{cases} \frac{y^{\alpha+\lambda_2-1}}{x^{\lambda+\alpha}}, & 0 < y < x, \\ \frac{x^\alpha}{y^{1+\lambda_1+\alpha}}, & y \geq x, \end{cases}$$

for $\lambda_2 \leq 1-\alpha$ ($\lambda_1 > -\alpha$), $k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}}$ is decreasing for $y > 0$ and strictly decreasing for y large enough. Since

$$k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}} = \begin{cases} \frac{x^{\alpha+\lambda_1-1}}{y^{\lambda+\alpha}}, & 0 < x < y, \\ \frac{y^\alpha}{x^{1+\lambda_2+\alpha}}, & x \geq y, \end{cases}$$

for $\lambda_1 \leq 1-\alpha$ ($\lambda_2 > -\alpha$), $k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}$ is decreasing for $x > 0$ and strictly decreasing for x large enough.

In other words, for $-\alpha < \lambda_1, \lambda_2 \leq 1-\alpha$, $k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}}$ ($k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}$) is decreasing for $y > 0$ ($x > 0$) and strictly decreasing for $y(x)$ large enough, satisfying $k(\lambda_1) \in \mathbf{R}_+$.

Lemma 1 *If $g(t)$ (> 0) is decreasing in $\mathbf{R}_+$, strictly decreasing in $[n_0, \infty)$ ($n_0 \in \mathbf{N}$), and satisfying $\int_0^\infty g(t)\,dt \in \mathbf{R}_+$, then we have*

$$\int_1^\infty g(t)\,dt < \sum_{n=1}^\infty g(n) < \int_0^\infty g(t)\,dt. \tag{10}$$

Proof Since

$$\begin{aligned} &\int_n^{n+1} g(t)\,dt \leq g(n) \leq \int_{n-1}^n g(t)\,dt \quad (n = 1, \ldots, n_0), \\ &\int_{n_0+1}^{n_0+2} g(t)\,dt < g(n_0+1) < \int_{n_0}^{n_0+1} g(t)\,dt, \end{aligned}$$

it follows that

$$0 < \int_1^{n_0+2} g(t)\,dt < \sum_{n=1}^{n_0+1} g(n) < \sum_{n=1}^{n_0+1}\int_{n-1}^n g(t)\,dt = \int_0^{n_0+1} g(t)\,dt < \infty.$$

In the same way, we have

$$0 < \int_{n_0+2}^\infty g(t)\,dt \leq \sum_{n=n_0+2}^\infty g(n) \leq \int_{n_0+1}^\infty g(t)\,dt < \infty.$$

Adding these two inequalities, we have (10). □

Lemma 2 *Let $-\alpha < \lambda_1, \lambda_2 \leq 1-\alpha$, $\lambda_1 + \lambda_2 = \lambda$, and $k(\lambda_1)$ be as in* (9). *Define the following weight coefficients*:

$$\omega(\lambda_2, m) := \sum_{n=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \frac{U_m^{\lambda_1}\upsilon_n}{V_n^{1-\lambda_2}}, \quad m \in \mathbf{N}, \tag{11}$$

$$\varpi(\lambda_1,n) := \sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \frac{V_n^{\lambda_2}\mu_m}{U_m^{1-\lambda_1}}, \quad n \in \mathbf{N}. \tag{12}$$

Then, we have the following inequalities:

$$\omega(\lambda_2,m) < k(\lambda_1) \quad (-\alpha < \lambda_2 \le 1-\alpha, \lambda_1 > -\alpha; m \in \mathbf{N}), \tag{13}$$

$$\varpi(\lambda_1,n) < k(\lambda_1) \quad (-\alpha < \lambda_1 \le 1-\alpha, \lambda_2 > -\alpha; n \in \mathbf{N}). \tag{14}$$

Proof We set $\mu(t) := \mu_m$, $t \in (m-1,m]$ ($m \in \mathbf{N}$); $\upsilon(t) := \upsilon_n$, $t \in (n-1,n]$ ($n \in \mathbf{N}$), and

$$U(x) := \int_0^x \mu(t)\,dt \quad (x \ge 0), \qquad V(y) := \int_0^y \upsilon(t)\,dt \quad (y \ge 0). \tag{15}$$

Then by (3) it follows that $U(m) = U_m$, $V(n) = V_n$ ($m,n \in \mathbf{N}$). For $x \in (m-1,m)$, $U'(x) = \mu(x) = \mu_m$ ($m \in \mathbf{N}$); for $y \in (n-1,n)$, $V'(y) = \upsilon(y) = \upsilon_n$ ($n \in \mathbf{N}$). Since $V(y)$ is strictly increasing in $(n-1,n]$, $-\alpha < \lambda_2 \le 1-\alpha$, $\lambda_1 > -\alpha$, in view of Example 1 and Lemma 1, we find

$$\begin{aligned} \omega(\lambda_2,m) &= \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}} V'(y)\,dy \\ &< \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{(\min\{U_m,V(y)\})^{\alpha}}{(\max\{U_m,V(y)\})^{\lambda+\alpha}} \frac{U_m^{\lambda_1}}{V^{1-\lambda_2}(y)} V'(y)\,dy. \end{aligned}$$

Setting $t = \frac{V(y)}{U_m}$, we obtain $V'(y)\,dy = U_m\,dt$ and

$$\begin{aligned} \omega(\lambda_2,m) &< \sum_{n=1}^{\infty} \int_{\frac{V(n-1)}{U_m}}^{\frac{V(n)}{U_m}} \frac{(\min\{1,t\})^{\alpha}}{(\max\{1,t\})^{\lambda+\alpha}} t^{\lambda_2-1}\,dt \\ &= \int_0^{\frac{V(\infty)}{U_m}} \frac{(\min\{1,t\})^{\alpha}}{(\max\{1,t\})^{\lambda+\alpha}} t^{\lambda_2-1}\,dt \\ &\le \int_0^{\infty} \frac{(\min\{1,t\})^{\alpha}}{(\max\{1,t\})^{\lambda+\alpha}} t^{\lambda_2-1}\,dt = k(\lambda_1). \end{aligned} \tag{16}$$

Hence, we have (13). In the same way, we have (14). □

Lemma 3 *Let* $-\alpha < \lambda_1, \lambda_2 \le 1-\alpha$, $\lambda_1 + \lambda_2 = \lambda$, $\lambda_1 + \lambda_2 = \lambda$, $k(\lambda_1)$ *be as in* (9), $m_0, n_0 \in \mathbf{N}$, $\mu_m \ge \mu_{m+1}$ ($m \in \{m_0, m_0+1, \ldots\}$), $\upsilon_n \ge \upsilon_{n+1}$ ($n \in \{n_0, n_0+1, \ldots\}$), $U(\infty) = V(\infty) = \infty$. *Then*

(i) *for* $m,n \in \mathbf{N}$, *we have*

$$k(\lambda_1)\bigl(1-\theta(\lambda_2,m)\bigr) < \omega(\lambda_2,m) \quad (-\alpha < \lambda_2 \le 1-\alpha, \lambda_1 > -\alpha), \tag{17}$$

$$k(\lambda_1)\bigl(1-\vartheta(\lambda_1,n)\bigr) < \varpi(\lambda_1,n) \quad (-\alpha < \lambda_1 \le 1-\alpha, \lambda_2 > -\alpha), \tag{18}$$

where, $\theta(\lambda_2,m) = O(\frac{1}{U_m^{\lambda_2+\alpha}}) \in (0,1)$, $\vartheta(\lambda_1,n) = O(\frac{1}{V_n^{\lambda_1+\alpha}}) \in (0,1)$;

(ii) *for any* $a > 0$, *we have*

$$\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+a}} = \frac{1}{a}\left(\frac{1}{U_{m_0}^{a}} + aO(1)\right), \tag{19}$$

$$\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+a}} = \frac{1}{a}\left(\frac{1}{V_{n_0}^a} + a\widetilde{O}(1)\right). \tag{20}$$

Proof Since $\upsilon_n \ge \upsilon_{n+1}$ ($n \ge n_0$), $-\alpha < \lambda_2 \le 1-\alpha$, $\lambda_1 > -\alpha$, and $V(\infty) = \infty$, by Lemma 1 we have

$$\begin{aligned}
\omega(\lambda_2, m) &\ge \sum_{n=n_0}^{\infty}\frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}}\frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}}\upsilon_{n+1} \\
&= \sum_{n=n_0}^{\infty}\int_n^{n+1}\frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}}\frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}}V'(y)\,dy \\
&> \sum_{n=n_0}^{\infty}\int_n^{n+1}\frac{(\min\{U_m, V(y)\})^\alpha}{(\max\{U_m, V(y)\})^{\lambda+\alpha}}\frac{U_m^{\lambda_1}}{V^{1-\lambda_2}(y)}V'(y)\,dy \\
&= \sum_{n=n_0}^{\infty}\int_{\frac{V(n)}{U_m}}^{\frac{V(n+1)}{U_m}}\frac{(\min\{1, t\})^\alpha}{(\max\{1, t\})^{\lambda+\alpha}}t^{\lambda_2-1}\,dt \\
&= \int_{\frac{V(n_0)}{U_m}}^{\infty}\frac{(\min\{1, t\})^\alpha}{(\max\{1, t\})^{\lambda+\alpha}}t^{\lambda_2-1}\,dt = k(\lambda_1)\bigl(1-\theta(\lambda_2, m)\bigr),
\end{aligned}$$

where

$$\theta(\lambda_2, m) := \frac{1}{k(\lambda_1)}\int_0^{\frac{V(n_0)}{U_m}}\frac{(\min\{1, t\})^\alpha}{(\max\{1, t\})^{\lambda+\alpha}}t^{\lambda_2-1}\,dt \in (0,1). \tag{21}$$

For $U_m > V(n_0)$, we obtain

$$\begin{aligned}
0 < \theta(\lambda_2, m) &= \frac{1}{k(\lambda_1)}\int_0^{\frac{V(n_0)}{U_m}} t^{\lambda_2+\alpha-1}\,dt \\
&= \frac{1}{(\lambda_2+\alpha)k(\lambda_1)}\left(\frac{V_{n_0}}{U_m}\right)^{\lambda_2+\alpha},
\end{aligned}$$

and then $\theta(\lambda_2, m) = O(\frac{1}{U_m^{\lambda_2+\alpha}})$. Hence, we have (17).

In the same way, since $\mu_m \ge \mu_{m+1}$ ($m \ge m_0$), $-\alpha < \lambda_1 \le 1-\alpha$, $\lambda_2 > -\alpha$, and $U(\infty) = \infty$, we have

$$\begin{aligned}
\varpi(\lambda_1, n) &\ge \sum_{m=m_0}^{\infty}\frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}}\frac{V_n^{\lambda_2}\mu_{m+1}}{U_m^{1-\lambda_1}} \\
&= \sum_{m=m_0}^{\infty}\int_m^{m+1}\frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}}\frac{V_n^{\lambda_2}U'(x)}{U_m^{1-\lambda_1}}\,dx \\
&> \sum_{m=m_0}^{\infty}\int_m^{m+1}\frac{(\min\{U(x), V_n\})^\alpha}{(\max\{U(x), V_n\})^{\lambda+\alpha}}\frac{V_n^{\lambda_2}U'(x)}{U^{1-\lambda_1}(x)}\,dx \\
&\overset{t=U(x)/V_n}{=} \sum_{m=m_0}^{\infty}\int_{\frac{U(m)}{V_n}}^{\frac{U(m+1)}{V_n}}\frac{(\min\{t, 1\})^\alpha}{(\max\{t, 1\})^{\lambda+\alpha}}t^{\lambda_1-1}\,dt \\
&= \int_{\frac{U(m_0)}{V_n}}^{\infty}\frac{(\min\{t, 1\})^\alpha}{(\max\{t, 1\})^{\lambda+\alpha}}t^{\lambda_1-1}\,dt = k(\lambda_1)\bigl(1-\vartheta(\lambda_1, n)\bigr),
\end{aligned}$$

where

$$\vartheta(\lambda_1,n) := \frac{1}{k(\lambda_1)}\int_0^{\frac{U(m_0)}{V_n}} \frac{(\min\{t,1\})^\alpha}{(\max\{t,1\})^{\lambda+\alpha}} t^{\lambda_1-1}\,dt \in (0,1). \tag{22}$$

For $V_n > U(m_0)$, we obtain

$$\vartheta(\lambda_1,n) = \frac{1}{k(\lambda_1)}\int_0^{\frac{U(m_0)}{V_n}} t^{\lambda_1+\alpha-1}\,dt = \frac{1}{(\lambda_1+\alpha)k(\lambda_1)}\left(\frac{U(m_0)}{V_n}\right)^{\lambda_1+\alpha}.$$

Hence, we have (18).

For $a > 0$, we find

$$\begin{aligned}\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+a}} &= \sum_{m=1}^{m_0}\frac{\mu_m}{U_m^{1+a}} + \sum_{m=m_0+1}^\infty \frac{\mu_m}{U_m^{1+a}}\\ &= \sum_{m=1}^{m_0}\frac{\mu_m}{U_m^{1+a}} + \sum_{m=m_0+1}^\infty \int_{m-1}^m \frac{U'(x)}{U_m^{1+a}}\,dx\\ &< \sum_{m=1}^{m_0}\frac{\mu_m}{U_m^{1+a}} + \sum_{m=m_0+1}^\infty \int_{m-1}^m \frac{U'(x)}{U^{1+a}(x)}\,dx\\ &= \sum_{m=1}^{m_0}\frac{\mu_m}{U_m^{1+a}} + \int_{m_0}^\infty \frac{dU(x)}{U^{1+a}(x)} = \sum_{m=1}^{m_0}\frac{\mu_m}{U_m^{1+a}} + \frac{1}{aU_{m_0}^a} = \frac{1}{a}\left(\frac{1}{U_{m_0}^a} + a\sum_{m=1}^{m_0}\frac{\mu_m}{U_m^{1+a}}\right),\\ \sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+a}} &\ge \sum_{m=m_0}^\infty \frac{\mu_{m+1}}{U_m^{1+a}} = \sum_{m=m_0}^\infty \int_m^{m+1}\frac{U'(x)}{U_m^{1+a}}\,dx\\ &> \sum_{m=m_0}^\infty \int_m^{m+1}\frac{U'(x)\,dx}{U^{1+a}(x)} = \int_{m_0}^\infty \frac{dU(x)}{U^{1+a}(x)} = \frac{1}{aU_{m_0}^a}.\end{aligned}$$

Hence, we have (19). In the same way, have (20). □

3 Equivalent inequalities and operator expressions

Theorem 4 *If* $-\alpha < \lambda_1, \lambda_2 \le 1-\alpha$, $\lambda_1+\lambda_2 = \lambda$, $k(\lambda_1)$ *is as in* (9), *then for* $p > 1$, $0 < \|a\|_{p,\Phi_\lambda}, \|b\|_{q,\Psi_\lambda} < \infty$, *we have the following equivalent inequalities*:

$$I := \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{(\min\{U_m,V_n\})^\alpha a_m b_n}{(\max\{U_m,V_n\})^{\lambda+\alpha}} < k(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{23}$$

$$J := \left\{\sum_{n=1}^\infty \frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^\infty \frac{(\min\{U_m,V_n\})^\alpha a_m}{(\max\{U_m,V_n\})^{\lambda+\alpha}}\right]^p\right\}^{\frac{1}{p}} < k(\lambda_1)\|a\|_{p,\Phi_\lambda}. \tag{24}$$

Proof By Hölder's inequality with weight (see [25]) we have

$$\begin{aligned}&\left[\sum_{m=1}^\infty \frac{(\min\{U_m,V_n\})^\alpha a_m}{(\max\{U_m,V_n\})^{\lambda+\alpha}}\right]^p\\ &\quad= \left[\sum_{m=1}^\infty \frac{(\min\{U_m,V_n\})^\alpha}{(\max\{U_m,V_n\})^{\lambda+\alpha}}\left(\frac{U_m^{\frac{1-\lambda_1}{q}} a_m}{V_n^{\frac{1-\lambda_2}{p}}\mu_m^{\frac{1}{q}}}\right)\left(\frac{V_n^{\frac{1-\lambda_2}{p}}\mu_m^{\frac{1}{q}}}{U_m^{\frac{1-\lambda_1}{q}}}\right)\right]^p\end{aligned}$$

$$\begin{aligned}
&\le \sum_{m=1}^{\infty} \frac{(\min\{U_m, V_n\})^{\alpha}}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \left(\frac{U_m^{(1-\lambda_1)p/q}}{V_n^{1-\lambda_2} \mu_m^{p/q}} a_m^p \right) \\
&\quad \times \left[\sum_{m=1}^{\infty} \frac{(\min\{U_m, V_n\})^{\alpha}}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \frac{V_n^{(1-\lambda_2)(q-1)} \mu_m}{U_m^{1-\lambda_1}} \right]^{p-1} \\
&= \frac{V_n^{1-p\lambda_2}}{(\varpi(\lambda_1, n))^{1-p} \upsilon_n} \sum_{m=1}^{\infty} \frac{(\min\{U_m, V_n\})^{\alpha}}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \frac{U_m^{(1-\lambda_1)(p-1)} \upsilon_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p.
\end{aligned} \tag{25}$$

In view of (14), we find

$$\begin{aligned}
J &\le \big(k(\lambda_1)\big)^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{U_m, V_n\})^{\alpha}}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \frac{U_m^{(1-\lambda_1)(p-1)} \upsilon_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}} \\
&= \big(k(\lambda_1)\big)^{\frac{1}{q}} \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(\min\{U_m, V_n\})^{\alpha}}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \frac{U_m^{(1-\lambda_1)(p-1)} \upsilon_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}} \\
&= \big(k(\lambda_1)\big)^{\frac{1}{q}} \left[\sum_{m=1}^{\infty} \omega(\lambda_2, m) \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}}.
\end{aligned} \tag{26}$$

Then by (13) we have (24).

By Hölder's inequality we have

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} \left[\frac{\upsilon_n^{\frac{1}{p}}}{V_n^{\frac{1}{p}-\lambda_2}} \sum_{m=1}^{\infty} \frac{(\min\{U_m, V_n\})^{\alpha} a_m}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \right] \left(\frac{V_n^{\frac{1}{p}-\lambda_2}}{\upsilon_n^{\frac{1}{p}}} b_n \right) \\
&\le J \|b\|_{q,\Psi_\lambda}.
\end{aligned} \tag{27}$$

Then by (24) we have (23).

On the other hand, assuming that (23) is valid, we set

$$b_n := \frac{\upsilon_n}{V_n^{1-p\lambda_2}} \left[\sum_{m=1}^{\infty} \frac{(\min\{U_m, V_n\})^{\alpha} a_m}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \right]^{p-1}, \quad n \in \mathbf{N}.$$

Then we find $J^p = \|b\|_{q,\Psi_\lambda}^q$. If $J = 0$, then (24) is trivially valid; if $J = \infty$, then by (26) and (13) it is impossible. Suppose that $0 < J < \infty$. By (23) it follows that

$$\|b\|_{q,\Psi_\lambda}^q = J^p = I < k(\lambda_1) \|a\|_{p,\Phi_\lambda} \|b\|_{q,\Psi_\lambda}, \tag{28}$$

$$\|b\|_{q,\Psi_\lambda}^{q-1} = J < k_s(\lambda_1) \|a\|_{p,\Phi_\lambda}, \tag{29}$$

and then (24) follows, which is equivalent to (23). □

Theorem 5 *With the assumptions of Theorem* 4, *if* $m_0, n_0 \in \mathbf{N}$, $\mu_m \ge \mu_{m+1}$ ($m \in \{m_0, m_0 + 1, \ldots\}$), $\upsilon_n \ge \upsilon_{n+1}$ ($n \in \{n_0, n_0 + 1, \ldots\}$), $U(\infty) = V(\infty) = \infty$, *then the constant factor* $k(\lambda_1)$ *in* (23) *and* (24) *is the best possible.*

Proof For $\varepsilon \in (0, p(\lambda_1+\alpha))$, we set $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ ($\in (-\alpha, 1-\alpha)$), $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ ($> -\alpha$), and $\widetilde{a} = \{\widetilde{a}_m\}_{m=1}^\infty$, $\widetilde{b} = \{\widetilde{b}_n\}_{n=1}^\infty$,

$$\widetilde{a}_m := U_m^{\widetilde{\lambda}_1 - 1}\mu_m = U_m^{\lambda_1 - \frac{\varepsilon}{p} - 1}\mu_m, \qquad \widetilde{b}_n = V_n^{\widetilde{\lambda}_2 - \varepsilon - 1}\upsilon_n = V_n^{\lambda_2 - \frac{\varepsilon}{q} - 1}\upsilon_n. \tag{30}$$

Then by (19), (20), and (18) we have

$$\begin{aligned}
\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda} &= \left(\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O(1)\right)^{\frac{1}{p}} \left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon \widetilde{O}(1)\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &:= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \widetilde{a}_m \widetilde{b}_n \\
&= \sum_{n=1}^\infty \left[\sum_{m=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}} \frac{V_n^{\widetilde{\lambda}_2}\mu_m}{U_m^{1-\widetilde{\lambda}_1}}\right] \frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \sum_{n=1}^\infty \varpi(\widetilde{\lambda}_1, n)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \ge k(\widetilde{\lambda}_1)\sum_{n=1}^\infty \bigl(1 - \vartheta(\widetilde{\lambda}_1, n)\bigr)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= k(\widetilde{\lambda}_1)\left(\sum_{n=1}^\infty \frac{\upsilon_n}{V_n^{\varepsilon+1}} - \sum_{n=1}^\infty O\left(\frac{\upsilon_n}{V_n^{\frac{\varepsilon}{q}+\lambda_1+\alpha+1}}\right)\right) \\
&= \frac{1}{\varepsilon}k(\widetilde{\lambda}_1)\left[\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\bigl(\widetilde{O}(1) - O(1)\bigr)\right].
\end{aligned}$$

If there exists a positive constant $K \le k(\lambda_1)$ such that (23) is valid when replacing $k(\lambda_1)$ to K, then, in particular, we have $\varepsilon\widetilde{I} < \varepsilon K\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda}$, namely,

$$k(\widetilde{\lambda}_1)\left[\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\bigl(\widetilde{O}(1) - O(1)\bigr)\right] < K\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O(1)\right)^{\frac{1}{p}} \left(\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.$$

It follows that $k(\lambda_1) \le K$ ($\varepsilon \to 0^+$). Hence, $K = k(\lambda_1)$ is the best possible constant factor of (23).

The constant factor $k(\lambda_1)$ in (24) is still the best possible. Otherwise, we would reach a contradiction by (27) that the constant factor in (23) is not the best possible. □

For $p > 1$, we find $\Psi_\lambda^{1-p}(n) = \frac{\upsilon_n}{V_n^{1-p\lambda_2}}$ and define the following normed spaces:

$$\begin{aligned}
l_{p,\Phi_\lambda} &:= \bigl\{a = \{a_m\}_{m=1}^\infty; \|a\|_{p,\Phi_\lambda} < \infty\bigr\}, \\
l_{q,\Psi_\lambda} &:= \bigl\{b = \{b_n\}_{n=1}^\infty; \|b\|_{q,\Psi_\lambda} < \infty\bigr\}, \\
l_{p,\Psi_\lambda^{1-p}} &:= \bigl\{c = \{c_n\}_{n=1}^\infty; \|c\|_{p,\Psi_\lambda^{1-p}} < \infty\bigr\}.
\end{aligned}$$

Assuming that $a = \{a_m\}_{m=1}^\infty \in l_{p,\Phi_\lambda}$ and setting

$$c = \{c_n\}_{n=1}^\infty, \qquad c_n := \sum_{m=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}} a_m, \quad n \in \mathbf{N},$$

we can rewrite (24) as

$$\|c\|_{p,\Psi_\lambda^{1-p}} < k(\lambda_1)\|a\|_{p,\Phi_\lambda} < \infty,$$

namely, $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 1 Define a Hardy-Hilbert-type operator $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ as follows: For any $a = \{a_m\}_{m=1}^\infty \in l_{p,\Phi_\lambda}$, there exists a unique representation $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$. Define the formal inner product of Ta and $b = \{b_n\}_{n=1}^\infty \in l_{q,\Psi_\lambda}$ as follows:

$$(Ta, b) := \sum_{n=1}^\infty \left[\sum_{m=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}} a_m\right] b_n. \tag{31}$$

Then we can rewrite (23) and (24) as follows:

$$(Ta, b) < k(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{32}$$

$$\|Ta\|_{p,\Psi_\lambda^{1-p}} < k(\lambda_1)\|a\|_{p,\Phi_\lambda}. \tag{33}$$

Define the norm of the operator T as follows:

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}} \frac{\|Ta\|_{p,\Psi_\lambda^{1-p}}}{\|a\|_{p,\Phi_\lambda}}.$$

Then by (31) we find $\|T\| \le k(\lambda_1)$. Since by Theorem 5 the constant factor in (31) is the best possible, we have

$$\|T\| = k(\lambda_1) = \frac{\lambda + 2\alpha}{(\lambda_1 + \alpha)(\lambda_2 + \alpha)}.$$

4 Some equivalent reverse inequalities

Theorem 6 *If* $-\alpha < \lambda_1, \lambda_2 \le 1 - \alpha$, $\lambda_1 + \lambda_2 = \lambda$, $k(\lambda_1)$ *is as in* (9), $m_0, n_0 \in \mathbf{N}$, $\mu_m \ge \mu_{m+1}$ ($m \in \{m_0, m_0 + 1, \ldots\}$), $\upsilon_n \ge \upsilon_{n+1}$ ($n \in \{n_0, n_0 + 1, \ldots\}$), $U(\infty) = V(\infty) = \infty$, *then for* $0 < p < 1$, $0 < \|a\|_{p,\Phi_\lambda}, \|b\|_{q,\Psi_\lambda} < \infty$, *we have the following equivalent inequalities with the best possible constant factor* $k(\lambda_1)$:

$$I = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}} a_m b_n > k(\lambda_1)\|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{34}$$

$$J = \left\{\sum_{n=1}^\infty \frac{\upsilon_n}{V_n^{1-p\lambda_2}} \left[\sum_{m=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha a_m}{(\max\{U_m, V_n\})^{\lambda+\alpha}}\right]^p\right\}^{\frac{1}{p}} > k(\lambda_1)\|a\|_{p,\widetilde{\Phi}_\lambda}. \tag{35}$$

Proof By the reverse Hölder's inequality and (14), we have the reverses of (25), (26), and (27). Then by (17) we have (35). By (35) and the reverse of (27) we have (34).

On the other hand, assuming that (34) is valid, we set b_n as in Theorem 4. Then we find $J^p = \|b\|_{q,\Psi_\lambda}^q$. If $J = \infty$, then (35) is trivially valid; if $J = 0$, then by reverse of (26) and (17) it is impossible. Suppose that $0 < J < \infty$. By (34) it follows that

$$\|b\|_{q,\Psi_\lambda}^{q} = J^p = I > k_s(\lambda_1)\|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{36}$$

$$\|b\|_{q,\Psi_\lambda}^{q-1} = J > k_s(\lambda_1)\|a\|_{p,\widetilde{\Phi}_\lambda}, \tag{37}$$

and then (35) follows, which is equivalent to (34).

For $\varepsilon \in (0, p(\lambda_1 + \alpha))$, we set $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$, $\widetilde{a}_m$, and $\widetilde{b}_n$ as (30). Then by (19), (20), and (14) we find

$$\begin{aligned}
\|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\Psi_\lambda} &= \left[\sum_{m=1}^{\infty}\bigl(1-\theta(\lambda_2,m)\bigr)\frac{\mu_m}{U_m^{1+\varepsilon}}\right]^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \left(\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}} - \sum_{m=1}^{\infty}O\left(\frac{\mu_m}{U_m^{1+\lambda_2+\alpha+\varepsilon}}\right)\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left[\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon\bigl(O(1) - O_1(1)\bigr)\right]^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}}\widetilde{a}_m\widetilde{b}_n \\
&= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}}\frac{V_n^{\widetilde{\lambda}_2}\mu_m}{U_m^{1-\widetilde{\lambda}_1}}\right]\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \sum_{n=1}^{\infty}\varpi(\widetilde{\lambda}_1,n)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \le k(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \frac{1}{\varepsilon}k(\widetilde{\lambda}_1)\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\widetilde{O}(1)\right).
\end{aligned}$$

If there exists a constant $K \ge k(\lambda_1)$ such that (34) is valid when replacing $k(\lambda_1)$ to K, then, in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{a}\|_{p,\widetilde{\Phi}_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda}$, namely,

$$k(\widetilde{\lambda}_1)\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\widetilde{O}(1)\right) > K\left[\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon\bigl(O(1) - O_1(1)\bigr)\right]^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.$$

It follows that $k(\lambda_1) \ge K$ ($\varepsilon \to 0^+$). Hence, $K = k(\lambda_1)$ is the best possible constant factor of (34).

The constant factor $k(\lambda_1)$ in (35) is still the best possible. Otherwise, we would reach a contradiction by the reverse of (27) that the constant factor in (34) is not the best possible. □

Theorem 7 *With the assumptions of Theorem* 6, *if* $p < 0$, *then we have the following equivalent inequalities with the best possible constant factor* $k(\lambda_1)$:

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}}a_m b_n > k(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda}, \tag{38}$$

$$J_1 := \left\{ \sum_{n=1}^{\infty} \frac{V_n^{p\lambda_2-1}\upsilon_n}{(1-\vartheta(\lambda_1,n))^{p-1}} \left[\sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha} a_m}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \right]^p \right\}^{\frac{1}{p}}$$
$$> k(\lambda_1)\|a\|_{p,\Phi_\lambda}. \tag{39}$$

Proof By the reverse Hölder inequality with weight, since $p < 0$, by (18) we have

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} a_m \right]^p \\
&= \left[\sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \left(\frac{U_m^{(1-\lambda_1)/q}}{V_n^{(1-\lambda_2)/p}\mu_m^{1/q}} a_m \right) \left(\frac{V_n^{(1-\lambda_2)/p}\mu_m^{1/q}}{U_m^{(1-\lambda_1)/q}} \right) \right]^p \\
&\le \sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \frac{U_m^{(1-\lambda_1)p/q}}{V_n^{1-\lambda_2}\mu_m^{p/q}} a_m^p \\
&\quad \times \left[\sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \frac{V_n^{(1-\lambda_2)(q-1)}\mu_m}{U_m^{1-\lambda_1}} \right]^{p-1} \\
&= \frac{V_n^{1-p\lambda_2}}{(\varpi(\lambda_1,n))^{1-p}} \sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \frac{U_m^{(1-\lambda_1)(p-1)}}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p \\
&\le \frac{(k(\lambda_1))^{p-1}V_n^{1-p\lambda_2}}{(1-\vartheta(\lambda_1,n))^{1-p}\upsilon_n} \sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p,
\end{aligned}$$

$$\begin{aligned}
J_1 &\ge \left(k(\lambda_1)\right)^{\frac{1}{q}} \left\{ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p \right\}^{\frac{1}{p}} \\
&= \left(k(\lambda_1)\right)^{\frac{1}{q}} \left\{ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} a_m \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p \right\}^{\frac{1}{p}} \\
&= \left(k(\lambda_1)\right)^{\frac{1}{q}} \left\{ \sum_{m=1}^{\infty} \omega(\lambda_2,m) \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right\}^{\frac{1}{p}}.
\end{aligned} \tag{40}$$

Then by (13) we have (39).

By the reverse Hölder inequality we have

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} \frac{V_n^{\lambda_2-\frac{1}{p}}\upsilon_n^{1/p}}{(1-\vartheta(\lambda_1,n))^{1/q}} \left[\sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} a_m \right] \\
&\quad \times \left[\left(1-\vartheta(\lambda_1,n)\right)^{\frac{1}{q}} \frac{V_n^{\frac{1}{p}-\lambda_2}}{\upsilon_n^{1/p}} b_n \right] \ge J_1\|b\|_{q,\widetilde{\Psi}_\lambda}.
\end{aligned} \tag{41}$$

Then by (39) we have (38).

On the other hand, assuming that (38) is valid, we set b_n as follows:

$$b_n := \frac{V_n^{p\lambda_2-1}\upsilon_n}{(1-\vartheta(\lambda_1,n))^{p-1}} \left[\sum_{m=1}^{\infty} \frac{(\min\{U_m,V_n\})^{\alpha}}{(\max\{U_m,V_n\})^{\lambda+\alpha}} a_m \right]^{p-1}, \quad n \in \mathbf{N}.$$

Then we find $J_1^p = \|b\|_{q,\widetilde{\Psi}_\lambda}^q$. If $J_1 = \infty$, then (39) is trivially valid; if $J_1 = 0$, then by (40) and (13) it is impossible. Suppose that $0 < J_1 < \infty$. By (38) it follows that

$$\|b\|_{q,\widetilde{\Psi}_\lambda}^q = J_1^p = I > k_s(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda},$$
$$\|b\|_{q,\widetilde{\Psi}_\lambda}^{q-1} = J_1 > k_s(\lambda_1)\|a\|_{p,\Phi_\lambda},$$

and then (39) follows, which is equivalent to (38).

For $\varepsilon \in (0, q(\lambda_2 + \alpha))$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ $(> -\alpha)$, $\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ $(\in (-\alpha, 1-\alpha))$, and

$$\widetilde{a}_m := U_m^{\widetilde{\lambda}_1 - 1 - \varepsilon}\mu_m = U_m^{\lambda_1 - \frac{\varepsilon}{p} - 1}\mu_m, \qquad \widetilde{b}_n = V_n^{\widetilde{\lambda}_2 - 1}\upsilon_n = V_n^{\lambda_2 - \frac{\varepsilon}{q} - 1}\upsilon_n.$$

Then by (19), (20), and (13) we have

$$\begin{aligned}
\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\widetilde{\Psi}_\lambda} &= \left(\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}}\left[\sum_{n=1}^\infty \bigl(1 - \vartheta(\lambda_1, n)\bigr)\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right]^{\frac{1}{q}} \\
&= \left(\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty \frac{\upsilon_n}{V_n^{1+\varepsilon}} - \sum_{n=1}^\infty O\left(\frac{\upsilon_n}{V_n^{1+\lambda_1+\alpha+\varepsilon}}\right)\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O(1)\right)^{\frac{1}{p}}\left[\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\bigl(\widetilde{O}(1) - O_1(1)\bigr)\right]^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^\infty\sum_{m=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}}\widetilde{a}_m\widetilde{b}_n \\
&= \sum_{m=1}^\infty\left[\sum_{n=1}^\infty \frac{(\min\{U_m, V_n\})^\alpha}{(\max\{U_m, V_n\})^{\lambda+\alpha}}\frac{U_m^{\widetilde{\lambda}_1}\upsilon_n}{V_n^{1-\widetilde{\lambda}_2}}\right]\frac{\mu_m}{U_m^{1+\varepsilon}} \\
&= \sum_{m=1}^\infty \omega(\widetilde{\lambda}_2, m)\frac{\mu_m}{U_m^{1+\varepsilon}} \le k(\widetilde{\lambda}_1)\sum_{n=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}} \\
&= \frac{1}{\varepsilon}k(\widetilde{\lambda}_1)\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O(1)\right).
\end{aligned}$$

If there exists a constant $K \ge k(\lambda_1)$ such that (38) is valid when replacing $k(\lambda_1)$ to K, then, in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\widetilde{\Psi}_\lambda}$, namely,

$$k(\widetilde{\lambda}_1)\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O(1)\right) > K\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O(1)\right)^{\frac{1}{p}}\left[\frac{1}{V_{n_0}^\varepsilon} + \varepsilon\bigl(\widetilde{O}(1) - O_1(1)\bigr)\right]^{\frac{1}{q}}.$$

It follows that $k(\lambda_1) \ge K$ $(\varepsilon \to 0^+)$. Hence, $K = k(\lambda_1)$ is the best possible constant factor of (38).

The constant factor $k(\lambda_1)$ in (39) is still the best possible. Otherwise, we would reach a contradiction by (41) that the constant factor in (38) is not the best possible. □

Remark 1 (i) For $\alpha = 0$ and $0 < \lambda_1, \lambda_2 \leq 1$ in (23) and (24), we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\max\{U_m, V_n\})^{\lambda}} < \frac{\lambda}{\lambda_1\lambda_2}\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{42}$$

$$\left\{\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty}\frac{a_m}{(\max\{U_m, V_n\})^{\lambda}}\right]^p\right\}^{\frac{1}{p}} < \frac{\lambda}{\lambda_1\lambda_2}\|a\|_{p,\Phi_\lambda}; \tag{43}$$

(ii) for $\alpha = -\lambda$ and $-1 \leq \lambda_1, \lambda_2 < 0$ in (23) and (24), we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(\min\{U_m, V_n\})^{\lambda}} < \frac{(-\lambda)}{\lambda_1\lambda_2}\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{44}$$

$$\left\{\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty}\frac{a_m}{(\min\{U_m, V_n\})^{\lambda}}\right]^p\right\}^{\frac{1}{p}} < \frac{(-\lambda)}{\lambda_1\lambda_2}\|a\|_{p,\Phi_\lambda}; \tag{45}$$

(iii) for $\lambda = 0$, $|\lambda_1| < \alpha$ $(0 < \alpha \leq \frac{1}{2})$; $|\lambda_1| < 1 - \alpha$ $(\frac{1}{2} < \alpha \leq 1)$, $\lambda_2 = -\lambda_1$ in (23) and (24), we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{\min\{U_m, V_n\}}{\max\{U_m, V_n\}}\right)^{\alpha} a_m b_n < \frac{2\alpha}{\alpha^2 - \lambda_1^2}\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{46}$$

$$\left\{\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+p\lambda_1}}\left[\sum_{m=1}^{\infty}\left(\frac{\min\{U_m, V_n\}}{\max\{U_m, V_n\}}\right)^{\alpha} a_m\right]^p\right\}^{\frac{1}{p}} < \frac{2\alpha}{\alpha^2 - \lambda_1^2}\|a\|_{p,\Phi_\lambda}. \tag{47}$$

In view of Theorem 5, the constant factors in these inequalities with the particular kernels are all the best possible.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment, and drafted the manuscript. YS participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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