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On characterizations of Bloch spaces and Besov spaces of pluriharmonic mappings

Xi Fu[1] and Xiaoyou Liu[2*]

*Correspondence:
liuxiaoyou2002@hotmail.com
[2]School of Mathematics and Physics, University of South China, Hengyang, Hunan 421001, People's Republic of China
Full list of author information is available at the end of the article

Abstract

We characterize the Bloch spaces and Besov spaces of pluriharmonic mappings on the unit ball of $\mathbb{C}^n$ by using the following quantity: $\sup_{\rho(z,w)<r,z\neq w}\frac{(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|\hat{D}^{(m)}f(z)-\hat{D}^{(m)}f(w)|}{|z-w|}$, where $\alpha+\beta=n+1$, $\hat{D}^{(m)}=\frac{\partial^m}{\partial z^m}+\frac{\partial^m}{\partial\bar{z}^m}$, $|m|=n$. This generalizes the main results of (Yoneda in Proc. Edinb. Math. Soc. 45:229-239, 2002) in the higher dimensional case.

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1 Introduction

Let $\mathbb{C}^n=\{z=(z_1,\ldots,z_n):z_1,\ldots,z_n\in\mathbb{C}\}$ denote the n dimensional complex vector space. For $a=(a_1,\ldots,a_n)\in\mathbb{C}^n$, we define the Euclidean inner product $\langle\cdot,\cdot\rangle$ by

$$\langle z,a\rangle=z_1\bar{a}_1+\cdots+z_n\bar{a}_n,$$

where $\bar{a}_k$ ($k\in\{1,\ldots,n\}$) denotes the complex conjugate of a_k. Then the Euclidean length of z is defined by

$$|z|=\langle z,z\rangle^{\frac{1}{2}}=\left(|z_1|^2+\cdots+|z_n|^2\right)^{\frac{1}{2}}.$$

Denote a ball in $\mathbb{C}^n$ with center a and radius $r>0$ by

$$\mathbb{B}^n(a,r)=\left\{z\in\mathbb{C}^n:|z-a|<r\right\}.$$

In particular, we let $\mathbb{B}^n$ denote the unit ball $\mathbb{B}^n(0,1)$ and let $\mathbb{D}$ be the unit disk in $\mathbb{C}$.

A complex-valued function f of $\mathbb{B}^n$ into $\mathbb{C}$ is called *pluriharmonic* if there are two holomorphic functions h and g, such that $f=h+\bar{g}$. We denote by $\mathcal{P}(\mathbb{B}^n)$ the class of all pluriharmonic mappings on the unit ball of $\mathbb{C}^n$.

Let $f=h+\bar{g}\in\mathcal{P}(\mathbb{B}^n)$. For a multi-index $m=(m_1,\ldots,m_n)$, we employ the notations

$$\nabla f(z)=\left(\frac{\partial f}{\partial z_1},\ldots,\frac{\partial f}{\partial z_n}\right),\qquad\overline{\nabla}f(z)=\left(\frac{\partial f}{\partial\bar{z}_1},\ldots,\frac{\partial f}{\partial\bar{z}_n}\right),$$

$$\partial^m f=\frac{\partial^m f}{\partial z^m}=\frac{\partial^{|m|}f}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}},\qquad\overline{\partial}^m f=\frac{\partial^m f}{\partial\bar{z}^m}=\frac{\partial^{|m|}f}{\partial\bar{z}_1^{m_1}\cdots\partial\bar{z}_n^{m_n}},$$

$$\hat{D}^{(m)}f = \partial^m f + \overline{\partial}^m f = \partial^m h + \overline{\partial}^m g,$$

where $|m| = m_1 + \cdots + m_n$. Obviously, if $f \in \mathcal{P}(\mathbb{B}^n)$, then so does $\hat{D}^{(m)}f$.

Similar to the planar case, the *Bloch space* $\mathcal{PB}(\mathbb{B}^n)$ of $\mathcal{P}(\mathbb{B}^n)$ consists of all mappings $f \in \mathcal{P}(\mathbb{B}^n)$ such that

$$\|f\| = \sup_{z\in\mathbb{B}^n}\left(1-|z|^2\right)\left(\left|\nabla f(z)\right| + \left|\overline{\nabla} f(z)\right|\right) < \infty;$$

the *little Bloch space* $\mathcal{PB}_0(\mathbb{B}^n)$ consists of all mappings $f \in \mathcal{PB}(\mathbb{B}^n)$ such that

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)\left(\left|\nabla f(z)\right| + \left|\overline{\nabla} f(z)\right|\right) = 0.$$

Let $d\lambda(z) = (1-|z|^2)^{-n-1}\,dv(z)$, where dv is the normalized Lebesgue measure of $\mathbb{B}^n$. For $1 \le p < \infty$, the *Besov space* $\mathcal{B}_p$ of $\mathcal{P}(\mathbb{B}^n)$ consists of all mappings $f \in \mathcal{P}(\mathbb{B}^n)$ such that $(1-|z|^2)(|\nabla f(z)| + |\overline{\nabla} f(z)|) \in L^p(\mathbb{B}^n, d\lambda)$, *i.e.*

$$\|f\|_{L^p(d\lambda(z))} = \int_{\mathbb{B}^n}\left(\left(1-|z|^2\right)\left(\left|\nabla f(z)\right| + \left|\overline{\nabla} f(z)\right|\right)\right)^p d\lambda(z) < \infty.$$

For a planar harmonic mapping f in $\mathbb{D}$, Colonna [1] proved that $f \in \mathcal{PB}(\mathbb{D})$ if and only if the Lipschitz number

$$\beta_f = \sup_{z,w\in\mathbb{D}, z\neq w}\frac{|f(z)-f(w)|}{\operatorname{arctanh}\left|\frac{z-w}{1-\bar{z}w}\right|} < \infty.$$

Let

$$l = \sup_{w\in D(z,r), z\neq w}\frac{(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|\hat{D}^{(n-1)}f(z) - \hat{D}^{(n-1)}f(w)|}{|z-w|},$$

where $D(z,r)$ is the Bergman disc with center $z \in \mathbb{D}$ and radius r, $n \ge 1$ an integer and $\alpha + \beta = n$. By means of it, Yoneda [2] characterized the spaces $\mathcal{PB}(\mathbb{D})$ and $\mathcal{B}_p$ as follows.

Theorem A *Let $n \ge 1$ be an integer and $f \in \mathcal{P}(\mathbb{D})$. Then $f \in \mathcal{PB}(\mathbb{D})$ if and only if l is bounded.*

Theorem B *Let $n \ge 1$ be an integer and $f \in \mathcal{P}(\mathbb{D})$. Then $f \in \mathcal{B}_p$ if and only if*

$$\int_{\mathbb{D}} l^p\, d\lambda(z) < \infty.$$

In this article, we consider the corresponding problems in higher dimensional setting. We refer to [3–7] for the related topics for holomorphic or harmonic functions. See [8–12] for various characterizations of the Bloch, little Bloch, and Besov spaces in the unit ball of $\mathbb{C}^n$. In Section 2, we recall some basic facts for pluriharmonic mappings. Our main results are Theorems 1-4, whose proofs will be presented in Sections 3 and 4.

2 Preliminaries

Let $\mathrm{Aut}(\mathbb{B}^n)$ denote the group of biholomorphic mappings of $\mathbb{B}^n$ onto itself. It is well known that $\mathrm{Aut}(\mathbb{B}^n)$ is generated by the unitary operators on $\mathbb{B}^n$ and the involutions ϕ_a of the form

$$\phi_a(z) = \frac{a - P_a z - (1-|a|^2)^{\frac{1}{2}} Q_a z}{1 - \langle z, a\rangle},$$

where $a, z \in \mathbb{B}^n$,

$$P_a z = \frac{a\langle z, a\rangle}{\langle a, a\rangle}, \qquad Q_a z = z - P_a z.$$

For $z, w \in \mathbb{B}^n$, we define $\rho(z,w) = |\phi_z(w)|$. It is known that ρ is a distance function on $\mathbb{B}^n$, and we call it *pseudo-hyperbolic metric* (*cf.* [6, 12]). For $r \in (0,1)$, the *pseudo-hyperbolic ball* with center z and radius r is given by

$$E(z,r) = \bigl\{ w \in \mathbb{B} : \bigl|\phi_z(w)\bigr| < r \bigr\}.$$

Clearly, $E(z,r) = \phi_z(\mathbb{B}(0,r))$.

Lemma 1 ([12]) *Let* $0 < r < 1$ *and* $w \in E(z,r)$. *Then*

$$1 - |z|^2 \asymp 1 - |w|^2 \asymp \bigl|1 - \langle z, w\rangle\bigr| \asymp \bigl|E(z,r)\bigr|^{\frac{1}{n+1}},$$

where $|E(z,r)|$ *is the normalized volume of* $E(z,r)$, $A \asymp B$ *means that there is a constant* $C > 0$ *such that* $B/C \le A \le BC$.

The following lemma is crucial [13].

Lemma 2 *Suppose that* $f : \overline{\mathbb{B}}^n(a,r) \to \mathbb{C}$ *is continuous and pluriharmonic in* $\mathbb{B}^n(a,r)$. *Then there exists* $C > 0$ *such that*

$$\bigl|\nabla f(a)\bigr| + \bigl|\overline{\nabla} f(a)\bigr| \le \frac{C}{r} \int_{\partial\mathbb{B}^n} \bigl|f(a + r\zeta) - f(a)\bigr|\, d\sigma(\zeta).$$

Let h be a holomorphic function in $\mathbb{B}^n$. We say that $h \in \mathcal{B}$ if

$$\sup_{z\in\mathbb{B}^n} \bigl(1 - |z|^2\bigr)\bigl|\nabla h(z)\bigr| < \infty;$$

similarly, $h \in \mathcal{B}_0$ if $h \in \mathcal{B}$ and

$$\lim_{|z|\to 1^-} \bigl(1 - |z|^2\bigr)\bigl|\nabla h(z)\bigr| = 0.$$

It is obvious that a pluriharmonic mapping $f = h + \bar{g} \in \mathcal{P}(\mathbb{B}^n)$ (resp. $\mathcal{PB}_0(\mathbb{B}^n)$) if and only if both $h, g \in \mathcal{B}$ (resp. $\mathcal{B}_0$).

The following is a characterization of the space $\mathcal{B}$ (resp. $\mathcal{B}_0$).

Lemma 3 ([12]) *Let h be holomorphic in $\mathbb{B}^n$ and N a positive integer. Then $h \in \mathcal{B}$ (resp. $\mathcal{B}_0$) if and only if*

$$\sup_{z\in\mathbb{B}^n}\left(1-|z|^2\right)^N\left|\frac{\partial^m h(z)}{\partial z^m}\right| < \infty \qquad \left(resp.\ \lim_{|z|\to 1^-}\left(1-|z|^2\right)^N\left|\frac{\partial^m h(z)}{\partial z^m}\right| = 0\right)$$

for all values of the multi-index m with $|m| = N$.

Corollary 1 *Let $f = h + \bar{g}$ be a pluriharmonic mapping in $\mathbb{B}^n$ and N a positive integer. Then $f \in \mathcal{PB}(\mathbb{B}^n)$ (resp. $\mathcal{PB}_0(\mathbb{B}^n)$) if and only if*

$$\sup_{z\in\mathbb{B}^n}\left(1-|z|^2\right)^N\left(\left|\partial^m f\right| + \left|\overline{\partial}^m f\right|\right) = \sup_{z\in\mathbb{B}^n}\left(1-|z|^2\right)^N\left(\left|\partial^m h\right| + \left|\overline{\partial}^m g\right|\right) < \infty,$$

respectively,

$$\lim_{|z|\to 1^-}\left(1-|z|^2\right)^N\left(\left|\partial^m f\right| + \left|\overline{\partial}^m f\right|\right) = \lim_{|z|\to 1^-}\left(1-|z|^2\right)^N\left(\left|\partial^m h\right| + \left|\overline{\partial}^m g\right|\right) \to 0$$

for all values of the multi-index m with $|m| = N$.

As an application of Lemma 3, we obtain the following.

Lemma 4 *Let h be holomorphic in $\mathbb{B}^n$. Then $h \in \mathcal{B}$ if and only if for each $j \in \{1,\dots,n\}$,*

$$L = \sup_{z,w\in\mathbb{B}^n, z\neq w}\frac{(1-|z|^2)(1-|w|^2)}{|z-w|}\left|\frac{\partial h(z)}{\partial z_j} - \frac{\partial h(w)}{\partial z_j}\right| < \infty.$$

Proof Fixing a point w and letting

$$z = w + \xi\overline{\nabla\left(\frac{\partial h}{\partial z_j}\right)(w)} \to w$$

with $\xi \in \mathbb{C}$, we have

$$\left(1-|w|^2\right)^2\left|\nabla\left(\frac{\partial h}{\partial z_j}\right)(w)\right| \le L,$$

for each $j \in \{1,\dots,n\}$. By Lemma 3, we see that $h \in \mathcal{B}$.

For the converse, we assume that $h \in \mathcal{B}$. Let $h_j(z) = \frac{\partial h(z)}{\partial z_j}$, then for each $j \in \{1,\dots,n\}$,

$$\begin{aligned}\left|h_j(z) - h_j(w)\right| &= \left|\int_0^1 \frac{dh_j}{ds}\left(sz + (1-s)w\right)ds\right| \\ &\le \sum_{k=1}^n\left|(z_k - w_k)\int_0^1 \frac{\partial h_j}{\partial z_k}\left(sz + (1-s)w\right)ds\right| \\ &\le \sqrt{n}|z-w|\int_0^1\left|\nabla h_j\left(sz + (1-s)w\right)\right|ds \\ &\le C|z-w|\int_0^1\frac{ds}{(1-|sz+(1-s)w|^2)^2}.\end{aligned}$$

It follows from [7] that there exists $0 < C_1 < \infty$ such that

$$\int_0^1 \frac{ds}{(1-|sz+(1-s)w|^2)^2} \le \frac{C_1}{(1-|z|^2)(1-|w|^2)}.$$

This implies that

$$\sup_{z,w\in\mathbb{B}^n, z\neq w} \frac{(1-|z|^2)(1-|w|^2)}{|z-w|} \left|h_j(z)-h_j(w)\right| < \infty.$$

So the result follows. □

3 The Bloch space for pluriharmonic mappings

In this section, we give some characterizations of the spaces $\mathcal{PB}(\mathbb{B}^n)$ and $\mathcal{PB}_0(\mathbb{B}^n)$ which can be viewed as the generalizations of Yoneda's results in the higher dimensional case.

Theorem 1 *Let $f \in \mathcal{P}(\mathbb{B}^n)$, $N \ge 0$ be an integer and $0 < r < 1$. Then $f \in \mathcal{PB}(\mathbb{B}^n)$ if and only if*

$$L_f = \sup_{z\in\mathbb{B}^n, \rho(z,w)<r, z\neq w} \frac{(1-|z|^2)^\alpha(1-|w|^2)^\beta |\hat{D}^{(m)}f(z)-\hat{D}^{(m)}f(w)|}{|z-w|} < \infty$$

for all values of the multi-index m with $|m| = N$, where $\alpha + \beta = N + 1$.

Proof First we prove the sufficiency. Let $f(z) \in \mathcal{P}(\mathbb{B}^n)$, then for each multi-index m with $|m| = N$, $\hat{D}^{(m)}f(z)$ is also pluriharmonic. According to Lemma 2, for $z \in \mathbb{B}^n$ and $r \in (0,1)$,

$$\left|\nabla\left(\hat{D}^{(m)}f\right)(z)\right| + \left|\overline{\nabla}\left(\hat{D}^{(m)}f\right)(z)\right| \le \frac{C}{(1-|z|^2)} \int_{\partial\mathbb{B}^n} \left|\left(\hat{D}^{(m)}f\right)(z+\varrho\zeta) - \left(\hat{D}^{(m)}f\right)(z)\right| d\sigma(\zeta),$$

where $\varrho = \frac{r(1-|z|^2)}{2}$. By a simple computation, we see that $\mathbb{B}^n(z,\varrho) \subset E(z,r)$, so

$$\left|\nabla\left(\hat{D}^{(m)}f\right)(z)\right| + \left|\overline{\nabla}\left(\hat{D}^{(m)}f\right)(z)\right| \le \frac{C}{(1-|z|^2)} \sup_{w\in E(z,r)} \left|\left(\hat{D}^{(m)}f\right)(z) - \left(\hat{D}^{(m)}f\right)(w)\right|.$$

Since for each $w \in E(z,r)$, $w \neq z$,

$$\frac{(1-|z|^2)^{\frac{1}{2}}(1-|w|^2)^{\frac{1}{2}}}{|z-w|} \ge \frac{(1-r^2)^{\frac{1}{2}}}{r},$$

by Lemma 1, we can deduce that

$$\frac{(1-|z|^2)^\alpha(1-|w|^2)^\beta}{|z-w|} \ge C_1\left(1-|z|^2\right)^N.$$

Therefore, there exists a positive constant C_2 such that

$$\left(1-|z|^2\right)^{N+1}\left(\left|\nabla\left(\hat{D}^{(m)}f\right)\right| + \left|\overline{\nabla}\left(\hat{D}^{(m)}f\right)\right|\right) \le C_2 L_f,$$

from which we see that $f \in \mathcal{PB}(\mathbb{B}^n)$.

Now we prove the necessity. Let $w \in E(z,r)$, $w \neq z$. Then for each multi-index m with $|m| = N$, we have

$$\begin{aligned}
\bigl|\bigl(\hat{D}^{(m)}f\bigr)(z) - \bigl(\hat{D}^{(m)}f\bigr)(w)\bigr| &= \left|\int_0^1 \frac{d(\hat{D}^{(m)}f)}{ds}\bigl(sz + (1-s)w\bigr)\,ds\right| \\
&\leq \sum_{k=1}^{n}\left|(z_k - w_k)\int_0^1 \frac{\partial(\hat{D}^{(m)}f)}{\partial z_k}\bigl(sz + (1-s)w\bigr)\,ds\right| \\
&\quad + \sum_{k=1}^{n}\left|(\bar{z}_k - \bar{w}_k)\int_0^1 \frac{\partial(\hat{D}^{(m)}f)}{\partial \bar{z}_k}\bigl(sz + (1-s)w\bigr)\,ds\right| \\
&\leq \sqrt{n}|z-w|\int_0^1 \bigl(\bigl|\nabla\bigl(\hat{D}^{(m)}f\bigr)\bigl(sz + (1-s)w\bigr)\bigr| \\
&\quad + \bigl|\overline{\nabla}\bigl(\hat{D}^{(m)}f\bigr)\bigl(sz + (1-s)w\bigr)\bigr|\bigr)\,ds \\
&\leq C|z-w|\int_0^1 \frac{ds}{(1-|sz + (1-s)w|)^{N+1}}.
\end{aligned}$$

By Lemma 1 we infer that there exists $\iota > 0$ such that $1 - |w| = \iota(1 - |z|)$ and

$$\begin{aligned}
\frac{|(\hat{D}^{(m)}f)(z) - (\hat{D}^{(m)}f)(w)|}{|z-w|} &\leq C\int_0^1 \frac{ds}{(s(1-|z|) + (1-s)(1-|w|))^{N+1}} \\
&\leq \frac{C'}{(1-|z|^2)^{N+1}}\int_0^1 \frac{ds}{[s + \iota(1-s)]^{N+1}} \\
&\leq \frac{C''}{(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}}.
\end{aligned}$$

Thus,

$$L_f = \sup_{z\in\mathbb{B}^n, \rho(z,w)<r, z\neq w} \frac{(1-|z|^2)^{\alpha}(1-|w|^2)^{\beta}|\hat{D}^{(m)}f(z) - \hat{D}^{(m)}f(w)|}{|z-w|} < \infty.$$

So the proof is complete. □

Theorem 2 *Let $f \in \mathcal{P}(\mathbb{B}^n)$ and $N = 1, 2$. Then $f \in \mathcal{PB}(\mathbb{B}^n)$ if and only if*

$$\sup_{z,w\in\mathbb{B}^n, z\neq w} \bigl(1-|z|^2\bigr)^{\frac{N}{2}}\bigl(1-|w|^2\bigr)^{\frac{N}{2}}\left|\frac{(\hat{D}^{(m)}f)(z) - (\hat{D}^{(m)}f)(w)}{z-w}\right| < \infty$$

for all multi-index with $|m| = N - 1$.

Proof The sufficiency follows from Theorem 1. We only need to prove the necessity. When $N = 1$, we refer to [8, 11]. Now we prove $N = 2$. Let $f = h + \bar{g}$. Then for each $j \in \{1, \ldots, n\}$,

$$\begin{aligned}
&\sup_{z,w\in\mathbb{B}^n, z\neq w} \frac{(1-|z|^2)(1-|w|^2)}{|z-w|}\left|\frac{\partial f(z)}{\partial z_j} + \frac{\partial f(z)}{\partial \bar{z}_j} - \frac{\partial f(w)}{\partial z_j} - \frac{\partial f(w)}{\partial \bar{z}_j}\right| \\
&\quad\leq \sup_{z,w\in\mathbb{B}^n, z\neq w} \frac{(1-|z|^2)(1-|w|^2)}{|z-w|}\left(\left|\frac{\partial h(z)}{\partial z_j} - \frac{\partial h(w)}{\partial z_j}\right| + \left|\frac{\partial g(z)}{\partial z_j} - \frac{\partial g(w)}{\partial z_j}\right|\right).
\end{aligned}$$

Since $f \in \mathcal{PB}(\mathbb{B}^n)$, $h, g \in \mathcal{B}$, by Lemma 4,

$$\begin{aligned}
&\sup_{z,w\in\mathbb{B}^n, z\neq w} \frac{(1-|z|^2)(1-|w|^2)}{|z-w|}\left|\frac{\partial h(z)}{\partial z_j} - \frac{\partial h(w)}{\partial z_j}\right| < \infty, \\
&\quad\leq \sup_{z,w\in\mathbb{B}^n, z\neq w} \frac{(1-|z|^2)(1-|w|^2)}{|z-w|}\left|\frac{\partial g(z)}{\partial z_j} - \frac{\partial g(w)}{\partial z_j}\right| < \infty.
\end{aligned}$$

This completes the proof. □

Theorem 3 *Let $f \in \mathcal{PB}(\mathbb{B}^n)$, $N \geq 0$ be an integer and $0 < r < 1$. Then $f \in \mathcal{PB}_0(\mathbb{B}^n)$ if and only if*

$$\lim_{|z|\to 1^-} \sup_{z\in\mathbb{B}^n, \rho(z,w)<r, z\neq w} \frac{(1-|z|^2)^\alpha (1-|w|^2)^\beta |\hat{D}^{(m)} f(z) - \hat{D}^{(m)} f(w)|}{|z-w|} = 0 \tag{1}$$

for all values of the multi-index m with $|m| = N$, where $\alpha + \beta = N + 1$.

Proof Sufficiency. Assume that (1) holds. Then for any $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\sup_{z\in\mathbb{B}^n, \rho(z,w)<r, z\neq w} \frac{(1-|z|^2)^\alpha (1-|w|^2)^\beta |\hat{D}^{(m)} f(z) - \hat{D}^{(m)} f(w)|}{|z-w|} < \epsilon$$

whenever $\delta < |z| < 1$. It follows from an argument similar to the proof of Theorem 1, that we have

$$\begin{aligned}
&\bigl(1-|z|^2\bigr)^{N+1}\bigl(\bigl|\nabla\bigl(\hat{D}^{(m)} f\bigr)\bigr| + \bigl|\overline{\nabla}\bigl(\hat{D}^{(m)} f\bigr)\bigr|\bigr) \\
&\quad\leq C \sup_{z\in\mathbb{B}^n, \rho(z,w)<r, z\neq w} \frac{(1-|z|^2)^\alpha (1-|w|^2)^\beta |\hat{D}^{(m)} f(z) - \hat{D}^{(m)} f(w)|}{|z-w|} < C\epsilon,
\end{aligned}$$

whenever $\delta < |z| < 1$. Hence

$$\lim_{|z|\to 1^-} \bigl(1-|z|^2\bigr)^{N+1}\bigl(\bigl|\nabla\bigl(\hat{D}^{(m)} f\bigr)\bigr| + \bigl|\overline{\nabla}\bigl(\hat{D}^{(m)} f\bigr)\bigr|\bigr) = 0,$$

from which we see that $f \in \mathcal{PB}_0(\mathbb{B}^n)$.

Necessity. For $\lambda \in (0,1)$, let $f_\lambda(z) = f(\lambda z)$. By Lemma 1 and the proof of Theorem 1, we see that for each multi-index m with $|m| = N$,

$$\begin{aligned}
&\frac{(1-|z|^2)^\alpha (1-|w|^2)^\beta |\hat{D}^{(m)}(f-f_\lambda)(z) - \hat{D}^{(m)}(f-f_\lambda)(w)|}{|z-w|} \\
&\quad\leq C_1\bigl(1-|\xi|^2\bigr)^{N+1}\bigl(\bigl|\nabla\hat{D}^{(m)}(f-f_\lambda)(\xi)\bigr| + \bigl|\overline{\nabla}\hat{D}^{(m)}(f-f_\lambda)(\xi)\bigr|\bigr)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{(1-|z|^2)^\alpha (1-|w|^2)^\beta |\hat{D}^{(m)} f_\lambda(z) - \hat{D}^{(m)} f_\lambda(w)|}{|z-w|} \\
&\quad\leq \frac{C_2\lambda}{(1-|\lambda|^2)^{N+1}}\bigl(1-|\eta|^2\bigr)^{N+1}\bigl(\bigl|\nabla\bigl(\hat{D}^{(m)} f_\lambda\bigr)(\eta)\bigr| + \bigl|\overline{\nabla}\bigl(\hat{D}^{(m)} f_\lambda\bigr)(\eta)\bigr|\bigr)
\end{aligned}$$

for all $z, w \in \mathbb{B}^n$, $\rho(z,w) < r$ and $\xi, \eta \in E(z,r)$. So

$$\begin{aligned} L_f &\leq C_1\big(1-|\xi|^2\big)^{N+1}\big(\big|\nabla\hat{D}^{(m)}(f-f_\lambda)(\xi)\big| + \big|\overline{\nabla}\hat{D}^{(m)}(f-f_\lambda)(\xi)\big|\big) \\ &\quad + \frac{C_2\lambda}{(1-|\lambda|^2)^{N+1}}\big(1-|\eta|^2\big)^{N+1}\big(\big|\nabla\big(\hat{D}^{(m)}f_\lambda\big)(\eta)\big| + \big|\overline{\nabla}\big(\hat{D}^{(m)}f_\lambda\big)(\eta)\big|\big). \end{aligned}$$

First letting $|z| \to 1^-$ and then letting $\lambda \to 1^-$, we obtain the desired result. □

From Theorem 2 and the proof of Theorem 3, we have the following.

Corollary 2 *Let $f \in \mathcal{PB}(\mathbb{B}^n)$ and $N = 1, 2$. Then $f \in \mathcal{PB}_0(\mathbb{B}^n)$ if and only if*

$$\lim_{|z|\to 1^-} \sup_{z,w\in\mathbb{B}^n, z\neq w} \big(1-|z|^2\big)^{\frac{N}{2}}\big(1-|w|^2\big)^{\frac{N}{2}}\left|\frac{(\hat{D}^{(m)}f)(z) - (\hat{D}^{(m)}f)(w)}{z-w}\right| = 0$$

for all multi-index with $|m| = N-1$.

4 The Besov space for pluriharmonic mappings

In order to state and prove our next result, we need the following lemmas.

Lemma 5 *Let $f \in \mathcal{P}(\mathbb{B}^n)$. Then $f \in \mathcal{B}_p$ if and only if*

$$\sup_{z\in\mathbb{B}^n}\big(1-|z|^2\big)^{N+1}\big(\big|\nabla\big(\hat{D}^{(m)}f\big)\big| + \big|\overline{\nabla}\big(\hat{D}^{(m)}f\big)\big|\big) \in L^p\big(\mathbb{B}^n, d\lambda\big)$$

for all values of the multi-index m with $|m| = N$, and $p(N+1) \geq n$.

Proof This follows from [12], Theorem 6.1. □

Lemma 6 *Let h be holomorphic in $\mathbb{B}^n$ and $0 < r < 1$. Then there exist constants $K > 0$, $r < r' < 1$ such that*

$$\sup_{z\in\mathbb{B}^n, \rho(z,w)<r, z\neq w}\left|\frac{h(z)-h(w)}{z-w}\right| \leq K\int_{E(z,r')}\big|\nabla h(u)\big|\, d\lambda(u).$$

Proof By the subharmonicity and Lemma 1, for each $w \in \mathbb{B}^n$, we have

$$\begin{aligned} \sup_{z\in\mathbb{B}^n, \rho(z,w)<r, z\neq w}\left|\frac{h(z)-h(w)}{z-w}\right| &\leq C\sup_{\zeta\in E(z,r)}\big|\nabla h(\zeta)\big| \\ &\leq \frac{C}{|E(z,r')|}\int_{E(z,r')}\big|\nabla h(\zeta)\big|\, dv(\zeta) \\ &\leq K\int_{E(z,r')}\big|\nabla h(\zeta)\big|\, d\lambda(\zeta) \end{aligned}$$

for some $r' > r$. □

Theorem 4 *Let $f \in \mathcal{P}(\mathbb{B}^n)$, $N \geq 0$ be an integer and $0 < r < 1$. Then $f \in \mathcal{B}_p$ if and only if*

$$K_f = \int_{\mathbb{B}^n}\left(\sup_{z\in\mathbb{B}^n, \rho(z,w)<r, z\neq w}\frac{(1-|z|^2)^\alpha(1-|w|^2)^\beta|\hat{D}^{(m)}f(z) - \hat{D}^{(m)}f(w)|}{|z-w|}\right)^p d\lambda(z) < \infty$$

for all values of the multi-index m with $|m| = N$, where $\alpha + \beta = N+1$, and $p(N+1) \geq n$.

Proof Let $f = h + \bar{g} \in \mathcal{P}(\mathbb{B}^n)$. Suppose that

$$\int_{\mathbb{B}^n}\left(\sup_{z\in\mathbb{B}^n,\rho(z,w)<r,z\neq w}\frac{(1-|z^2|)^\alpha(1-|w^2|)^\beta|\hat{D}^{(m)}f(z)-\hat{D}^{(m)}f(w)|}{|z-w|}\right)^p d\lambda(z)<\infty.$$

Let

$$L_f(z)=\lim_{z\to w}\sup\frac{(1-|z|^2)^\alpha(1-|w|^2)^\beta|\hat{D}^{(m)}f(z)-\hat{D}^{(m)}f(w)|}{|z-w|}.$$

It follows from the proof of Theorem 1 that we have

$$\left(1-|z|^2\right)^{N+1}\left(\left|\nabla\left(\hat{D}^{(m)}f\right)(z)\right|+\left|\overline{\nabla}\left(\hat{D}^{(m)}f\right)(z)\right|\right)\leq CL_f(z).$$

Since $L_f(z)\leq L_f$, we see that

$$\begin{aligned}&\int\left(1-|z|^2\right)^{(N+1)p}\left(\left|\nabla\left(\hat{D}^{(m)}f\right)\right|+\left|\overline{\nabla}\left(\hat{D}^{(m)}f\right)\right|\right)^p d\lambda(z)\\&\quad\leq C\int_{\mathbb{B}^n}\left(\sup_{z\in\mathbb{B}^n,\rho(z,w)<r,z\neq w}\frac{(1-|z^2|)^\alpha(1-|w^2|)^\beta|\hat{D}^{(m)}f(z)-\hat{D}^{(m)}f(w)|}{|z-w|}\right)^p d\lambda(z),\end{aligned}$$

which yields $f\in\mathcal{B}_p$.

To prove the necessity, we suppose that $f = h+\bar{g}\in\mathcal{B}_p$. By Lemmas 1 and 6, for each multi-index m,

$$\begin{aligned}L_f&\leq\sup_{z\in\mathbb{B}^n,\rho(z,w)<r,z\neq w}\frac{(1-|z|^2)^\alpha(1-|w|^2)^\beta(|\partial^m h(z)-\partial^m h(w)|+|\partial^m g(z)-\partial^m g(w)|)}{|z-w|}\\&\leq C\sup_{z\in\mathbb{B}^n,\rho(z,w)<r,z\neq w}\frac{(1-|z|^2)^{N+1}|\partial^m h(z)-\partial^m h(w)|}{|z-w|}\\&\quad+C\sup_{z\in\mathbb{B}^n,\rho(z,w)<r,z\neq w}\frac{(1-|z|^2)^{N+1}|\partial^m g(z)-\partial^m g(w)|}{|z-w|}\\&\leq C_1\int_{E(z,r')}\left(1-|u|^2\right)^{N+1}\left(\left|\nabla\left(\partial^m h\right)(u)\right|+\left|\nabla\left(\partial^m g\right)(u)\right|\right)d\lambda(u).\end{aligned}$$

Since

$$\int_{E(z,r')}d\lambda(u)<\infty,$$

by Hölder's inequality and Fubini's theorem, we can obtain

$$\begin{aligned}K_f&\leq C\int_{\mathbb{B}^n}\left(\int_{E(z,r')}\left(1-|u|^2\right)^{N+1}\left(\left|\nabla\left(\partial^m h\right)(u)\right|+\left|\nabla\left(\partial^m g\right)(u)\right|\right)d\lambda(u)\right)^p d\lambda(z)\\&\leq C\int_{\mathbb{B}^n}\left(\int_{E(z,r')}\left(1-|u|^2\right)^{(N+1)p}\left(\left|\nabla\left(\partial^m h\right)(u)\right|+\left|\nabla\left(\partial^m g\right)(u)\right|\right)^p d\lambda(u)\right)d\lambda(z)\\&\leq C'\int_{\mathbb{B}^n}\left(1-|u|^2\right)^{(N+1)p}\left(\left|\nabla\left(\partial^m h\right)(u)\right|+\left|\nabla\left(\partial^m g\right)(u)\right|\right)^p d\lambda(u).\end{aligned}$$

It follows from Lemma 5 that K_f is bounded. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shaoxing University, Shaoxing, Zhejiang 312000, People's Republic of China. [2]School of Mathematics and Physics, University of South China, Hengyang, Hunan 421001, People's Republic of China.

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