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Optimality and duality results for a nondifferentiable multiobjective fractional programming problem

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Abstract

The purpose of this paper is to study a class of nondifferentiable multiobjective fractional programming problems in which every component of objective functions contains a term involving the support function of a compact convex set. For a differentiable function, we introduce the definition of higher-order $(C, \alpha, \gamma, \rho, d)$ -convex function. A nontrivial example is also constructed which is in this class but not $(F, \alpha, \gamma, \rho, d)$ -convex. Based on the $(C, \alpha, \gamma, \rho, d)$ -convexity, sufficient optimality conditions for an efficient solution of the nondifferentiable multiobjective fractional programming problem are established. Further, a higher-order Mond-Weir type dual is formulated for this problem and appropriate duality results are proved under higher-order $(C, \alpha, \gamma, \rho, d)$ -assumptions.

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Keywords: duality results; multiobjective programming problem; support function; KKT conditions; efficient solution

1 Introduction

Higher-order duality is significant due to its computational importance as it provides higher bounds whenever an approximation is used. By introducing two different functions, $h: R^n \times R^n \rightarrow R$ and $k: R^n \times R^n \rightarrow R^m$, Mangasarian [1] formulated a higher-order dual for a nonlinear optimization problem, $\{\min f(x), \text{ subject to } g(x) \leq 0\}$. Inspired by this concept, many researchers have worked in this direction. Chen [2] has formulated higher-order multiobjective symmetric dual programs and established duality relations under higher-order F -convexity assumptions. A higher-order vector optimization problem and its dual have been studied by Kassem [3].

In the last several years, various optimality and duality results have been obtained for multiobjective fractional programming problems. In Chen [2], multiobjective fractional problem and its duality theorems have been considered under higher-order (F, α, ρ, d) -convexity. Later on, Suneja *et al.* [4] discussed higher-order Mond-Weir and Schaible type nondifferentiable dual programs and their duality theorems under higher-order (F, ρ, σ) -type I -assumptions. Recently, Ying [5] has studied higher-order multiobjective symmetric fractional problem and formulated its Mond-Weir type dual. Further, duality results are obtained under higher-order (F, α, ρ, d) -convexity.

Yuan *et al.* [6] introduced a class of functions called (C, α, ρ, d) -convex functions and derived duality theorems for a nondifferentiable minimax fractional programming problem under (C, α, ρ, d) -convexity. Chinchuluun *et al.* [7] later studied nonsmooth multiobjective fractional programming problems in the framework of (C, α, ρ, d) -convexity.

In this paper, we first introduce the definition of higher-order $(C, \alpha, \gamma, \rho, d)$ -convex with respect to a differentiable function $H : X \times R^n \rightarrow R$ ($X \subset R^n$), $p, s \in R^n$ and construct a nontrivial example which is higher-order $(C, \alpha, \gamma, \rho, d)$ -convex but not a $(F, \alpha, \gamma, \rho, d)$ -convex function. We prove that the ratio of higher-order $(C, \alpha, \gamma, \rho, d)$ -convex functions is also higher-order $(\bar{C}, \bar{\alpha}, \bar{\gamma}, \bar{\rho}, \bar{d})$ -convex. A sufficient optimality condition related to the efficient solution of a multiobjective nondifferentiable fractional problem has been established. Finally, we formulate a higher-order Mond-Weir type dual problem corresponding to the multiobjective nondifferentiable fractional programming problem and established usual duality relations under the aforesaid assumptions.

2 Preliminaries

Definition 2.1 A function $C : X \times X \times R^n \rightarrow R$ ($X \subset R^n$) is said to be convex on R^n iff, for any fixed $(x, u) \in X \times X$ and for any $x_1, x_2 \in R^n$,

$$C_{x,u}(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda C_{x,u}(x_1) + (1 - \lambda)C_{x,u}(x_2), \quad \forall \lambda \in (0, 1).$$

We now introduce the definition of higher-order $(C, \alpha, \gamma, \rho, d)$ -convex function. Let C be a convex function with respect to the third variable such that $C_{x,u}(0) = 0, \forall (x, u) \in X \times X$. Let $H : X \times R^n \rightarrow R, \phi : X \rightarrow R$ be differentiable functions on X . Assume that $\alpha, \gamma : X \times X \rightarrow R_+ \setminus \{0\}, \rho \in R, d : X \times X \rightarrow R_+$ satisfying $d(x, x_0) = 0 \Leftrightarrow x = x_0$ and $p, s \in R^n$.

Definition 2.2 The function ϕ is said to be higher-order (strictly) $(C, \alpha, \gamma, \rho, d)$ -convex at u with respect to H, p and s if for each $x \in X$,

$$\begin{aligned} \frac{1}{\alpha(x, u)} [\phi(x) - \phi(u)] &\geq (>) C_{x,u}(\nabla \phi(u) + \nabla_p H(u, p)) \\ &+ \frac{1}{\gamma(x, u)} [H(u, s) - s^T \nabla_s H(u, s)] + \frac{\rho d(x, u)}{\alpha(x, u)}. \end{aligned}$$

Remark 2.1 A differentiable function $f = (f_1, f_2, \dots, f_k) : X \rightarrow R^k$ is $(C, \alpha, \gamma, \rho, d)$ -convex if for all $i = 1, 2, \dots, k, f_i$ is $(C, \alpha_i, \gamma_i, \rho_i, d_i)$ -convex.

Remark 2.2

- (i) If $H(u, \cdot) = 0$, then Definition 2.2 becomes that of a (C, α, ρ, d) -convex function as given in [6, 8]. Further if $\alpha(x, u) = 1, \rho = 0$, and $C_{x,u}(a) = \eta^T(x, u)a$, for $\eta : X \times X \rightarrow R^n$, then $(C, \alpha, \gamma, \rho, d)$ -convexity reduces to invexity (see Hanson [9]).
- (ii) If $H(u, \cdot) = \frac{1}{2}(\cdot)^T \nabla^2 f(u)(\cdot), \alpha(x, u) = \gamma(x, u)$, and $p = s$, then Definition 2.2 reduces to the first expression of a second-order (C, α, ρ, d) -type I-convex function given in Gupta *et al.* [10].
- (iii) If $\alpha(x, u) = \gamma(x, u) = 1, \rho = 0, H(u, \cdot) = \frac{1}{2}(\cdot)^T \nabla^2 f(u)(\cdot), p = s$, and $C_{x,u}(a) = \eta^T(x, u)a$, where $\eta : X \times X \rightarrow R^n$, the above definition becomes that of η -bonvexity introduced by Pandey [11].

(iv) If C is sublinear with respect to third variable and $p = s$, then the above definition reduces to higher-order $(F, \alpha, \gamma, \rho, d)$ -convexity as given in Gulati and Saini [12]. Furthermore, if $\gamma(x, u) = \alpha(x, u)$, then Definition 2.2 reduces to higher-order (F, α, ρ, d) -convexity as in Ying [5]. Moreover, if $H(u, \cdot) = 0$, then Definition 2.2 becomes that of a (F, α, ρ, d) -convex function as introduced by Liang *et al.* [13].

Remark 2.3 Every $(F, \alpha, \gamma, \rho, d)$ -convex function is $(C, \alpha, \gamma, \rho, d)$ -convex. However, the converse need not be true. This is illustrated by the following example.

Example 2.1 Let $X = \{x : x \geq 1\} \subset R, f : X \rightarrow R, C : X \times X \times R \rightarrow R, H : X \times R \rightarrow R,$ and $d : X \times X \rightarrow R_+$ be defined as

$$f(x) = \frac{x^2 + x + 2}{x + 1}, \quad C_{x,u}(a) = a^2(x - u)^2, \quad H(u, \cdot) = \frac{(\cdot)}{u + 1}, \quad d(x, u) = (x - u)^2.$$

Clearly, the function C (defined above) is convex with respect to the third variable satisfying $C_{x,u}(0) = 0, \forall x, u \in X$. Also, $d(x, u) = 0 \Leftrightarrow x = u$.

Let $\rho = -1$ and $\alpha = \gamma = \frac{3}{4}$. Then at $u = 1$, for all $x \in X$, we have

$$\begin{aligned} \frac{1}{\alpha(x, u)} [f(x) - f(u)] &= \frac{4}{3} \left(\frac{x^2 - x}{x + 1} \right) \geq C_{x,u}(\nabla_x f(u) + \nabla_p H(u, p)) \\ &+ \frac{1}{\gamma(x, u)} [H(u, s) - s^T \nabla_s H(u, s)] + \frac{\rho d(x, u)}{\alpha(x, u)} = -\frac{1}{3} (x - 1)^2. \end{aligned}$$

Hence, f is a higher-order $(C, \alpha, \gamma, \rho, d)$ -convex function with respect to H, p , and s . But considering the same C, f is not higher-order $(F, \alpha, \gamma, \rho, d)$ -convex because C is not a sub-linear functional with respect to the third variable.

Definition 2.3 [14] Let C be a compact convex set in R^n . The support function of C is defined by

$$S(x|C) = \max \{x^T y : y \in C\}.$$

3 Problem formulation and optimality conditions

Consider the following nondifferentiable multiobjective programming problem:

$$\begin{aligned} \text{(MFP) Minimize } F(x) &= \left\{ \frac{f_1(x) + S(x|C_1)}{g_1(x) - S(x|D_1)}, \frac{f_2(x) + S(x|C_2)}{g_2(x) - S(x|D_2)}, \dots, \frac{f_k(x) + S(x|C_k)}{g_k(x) - S(x|D_k)} \right\} \\ \text{subject to } x \in X^0 &= \{x \in X \subset R^n : h_j(x) + S(x|E_j) \leq 0, j = 1, 2, \dots, m\}, \end{aligned}$$

where $f_i, g_i : X \rightarrow R (i = 1, 2, \dots, k)$ and $h_j : X \rightarrow R (j = 1, 2, \dots, m)$ are continuously differentiable functions. Assume that $f_i(\cdot) + S(\cdot|C_i) \geq 0$ and $g_i(\cdot) - S(\cdot|D_i) > 0; C_i, D_i,$ and E_j are compact convex sets in R^n and $S(x|C_i), S(x|D_i),$ and $S(x|E_j)$ denote the support functions of compact convex sets, $C_i, D_i,$ and $E_j, i = 1, 2, \dots, k, j = 1, 2, \dots, m,$ respectively.

Definition 3.1 [15] A point $x^0 \in X^0$ is a weakly efficient solution of (MFP), if there exists no $x \in X^0$ such that for every $i = 1, 2, \dots, k,$ $\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} < \frac{f_i(x^0) + S(x^0|C_i)}{g_i(x^0) - S(x^0|D_i)}.$

Definition 3.2 [16] A point $x^0 \in X^0$ is an efficient solution (or a Pareto optimal solution) of (MFP), if there exists no $x \in X^0$ such that for every $i = 1, 2, \dots, k$, $\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(x^0) + S(x^0|C_i)}{g_i(x^0) - S(x^0|D_i)}$ and for some $r = 1, 2, \dots, k$, $\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(x^0) + S(x^0|C_r)}{g_r(x^0) - S(x^0|D_r)}$.

Lemma 3.1 [17] If for a given $\bar{\lambda}_i > 0$ ($\bar{\lambda}_i \geq 0$ and $\bar{\lambda} \neq 0$), $i = 1, 2, \dots, k$, $\bar{x} \in X$ is an optimal solution for the following single-objective problem:

$$\begin{aligned}
 (\text{FP}_{\bar{\lambda}}) \quad & \text{minimize} \quad G(x) = \sum_{i=1}^k \bar{\lambda}_i \left(\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \right) \\
 & \text{subject to} \quad x \in X \subset R^n,
 \end{aligned}$$

then \bar{x} is an efficient solution (a weakly efficient solution) for (MFP).

Theorem 3.1 For some t , assume $f_t(\cdot) + (\cdot)^T z_t$ and $-(g_t(\cdot) - (\cdot)^T v_t)$ are $(C, \alpha_t, \gamma_t, \rho_t, d_t)$ -convex at $u \in X$ with respect to H_t, p , and s . Then $\left(\frac{f_t(\cdot) + (\cdot)^T z_t}{g_t(\cdot) - (\cdot)^T v_t}\right)$ is also higher-order $(\bar{C}, \bar{\alpha}_t, \bar{\gamma}_t, \bar{\rho}_t, \bar{d}_t)$ -convex at $u \in X$ with respect to \bar{H}_t, p , and s , where

$$\begin{aligned}
 \bar{\alpha}_t(x, u) &= \left(\frac{g_t(u) - u^T v_t}{g_t(x) - x^T v_t} \right) \alpha_t(x, u), & \bar{\gamma}_t(x, u) &= \gamma_t(x, u), \\
 \bar{\rho}_t &= \rho_t \left(1 + \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right), & \bar{H}_t(u, \cdot) &= \left(\frac{1}{(g_t(u) - u^T v_t)} + \frac{f_t(u) + u^T z_t}{(g_t(u) - u^T v_t)^2} \right) H_t(u, \cdot), \\
 \bar{d}_t(x, u) &= \left(\frac{d_t(x, u)}{g_t(x) - x^T v_t} \right),
 \end{aligned}$$

and

$$\begin{aligned}
 \bar{C}_{x,u}(a) &= \left(\frac{f_t(u) + u^T z_t + g_t(u) - u^T v_t}{(g_t(u) - u^T v_t)^2} \right) C_{x,u} \left(\frac{(g_t(u) - u^T v_t)^2 a}{f_t(u) + u^T z_t + g_t(u) - u^T v_t} \right), \\
 a &= \nabla \left(\frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) + \nabla_p \bar{H}_t(u, p).
 \end{aligned}$$

Proof For any $x, u \in X$,

$$\begin{aligned}
 \left(\frac{f_t(x) + x^T z_t}{g_t(x) - x^T v_t} - \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) &= \left(\frac{f_t(x) + x^T z_t - f_t(u) - u^T z_t}{g_t(x) - x^T v_t} \right) \\
 &\quad - [f_t(u) + u^T z_t] \left(\frac{g_t(x) - x^T v_t - g_t(u) + u^T v_t}{(g_t(x) - x^T v_t)(g_t(u) - u^T v_t)} \right).
 \end{aligned}$$

Since $f_t(\cdot) + (\cdot)^T z_t$ and $-(g_t(\cdot) - (\cdot)^T v_t)$ are $(C, \alpha_t, \gamma_t, \rho_t, d_t)$ -convex at $u \in X$ with respect to H_t, p , and s , we have

$$\begin{aligned}
 & \frac{1}{\alpha_t(x, u)} \left(\frac{f_t(x) + x^T z_t}{g_t(x) - x^T v_t} - \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) \\
 & \geq \frac{1}{(g_t(x) - x^T v_t)} \left\{ C_{x,u}(\nabla(f_t(u) + u^T z_t) + \nabla_p H_t(u, p)) \right. \\
 & \quad \left. + \frac{1}{\gamma_t(x, u)} [H_t(u, s) - s^T \nabla_s H_t(u, s)] + \frac{\rho_t d_t(x, u)}{\alpha_t(x, u)} \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{f_t(u) + u^T z_t}{(g_t(x) - x^T v_t)(g_t(u) - u^T v_t)} \left\{ C_{x,u}(-\nabla(g_t(u) - u^T v_t) + \nabla_p H_t(u, p)) \right. \\
 & \left. + \frac{1}{\gamma_t(x, u)} [H_t(u, s) - s^T \nabla_s H_t(u, s)] + \frac{\rho_t d_t(x, u)}{\alpha_t(x, u)} \right\},
 \end{aligned}$$

which implies

$$\begin{aligned}
 & \frac{1}{\alpha_t(x, u)} \left(\frac{f_t(x) + x^T z_t}{g_t(x) - x^T v_t} - \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) \\
 & \geq \left(\frac{f_t(u) + u^T z_t + g_t(u) - u^T v_t}{(g_t(x) - x^T v_t)(g_t(u) - u^T v_t)} \right) \left(\frac{g_t(u) - u^T v_t}{f_t(u) + u^T z_t + g_t(u) - u^T v_t} \right) \\
 & \quad \times \left\{ C_{x,u}(\nabla(f_t(u) + u^T z_t) + \nabla_p H_t(u, p)) \right. \\
 & \quad \left. + \frac{1}{\gamma_t(x, u)} [H_t(u, s) - s^T \nabla_s H_t(u, s)] + \frac{\rho_t d_t(x, u)}{\alpha_t(x, u)} \right\} \\
 & \quad + \left(\frac{f_t(u) + u^T z_t + g_t(u) - u^T v_t}{(g_t(x) - x^T v_t)(g_t(u) - u^T v_t)} \right) \left(\frac{f_t(u) + u^T z_t}{f_t(u) + u^T z_t + g_t(u) - u^T v_t} \right) \\
 & \quad \times \left\{ C_{x,u}(-\nabla(g_t(u) - u^T v_t) + \nabla_p H_t(u, p)) \right. \\
 & \quad \left. + \frac{1}{\gamma_t(x, u)} [H_t(u, s) - s^T \nabla_s H_t(u, s)] + \frac{\rho_t d_t(x, u)}{\alpha_t(x, u)} \right\}.
 \end{aligned}$$

This further yields

$$\begin{aligned}
 & \frac{1}{\alpha_t(x, u)} \left(\frac{f_t(x) + x^T z_t}{g_t(x) - x^T v_t} - \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) \\
 & \geq \left(\frac{f_t(u) + u^T z_t + g_t(u) - u^T v_t}{(g_t(x) - x^T v_t)(g_t(u) - u^T v_t)} \right) \\
 & \quad \times \left\{ C_{x,u} \left(\frac{(g_t(u) - u^T v_t)^2}{f_t(u) + u^T z_t + g_t(u) - u^T v_t} \left(\nabla \left(\frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) \right. \right. \right. \\
 & \quad \left. \left. + \left(\frac{1}{g_t(u) - u^T v_t} + \frac{f_t(u) + u^T z_t}{(g_t(u) - u^T v_t)^2} \right) \nabla_p H_t(u, p) \right) \right\} + \frac{1}{\gamma_t(x, u)(g_t(x) - x^T v_t)} \\
 & \quad \times \left[\left(1 + \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) (H_t(u, s) - s^T \nabla_s H_t(u, s)) \right] \\
 & \quad + \frac{\rho_t d_t(x, u)}{\alpha_t(x, u)(g_t(x) - x^T v_t)} \left(1 + \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right).
 \end{aligned}$$

Multiplying by $\left(\frac{g_t(x) - x^T v_t}{g_t(u) - u^T v_t}\right)$, the above inequality gives

$$\begin{aligned}
 & \frac{(g_t(x) - x^T v_t)}{\alpha_t(x, u)(g_t(u) - u^T v_t)} \left(\frac{f_t(x) + x^T z_t}{g_t(x) - x^T v_t} - \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) \\
 & \geq \left(\frac{f_t(u) + u^T z_t + g_t(u) - u^T v_t}{(g_t(u) - u^T v_t)^2} \right) \\
 & \quad \times \left\{ C_{x,u} \left(\frac{(g_t(u) - u^T v_t)^2}{f_t(u) + u^T z_t + g_t(u) - u^T v_t} \left(\nabla \left(\frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) \right. \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\frac{1}{g_t(u) - u^T v_t} + \frac{f_t(u) + u^T z_t}{(g_t(u) - u^T v_t)^2} \right) \nabla_p H_t(u, p) \Big) \Big\} \\
 & + \frac{1}{\gamma_t(x, u)(g_t(u) - u^T v_t)} \left[\left(1 + \frac{f_t(u) + u^T z_t}{(g_t(u) - u^T v_t)} \right) (H_t(u, s) - s^T \nabla_s H_t(u, s)) \right] \\
 & + \frac{\rho_t d_t(x, u)}{\alpha_t(x, u)(g_t(u) - u^T v_t)} \left(1 + \frac{f_t(u) + u^T z_t}{(g_t(u) - u^T v_t)} \right).
 \end{aligned}$$

Setting

$$\begin{aligned}
 \bar{\alpha}_t(x, u) &= \left(\frac{g_t(u) - u^T v_t}{g_t(x) - x^T v_t} \right) \alpha_t(x, u), & \bar{\gamma}_t(x, u) &= \gamma_t(x, u), \\
 \bar{\rho}_t &= \rho_t \left(1 + \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right), \\
 \bar{H}_t(u, \cdot) &= \left(\frac{1}{(g_t(u) - u^T v_t)} + \frac{f_t(u) + u^T z_t}{(g_t(u) - u^T v_t)^2} \right) H_t(u, \cdot), & \bar{d}_t(x, u) &= \left(\frac{d_t(x, u)}{g_t(x) - x^T v_t} \right), \\
 \bar{C}_{(x, u)}(a) &= \left(\frac{f_t(u) + u^T z_t + g_t(u) - u^T v_t}{(g_t(u) - u^T v_t)^2} \right) C_{x, u} \left(\frac{(g_t(u) - u^T v_t)^2 a}{f_t(u) + u^T z_t + g_t(u) - u^T v_t} \right),
 \end{aligned}$$

and

$$a = \nabla \left(\frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) + \nabla_p \bar{H}_t(u, p).$$

It follows that

$$\begin{aligned}
 & \frac{1}{\bar{\alpha}_t(x, u)} \left(\frac{f_t(x) + x^T z_t}{g_t(x) - x^T v_t} - \frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) \\
 & \geq \left\{ \bar{C}_{(x, u)} \left(\nabla \left(\frac{f_t(u) + u^T z_t}{g_t(u) - u^T v_t} \right) + \nabla_p \bar{H}_t(u, p) \right) \right. \\
 & \quad \left. + \frac{1}{\bar{\gamma}_t(x, u)} \left[\bar{H}_t(u, s) - s^T \nabla_s \bar{H}_t(u, s) \right] + \frac{\bar{\rho}_t \bar{d}_t(x, u)}{\bar{\alpha}_t(x, u)} \right\}.
 \end{aligned}$$

Hence, $\left(\frac{f_t(\cdot) + (\cdot)^T z_t}{g_t(\cdot) - (\cdot)^T v_t} \right)$ is higher-order $(\bar{C}, \bar{\alpha}_t, \bar{\gamma}_t, \bar{\rho}_t, \bar{d}_t)$ -convex at $u \in X$ with respect to \bar{H}_t, p , and s . □

Theorem 3.2 *Let $X \subset R^n$ be an open convex set. Let $\psi_i : X \rightarrow R$ be higher-order $(C, \alpha_i, \gamma_i, \rho_i, d_i)$ -convex at one point in X with respect to $\phi_i, \rho_i \geq 0$, and for all $x \in X, \phi_i(x, \cdot) = -(\cdot)^T \nabla \psi_i(x), i = 1, 2, \dots, k$; then only one of the following two cases holds:*

- (i) *there exists $x \in X$ such that $\psi_i(x) < 0, i = 1, 2, \dots, k$;*
- (ii) *there exists $\lambda \in R_+^k \setminus \{0\}$ such that $\sum_{i=1}^k \lambda_i \psi_i(x) \geq 0$, for all $x \in X$.*

Proof If (i) has a solution, that is, there exists $x \in X$ such that $\psi_i(x) < 0, i = 1, 2, \dots, k$, then for every $\lambda \in R_+^k \setminus \{0\}$, we have $\sum_{i=1}^k \lambda_i \psi_i(x) < 0$, which implies that (ii) does not hold.

If (i) has no solution, let $K = \psi(X) + \text{int}(R_+^k), \psi = (\psi_1, \psi_2, \dots, \psi_k)^T$, then $K \cap (-R_+^k) = \phi$. For any $z^1, z^2 \in K$ and $\beta \in (0, 1)$, there exist $x^1, x^2 \in X$ and $s^1, s^2 \in \text{int}(R_+^k)$ such that

$$\beta z^1 + (1 - \beta) z^2 = \beta \psi(x^1) + (1 - \beta) \psi(x^2) + \beta s^1 + (1 - \beta) s^2.$$

Since $u = \beta x^1 + (1 - \beta)x^2 \in X$ and $\phi_i(x, \cdot) = -(\cdot)^T \nabla \psi_i(x)$, it follows from the higher-order $(C, \alpha_i, \gamma_i, \rho_i, d_i)$ -convexity of ψ_i with respect to ϕ_i, p , and s that we have

$$\begin{aligned} & \frac{1}{\alpha_i(x^j, u)} [\psi_i(x^j) - \psi_i(u)] \\ & \geq C_{x,u} [\nabla \psi_i(u) + \nabla_p \phi_i(u, p)] + \frac{1}{\gamma_i(x^j, u)} \{ \phi_i(u, s) - s^T [\nabla_s \phi_i(u, s)] \} + \frac{\rho_i d(x^j, u)}{\alpha_i(x^j, u)} \\ & = \frac{1}{\alpha_i(x^j, u)} \rho_i d(x^j, u), \quad j = 1, 2. \end{aligned}$$

Thus, there exists $t \geq 0$ such that

$$\beta \psi_i(x^1) + (1 - \beta) \psi_i(x^2) = \psi_i(\beta x^1 + (1 - \beta)x^2) + t,$$

which implies that

$$\beta z^1 + (1 - \beta)z^2 = \beta \psi(x^1) + (1 - \beta)\psi(x^2) + \beta s^1 + (1 - \beta)s^2 + (t, t, \dots, t)^T.$$

Since R_+^k is a closed convex cone, we have $\beta z_1 + (1 - \beta)z_2 \in K$, that is, K is an open convex set. It follows from the convex separated theorem that there exists $\lambda \in R_+^k \setminus \{0\}$ such that, for all $x \in X$, $\sum_{i=1}^k \lambda_i \psi_i(x) \geq 0$. □

We now discuss some optimality conditions for the problem (MFP).

Theorem 3.3 *If $u \in X$ is a weakly efficient solution of (MFP), $f_i(\cdot) + (\cdot)^T z_i$, and $-(g_i(\cdot) - (\cdot)^T v_i)$ are higher-order $(C, \alpha_i, \gamma_i, \rho_i, d_i)$ -convex functions at $u \in X$ with respect to $\phi_i(u, \cdot) = -\frac{(g_i(u) - u^T v_i)^2}{(f_i(u) + u^T z_i + g_i(u) - u^T v_i)} \times (\cdot)^T \nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right)$, and $\rho_i \geq 0, i = 1, 2, \dots, k$. Then there exists $0 \leq \bar{\lambda} \in R^k, \bar{\lambda} \neq 0$ such that u is an optimal solution of $(FP_{\bar{\lambda}})$.*

Proof If $u \in X$ is a weakly efficient solution of (MFP), then there does not exist any $x \in X$ such that

$$\psi_j(x) = \left(\frac{f_j(x) + x^T z_j}{g_j(x) - x^T v_j} - \frac{f_j(u) + u^T z_j}{g_j(u) - u^T v_j} \right) < 0, \quad j = 1, 2, \dots, k.$$

Since $f_j(\cdot) + (\cdot)^T z_j$ and $-(g_j(\cdot) - (\cdot)^T v_j)$ are higher-order $(C, \alpha_j, \gamma_j, \rho_j, d_j)$ -convex at u with respect to ϕ_j, p and $s, j = 1, 2, \dots, k$, it follows from Theorem 3.1 that $\psi_j(x) = \left(\frac{f_j(x) + x^T z_j}{g_j(x) - x^T v_j} - \frac{f_j(u) + u^T z_j}{g_j(u) - u^T v_j} \right)$ is higher-order $(\bar{C}, \bar{\alpha}_j, \bar{\gamma}_j, \bar{\rho}_j, \bar{d}_j)$ -convex at u with respect to $\bar{\phi}_j, p$, and $s, j = 1, 2, \dots, k$, where

$$\begin{aligned} \bar{\alpha}_j(x, u) &= \left(\frac{g_j(x) - x^T v_j}{g_j(u) - u^T v_j} \right) \alpha_j(x, u), & \bar{\gamma}_j(x, u) &= \gamma_j(x, u), \\ \bar{\rho}_j &= \rho_j \left(1 + \frac{f_j(u) + u^T z_j}{g_j(u) - u^T v_j} \right), \\ \bar{\phi}_j(u, \cdot) &= \left(\frac{1}{g_j(u) - u^T v_j} + \frac{f_j(u) + u^T z_j}{(g_j(u) - u^T v_j)^2} \right) \phi_j(u, \cdot) = -(\cdot)^T \nabla \left(\frac{f_j(u) + u^T z_j}{g_j(u) - u^T v_j} \right) \\ &= -(\cdot)^T \nabla \psi_j(u), \end{aligned}$$

$$\bar{d}_j(x, u) = \left(\frac{d_j(x, u)}{g_j(x) - x^T v_j} \right),$$

and

$$\begin{aligned} \bar{C}_{x,u}(a) &= \frac{f_j(u) + u^T z_j + g_j(u) - u^T v_j}{(g_j(u) - u^T v_j)^2} C_{x,u} \left(\frac{(g_j(u) - u^T v_j)^2 a}{f_j(u) + u^T z_j + g_j(u) - u^T v_j} \right), \\ a &= \nabla \left(\frac{f_j(u) + u^T z_j}{g_j(u) - u^T v_j} \right) + \nabla_p \bar{\phi}_j(u, p). \end{aligned}$$

Using Theorem 3.2, there exists $0 \leq \bar{\lambda} \in R^k, \bar{\lambda} \neq 0$ such that $\sum_{j=1}^k \bar{\lambda}_j \psi_j(x) \geq 0$, that is,

$$\sum_{j=1}^k \bar{\lambda}_j \left(\frac{f_j(x) + x^T z_j}{g_j(x) - x^T v_j} \right) \geq \sum_{j=1}^k \bar{\lambda}_j \left(\frac{f_j(u) + u^T z_j}{g_j(u) - u^T v_j} \right),$$

which implies that u is an optimal solution of $(FP_{\bar{\lambda}})$. □

Theorem 3.4 (Necessary condition) [18] *Assume that \bar{x} is an efficient solution of (MFP) and the Slater constraint qualification is satisfied on X . Then there exist $\bar{\lambda} \in R^k, \bar{\mu} \in R^m, \bar{z}_i \in R^n, \bar{v}_i \in R^n$, and $\bar{w}_j \in R^n, i = 1, 2, \dots, k, j = 1, 2, \dots, m$, such that*

$$\begin{aligned} \sum_{i=1}^k \bar{\lambda}_i \nabla \left(\frac{f_i(\bar{x}) + \bar{x}^T \bar{z}_i}{g_i(\bar{x}) - \bar{x}^T \bar{v}_i} \right) + \sum_{j=1}^m \bar{\mu}_j \nabla (h_j(\bar{x}) + \bar{x}^T \bar{w}_j) &= 0, \\ \sum_{j=1}^m \bar{\mu}_j (h_j(\bar{x}) + \bar{x}^T \bar{w}_j) &= 0, \\ \bar{x}^T \bar{z}_i &= S(\bar{x}|C_i), \quad i = 1, 2, \dots, k, \\ \bar{x}^T \bar{v}_i &= S(\bar{x}|D_i), \quad i = 1, 2, \dots, k, \\ \bar{x}^T \bar{w}_j &= S(\bar{x}|E_j), \quad j = 1, 2, \dots, m, \\ \bar{z}_i \in C_i, \quad \bar{v}_i \in D_i, \quad \bar{w}_j \in E_j, \quad i = 1, 2, \dots, k, j = 1, 2, \dots, m, \\ \bar{\lambda}_i &> 0, \quad \bar{\mu}_j \geq 0, \quad i = 1, 2, \dots, k, j = 1, 2, \dots, m. \end{aligned}$$

Theorem 3.5 (Sufficient condition) *Let u be a feasible solution of (MFP). Assume that there exist $\lambda_i > 0, i = 1, 2, \dots, k$ and $\mu_j \geq 0, j = 1, 2, \dots, m$, such that*

$$\begin{aligned} \sum_{i=1}^k \lambda_i \nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{j=1}^m \mu_j \nabla (h_j(u) + u^T w_j) &= 0, \\ \sum_{j=1}^m \mu_j (h_j(u) + u^T w_j) &= 0, \\ u^T z_i &= S(u|C_i), \quad i = 1, 2, \dots, k, \\ u^T v_i &= S(u|D_i), \quad i = 1, 2, \dots, k, \\ u^T w_j &= S(u|E_j), \quad j = 1, 2, \dots, m, \end{aligned}$$

$$z_i \in C_i, \quad v_i \in D_i, \quad w_j \in E_j, \quad i = 1, 2, \dots, k, j = 1, 2, \dots, m.$$

Let for any $i = 1, 2, \dots, k, j = 1, 2, \dots, m,$

- (i) $(f_i(\cdot) + (\cdot)^T z_i)$ and $-(g_i(\cdot) - (\cdot)^T v_i)$ be higher-order $(C, \alpha_i, \gamma_i, \rho_i, d_i)$ -convex at u , with respect to H_i, p , and s ,
- (ii) $(h_j(\cdot) + (\cdot)^T w_j)$ be higher-order $(C, \xi_j, \delta_j, \eta_j, c_j)$ -convex at u , with respect to K_j, q , and r ,
- (iii) $\sum_{i=1}^k \lambda_i \bar{\rho}_i \frac{d_i(x,u)}{\bar{\alpha}_i(x,u)} + \sum_{j=1}^m \mu_j \eta_j \frac{c_j(x,u)}{\xi_j(x,u)} \geq 0$,
- (iv) $\bar{\gamma}_i(x, u) = \zeta(x, u), \xi_j(x, u) = \sigma(x, u)$, and $\delta_j(x, u) = \sigma'(x, u)$,
- (v) $\sum_{i=1}^k \lambda_i (\nabla_p \bar{H}_i(u, p)) + \sum_{j=1}^m \mu_j (\nabla_q K_j(u, q)) = 0, \sum_{i=1}^k \lambda_i (\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)) \geq 0$ and $\sum_{j=1}^m \mu_j (K_j(u, r) - r^T \nabla_r K_j(u, r)) \geq 0$,

where

$$\begin{aligned} \bar{\alpha}_i(x, u) &= \left(\frac{g_i(u) - u^T v_i}{g_i(x) - x^T v_i} \right) \alpha_i(x, u), \quad \bar{\gamma}_i(x, u) = \gamma_i(x, u), \quad \bar{\rho}_i = \rho_i \left(1 + \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right), \\ \bar{H}_i(u, \cdot) &= \left(\frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) H_i(u, \cdot), \quad \bar{d}_i(x, u) = \left(\frac{d_i(x, u)}{g_i(x) - x^T v_i} \right). \end{aligned}$$

Then u is an efficient solution of (MFP).

Proof Suppose u is not an efficient solution of (MFP), then there exists $x \in X^0$ such that

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)}, \quad \text{for all } i = 1, 2, \dots, k,$$

and

$$\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(u) + S(u|C_r)}{g_r(u) - S(u|D_r)}, \quad \text{for some } r = 1, 2, \dots, k,$$

which implies

$$\begin{aligned} \frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} &\leq \frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(u) + S(u|C_i)}{g_i(u) - S(u|D_i)} = \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}, \\ &\text{for all } i = 1, 2, \dots, k, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{f_r(x) + x^T z_r}{g_r(x) - x^T v_r} &\leq \frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(u) + S(u|C_r)}{g_r(u) - S(u|D_r)} = \frac{f_r(u) + u^T z_r}{g_r(u) - u^T v_r}, \\ &\text{for some } r = 1, 2, \dots, k. \end{aligned} \tag{2}$$

Since $\frac{\lambda_i}{\bar{\alpha}_i(x,u)} > 0, i = 1, 2, \dots, k$, the inequalities (1) and (2) give

$$\sum_{i=1}^k \frac{\lambda_i}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) < 0. \tag{3}$$

From Theorem 3.1, for each $i, 1 \leq i \leq k$, $\left(\frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}\right)$ is higher-order $(\bar{C}, \bar{\alpha}_i, \bar{\gamma}_i, \bar{\rho}_i, \bar{d}_i)$ -convex at $u \in X^0$ with respect to \bar{H}_i, p , and s , therefore

$$\begin{aligned} & \frac{1}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\ & \geq \bar{C}_{x,u} \left(\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \nabla_p \bar{H}_i(u, p) \right) \\ & \quad + \frac{1}{\bar{\gamma}_i(x, u)} [\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)] + \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \bar{\alpha}_i(x, u) &= \left(\frac{g_i(u) - u^T v_i}{g_i(x) - x^T v_i} \right) \alpha_i(x, u), & \bar{\gamma}_i(x, u) &= \gamma_i(x, u), \\ \bar{\rho}_i &= \rho_i \left(1 + \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right), \\ \bar{H}_i(u, \cdot) &= \left(\frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) H_i(u, \cdot), & \bar{d}_i(x, u) &= \left(\frac{d_i(x, u)}{g_i(x) - x^T v_i} \right), \end{aligned}$$

and

$$\begin{aligned} \bar{C}_{x,u}(a) &= \left(\frac{f_i(u) + u^T z_i + g_i(u) - u^T v_i}{(g_i(u) - u^T v_i)^2} \right) C_{x,u} \left(\frac{(g_i(u) - u^T v_i)^2 a}{f_i(u) + u^T z_i + g_i(u) - u^T v_i} \right), \\ a &= \nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \nabla_p \bar{H}_i(u, p). \end{aligned}$$

Also, by the higher-order $(C, \xi_j, \delta_j, \eta_j, c_j)$ -convexity of $(h_j(\cdot) + (\cdot)^T w_j)$ at u with respect to K_j, q , and $r, j = 1, 2, \dots, m$, we have

$$\begin{aligned} & \frac{1}{\xi_j(x, u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \\ & \geq C_{x,u} (\nabla(h_j(u) + u^T w_j) + \nabla_q K_j(u, q)) \\ & \quad + \frac{1}{\delta_j(x, u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] + \frac{\eta_j c_j(x, u)}{\xi_j(x, u)}. \end{aligned} \tag{5}$$

Let $\tau = \sum_{i=1}^k \lambda_i \frac{(f_i(u) + u^T z_i + g_i(u) - u^T v_i)}{(g_i(u) - u^T v_i)^2} + \sum_{j=1}^m \mu_j > 0$.

Adding the inequalities obtained by multiplying (4) by $\frac{\lambda_i}{\tau}$ and (5) by $\frac{\mu_j}{\tau}$, we get

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\ & \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \xi_j(x, u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \\ & \geq \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{(f_i(u) + u^T z_i + g_i(u) - u^T v_i)}{(g_i(u) - u^T v_i)^2} C_{x,u} \left[\left(\frac{(g_i(u) - u^T v_i)^2}{f_i(u) + u^T z_i + g_i(u) - u^T v_i} \right) \right] \end{aligned}$$

$$\begin{aligned} & \times \left(\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \left(\frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) \nabla_p H_i(u, p) \right) \\ & + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{1}{\bar{\gamma}_i(x, u)} \left(\frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) (H_i(u, s) - s^T \nabla_s H_i(u, s)) \Big] \\ & + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{\rho_i d_i(x, u)}{\bar{\alpha}_i(x, u)} \left(\frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) \\ & + \sum_{j=1}^m \frac{\mu_j}{\tau} C_{x, u} (\nabla(h_j(u) + u^T w_j) + \nabla_q K_j(u, q)) \\ & + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{1}{\delta_j(x, u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{\eta_j c_j(x, u)}{\xi_j(x, u)}. \end{aligned}$$

Further, using the convexity of C , we have

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\ & + \sum_{j=1}^m \frac{\mu_j}{\tau \xi_j(x, u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \\ & \geq C_{x, u} \left(\frac{1}{\tau} \left(\sum_{i=1}^k \lambda_i \nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{i=1}^k \lambda_i \nabla_p \bar{H}_i(u, p) \right) \right. \\ & \quad \left. + \sum_{j=1}^m \mu_j \nabla (h_j(u) + u^T w_j) + \sum_{j=1}^m \mu_j \nabla_q K_j(u, q) \right) \\ & + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{1}{\bar{\gamma}_i(x, u)} (\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)) + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)} \\ & + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{1}{\delta_j(x, u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{\eta_j c_j(x, u)}{\xi_j(x, u)}. \end{aligned}$$

It follows from hypotheses (iii)-(v) that

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\ & + \sum_{j=1}^m \frac{\mu_j}{\tau \sigma(x, u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \\ & \geq C_{x, u} \left(\frac{1}{\tau} \left(\sum_{i=1}^k \lambda_i \nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{j=1}^m \mu_j \nabla (h_j(u) + u^T w_j) \right) \right). \end{aligned}$$

Using the fact $C_{x, u}(0) = 0$ and $\sum_{j=1}^m \mu_j (h_j(u) + u^T w_j) = 0$, we get

$$\sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{i=1}^m \frac{\mu_j}{\tau \sigma(x, u)} [h_j(x) + x^T w_j] \geq 0.$$

Finally, using feasibility of the primal problem (MFP), we have

$$\sum_{i=1}^k \frac{\lambda_i}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \geq 0,$$

which contradicts (3). Therefore, u is an efficient solution of (MFP). □

4 Duality model

Consider the following higher-order Mond-Weir type dual (MFD) of (MFP):

$$\begin{aligned} \text{(MFD)} \quad & \text{maximize} \quad G(u) = \left\{ \frac{f_1(u) + u^T z_1}{g_1(u) - u^T v_1}, \frac{f_2(u) + u^T z_2}{g_2(u) - u^T v_2}, \dots, \frac{f_k(u) + u^T z_k}{g_k(u) - u^T v_k} \right\} \\ & \text{subject to} \\ & \sum_{i=1}^k \lambda_i \nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{j=1}^m \mu_j \nabla (h_j(u) + u^T w_j) \\ & \quad + \sum_{i=1}^k \lambda_i \nabla_p \bar{H}_i(u, p) + \sum_{j=1}^m \mu_j \nabla_q K_j(u, q) = 0, \end{aligned} \tag{6}$$

$$\sum_{j=1}^m \mu_j [h_j(u) + u^T w_j + K_j(u, r) - r^T \nabla_r K_j(u, r)] \geq 0, \tag{7}$$

$$\sum_{i=1}^k \lambda_i [\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)] \geq 0, \tag{8}$$

$$z_i \in C_i, \quad v_i \in D_i, \quad w_j \in E_j, \quad i = 1, 2, \dots, k, j = 1, 2, \dots, m,$$

$$\lambda_i > 0, \quad \mu_j \geq 0, \quad i = 1, 2, \dots, k, j = 1, 2, \dots, m.$$

We now discuss the duality results for the primal-dual pair (MFP) and (MFD).

Theorem 4.1 (Weak duality theorem) *Let $x \in X^0$ and $(u, z, v, \mu, \lambda, w, p, q, r, s)$ be feasible for (MFD). Suppose that:*

- (i) $(f_i(\cdot) + (\cdot)^T z_i)$ and $-(g_i(\cdot) - (\cdot)^T v_i)$ are higher-order $(C, \alpha_i, \gamma_i, \rho_i, d_i)$ -convex at u , with respect to H_i, p , and $s, i = 1, 2, \dots, k$,
- (ii) $(h_j(\cdot) + (\cdot)^T w_j)$ is higher-order $(C, \xi_j, \delta_j, \eta_j, c_j)$ -convex at u , with respect to K_j, q , and $r, j = 1, 2, \dots, m$,
- (iii) $\sum_{i=1}^k \lambda_i \bar{\rho}_i \frac{\bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)} + \sum_{j=1}^m \mu_j \eta_j \frac{c_j(x, u)}{\xi_j(x, u)} \geq 0$,
- (iv) $\bar{\gamma}_i(x, u) = \zeta(x, u)$ and $\xi_j(x, u) = \delta_j(x, u) = \sigma(x, u), i = 1, 2, \dots, k, j = 1, 2, \dots, m$,

where

$$\begin{aligned} \bar{\alpha}_i(x, u) &= \left(\frac{g_i(x) - x^T v_i}{g_i(u) - u^T v_i} \right) \alpha_i(x, u), \quad \bar{\rho}_i = \rho_i \left(1 + \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right), \\ \bar{d}_i(x, u) &= \left(\frac{d_i(x, u)}{g_i(x) - x^T v_i} \right). \end{aligned}$$

Then the following cannot hold:

$$\frac{f_i(x) + S(x|C_i)}{g_i(x) - S(x|D_i)} \leq \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i}, \quad \text{for all } i = 1, 2, \dots, k, \tag{9}$$

and

$$\frac{f_r(x) + S(x|C_r)}{g_r(x) - S(x|D_r)} < \frac{f_r(u) + u^T z_r}{g_r(u) - u^T v_r}, \quad \text{for some } r = 1, 2, \dots, k. \tag{10}$$

Proof Suppose that (9) and (10) hold, then using $\frac{\lambda_i}{\bar{\alpha}_i(x,u)} > 0, x^T z_i \leq S(x|C_i), x^T v_i \leq S(x|D_i), i = 1, 2, \dots, k$, we have

$$\sum_{i=1}^k \frac{\lambda_i}{\bar{\alpha}_i(x,u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) < 0. \tag{11}$$

From hypothesis (i) and Theorem 3.1, $\left(\frac{f_i(\cdot) + (\cdot)^T z_i}{g_i(\cdot) - (\cdot)^T v_i}\right)$ is higher-order $(\bar{C}, \bar{\alpha}_i, \bar{\gamma}_i, \bar{\rho}_i, \bar{d}_i)$ -convex at u with respect to \bar{H}_i, p , and $s, i = 1, 2, \dots, k$, therefore

$$\begin{aligned} & \frac{1}{\bar{\alpha}_i(x,u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\ & \geq \bar{C}_{x,u} \left(\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \nabla_p \bar{H}_i(u, p) \right) \\ & \quad + \frac{1}{\bar{\gamma}_i(x,u)} [\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)] + \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)}, \end{aligned} \tag{12}$$

and by the higher-order $(C, \xi_j, \delta_j, \eta_j, c_j)$ -convex of $(h_j(\cdot) + (\cdot)^T w_j)$ at u with respect to K_j, q , and $r, j = 1, 2, \dots, m$, we have

$$\begin{aligned} & \frac{1}{\xi_j(x,u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \\ & \geq C_{x,u} (\nabla (h_j(u) + u^T w_j) + \nabla_q K_j(u, q)) \\ & \quad + \frac{1}{\delta_j(x,u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] + \frac{\eta_j c_j(x, u)}{\xi_j(x, u)}. \end{aligned} \tag{13}$$

Let $\tau = \sum_{i=1}^k \lambda_i \frac{(f_i(u) + u^T z_i + g_i(u) - u^T v_i)}{(g_i(u) - u^T v_i)^2} + \sum_{j=1}^m \mu_j > 0$.

Multiply (12) by $\frac{\lambda_i}{\tau}$ and (13) by $\frac{\mu_j}{\tau}$ and add them, to get

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x,u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\ & \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \xi_j(x,u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \\ & \geq \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{(f_i(u) + u^T z_i + g_i(u) - u^T v_i)}{(g_i(u) - u^T v_i)^2} C_{x,u} \left[\left(\frac{(g_i(u) - u^T v_i)^2}{f_i(u) + u^T z_i + g_i(u) - u^T v_i} \right) \right. \\ & \quad \times \left(\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \left(\frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) \nabla_p H_i(u, p) \right) \\ & \quad \left. + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{1}{\bar{\gamma}_i(x,u)} \left(\frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) (H_i(u, s) - s^T \nabla_s H_i(u, s)) \right] \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{\rho_i d_i(x, u)}{\alpha_i(x, u)} \left(\frac{1}{g_i(u) - u^T v_i} + \frac{f_i(u) + u^T z_i}{(g_i(u) - u^T v_i)^2} \right) \\
 & + \sum_{j=1}^m \frac{\mu_j}{\tau} C_{x,u} (\nabla(h_j(u) + u^T w_j) + \nabla_q K_j(u, q)) \\
 & + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{1}{\delta_j(x, u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{\eta_j c_j(x, u)}{\xi_j(x, u)}.
 \end{aligned}$$

Further, using convexity on C , (6)-(8), and hypotheses (iii)-(iv), we have

$$\sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{i=1}^m \frac{\mu_j}{\tau \sigma(x, u)} [h_j(x) + x^T w_j] \geq 0.$$

Since x is a feasible solution of (MFP), it follows that

$$\sum_{i=1}^k \frac{\lambda_i}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \geq 0.$$

This contradicts (11). Hence we have the result. □

Theorem 4.2 (Strong duality theorem) *Assume \bar{u} is an efficient solution of (MFP) and let the Slater constraint qualification be satisfied on X . Also, if*

$$\begin{aligned}
 \bar{H}(\bar{u}, 0) = 0, \quad K(\bar{u}, 0) = 0, \quad \nabla_p \bar{H}(\bar{u}, 0) = 0, \quad \nabla_q K(\bar{u}, 0) = 0, \\
 \nabla_s \bar{H}(\bar{u}, 0) = 0, \quad \nabla_r K(\bar{u}, 0) = 0,
 \end{aligned}$$

then there exist $0 < \bar{\lambda} \in R^k$, $\bar{\mu} \in R^m$, $\bar{z}_i \in R^n$, \bar{v}_i , and $\bar{w}_j \in R^n$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, m$, such that $(\bar{u}, \bar{z}, \bar{v}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{s} = 0, \bar{r} = 0)$ is a feasible solution of (MFD) and the objective function values of (MFP) and (MFD) are equal. Furthermore, if the conditions of Theorem 4.1 hold for all feasible solutions of (MFP) and each feasible solution $(u', z', v', \mu', \lambda', w', p' = 0, q' = 0, r' = 0, s' = 0)$ of (MFD), then $(\bar{u}, \bar{z}, \bar{v}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{r} = 0, \bar{s} = 0)$ is an efficient solution of (MFD).

Proof Assume \bar{u} is an efficient solution of (MFP) and the Slater constraint qualification is satisfied on X . Then, from Theorem 3.4, there exist $\bar{\lambda}_i > 0$, $\bar{\mu} \in R^m$, $\bar{z}_i \in R^n$, $\bar{v}_i \in R^n$, and $\bar{w}_j \in R^n$, $i = 1, 2, \dots, k$, $j = 1, 2, \dots, m$, such that

$$\sum_{i=1}^k \bar{\lambda}_i \nabla \left(\frac{f_i(\bar{u}) + \bar{u}^T \bar{z}_i}{g_i(\bar{u}) - \bar{u}^T \bar{v}_i} \right) + \sum_{j=1}^m \bar{\mu}_j \nabla (h_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0, \tag{14}$$

$$\sum_{j=1}^m \bar{\mu}_j \nabla (h_j(\bar{u}) + \bar{u}^T \bar{w}_j) = 0, \tag{15}$$

$$\bar{u}^T \bar{z}_i = S(\bar{u}|C_i), \quad i = 1, 2, \dots, k, \tag{16}$$

$$\bar{u}^T \bar{v}_i = S(\bar{u}|D_i), \quad i = 1, 2, \dots, k, \tag{17}$$

$$\bar{u}^T \bar{w}_j = S(\bar{u}|E_j), \quad j = 1, 2, \dots, m, \tag{18}$$

$$\bar{z}_i \in C_i, \quad \bar{v}_i \in D_i, \quad \bar{w}_j \in E_j, \quad i = 1, 2, \dots, k, j = 1, 2, \dots, m, \tag{19}$$

$$\bar{\lambda}_i > 0, \quad \bar{\mu}_j \geq 0, \quad i = 1, 2, \dots, k, j = 1, 2, \dots, m. \tag{20}$$

Thus, $(\bar{u}, \bar{z}, \bar{v}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{r} = 0, \bar{s} = 0)$ is feasible for (MFD) and from (16)-(18), the objective function values of (MFP) and (MFD) are equal.

We now show that $(\bar{u}, \bar{z}, \bar{v}, \bar{\mu}, \bar{\lambda}, \bar{w}, \bar{p} = 0, \bar{q} = 0, \bar{r} = 0, \bar{s} = 0)$ is an efficient solution of (MFD). If not, then there exists $(u', z', v', \mu', \lambda', w', p', q', r', s')$ of (MFD) such that

$$\frac{f_i(\bar{u}) + \bar{u}^T \bar{z}_i}{g_i(\bar{u}) - \bar{u}^T \bar{v}_i} \leq \frac{f_i(u') + u'^T z'_i}{g_i(u') - u'^T v'_i}, \quad i = 1, 2, \dots, k,$$

and

$$\frac{f_r(\bar{u}) + \bar{u}^T \bar{z}_r}{g_r(\bar{u}) - \bar{u}^T \bar{v}_r} < \frac{f_r(u') + u'^T z'_r}{g_r(u') - u'^T v'_r}, \quad \text{for some } r = 1, 2, \dots, k.$$

This contradicts the weak duality theorem. Hence we have the result. □

Theorem 4.3 (Strict converse duality theorem) *Let x be a feasible solution for (MFP) and $(u, z, v, \mu, \lambda, w, p, q, r, s)$ be feasible for (MFD). Suppose that:*

- (i) $(f_i(\cdot) + (\cdot)^T z_i)$ and $-(g_i(\cdot) - (\cdot)^T v_i)$ are higher-order strictly $(C, \alpha_i, \gamma_i, \rho_i, d_i)$ -convex at u , with respect to H_i, p , and $s, i = 1, 2, \dots, k$,
- (ii) $(h_j(\cdot) + (\cdot)^T w_j)$ is higher-order $(C, \xi_j, \delta_j, \eta_j, c_j)$ -convex at u , with respect to K_j, q , and $r, j = 1, 2, \dots, m$,
- (iii) $\bar{\gamma}_i(x, u) = \zeta(x, u)$ and $\xi_j(x, u) = \delta_j(x, u) = \sigma(x, u), i = 1, 2, \dots, k, j = 1, 2, \dots, m$,
- (iv) $\sum_{i=1}^k \lambda_i \bar{\rho}_i \frac{\bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)} + \sum_{j=1}^m \mu_j \eta_j \frac{c_j(x, u)}{\xi_j(x, u)} \geq 0$,
- (v) $\sum_{i=1}^k \frac{\lambda_i}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \leq 0$,

where

$$\bar{\alpha}_i(x, u) = \left(\frac{g_i(x) - x^T v_i}{g_i(u) - u^T v_i} \right) \alpha_i(x, u), \quad \bar{\rho}_i = \rho_i \left(1 + \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right),$$

$$\bar{d}_i(x, u) = \left(\frac{d_i(x, u)}{g_i(x) - x^T v_i} \right).$$

Then $x = u$.

Proof Suppose that $x \neq u$ and exhibit a contradiction. Let $(u, z, v, \mu, \lambda, w, p, q, r, s)$ be feasible for (MFD), then

$$C_{x,u}(0) = C_{x,u} \left(\sum_{i=1}^k \lambda_i \nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{j=1}^m \mu_j \nabla (h_j(u) + u^T w_j) \right. \\ \left. + \sum_{i=1}^k \lambda_i \nabla_p \bar{H}_i(u, p) + \sum_{j=1}^m \mu_j \nabla_q K_j(u, q) \right) = 0. \tag{21}$$

Since $\left(\frac{f_i(\cdot)+(\cdot)^T z_i}{g_i(\cdot)-(\cdot)^T v_i}\right)$ is higher-order strictly $(\bar{C}, \bar{\alpha}_i, \bar{\gamma}_i, \bar{\rho}_i, \bar{d}_i)$ -convex at u with respect to \bar{H}_i, p , and s , we have

$$\begin{aligned} & \frac{1}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\ & > \bar{C}_{x,u} \left(\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \nabla_p \bar{H}_i(u, p) \right) \\ & \quad + \frac{1}{\bar{\gamma}_i(x, u)} [\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)] + \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)}. \end{aligned}$$

Let $\tau = \sum_{i=1}^k \lambda_i \frac{(f_i(u)+u^T z_i+g_i(u)-u^T v_i)}{(g_i(u)-u^T v_i)^2} + \sum_{j=1}^m \mu_j > 0$.

Multiplying by $\frac{\lambda_i}{\tau}$ in the above inequality, we get

$$\begin{aligned} & \frac{\lambda_i}{\tau} \left[\frac{1}{\bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \right. \\ & > \bar{C}_{x,u} \left(\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \nabla_p \bar{H}_i(u, p) \right) \\ & \quad \left. + \frac{1}{\bar{\gamma}_i(x, u)} [\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)] + \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)} \right]. \tag{22} \end{aligned}$$

By the higher-order $(C, \xi_j, \delta_j, \eta_j, c_j)$ -convex of $(h_j(\cdot) + (\cdot)^T w_j)$ at u with respect to K_j, q , and $r, j = 1, 2, \dots, m$, we obtain

$$\begin{aligned} \frac{1}{\xi_j(x, u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] & \geq C_{x,u} (\nabla (h_j(u) + u^T w_j) + \nabla_q K_j(u, q)) \\ & \quad + \frac{1}{\delta_j(x, u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] + \frac{\eta_j c_j(x, u)}{\xi_j(x, u)}. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{\mu_j}{\tau} \left[\frac{1}{\xi_j(x, u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \right. \\ & \geq C_{x,u} (\nabla (h_j(u) + u^T w_j) + \nabla_q K_j(u, q)) \\ & \quad \left. + \frac{1}{\delta_j(x, u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] + \frac{\eta_j c_j(x, u)}{\xi_j(x, u)} \right]. \tag{23} \end{aligned}$$

Taking summation over i in (22) and over j in (23), we get

$$\begin{aligned} & \sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\ & \quad + \sum_{j=1}^m \frac{\mu_j}{\tau \xi_j(x, u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \\ & > \sum_{i=1}^k \frac{\lambda_i}{\tau} \bar{C}_{x,u} \left(\nabla \left(\frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \nabla_p \bar{H}_i(u, p) \right) \\ & \quad + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{1}{\bar{\gamma}_i(x, u)} [\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)] \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)} + \sum_{j=1}^m \frac{\mu_j}{\tau} C_{x,u}(\nabla(h_j(u) + u^T w_j) + \nabla_q K_j(u, q)) \\
 & + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{1}{\delta_j(x, u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{\eta_j c_j(x, u)}{\xi_j(x, u)}.
 \end{aligned}$$

Further, using convexity on C and (21), we obtain

$$\begin{aligned}
 & \sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) \\
 & + \sum_{j=1}^m \frac{\mu_j}{\tau \xi_j(x, u)} [h_j(x) + x^T w_j - h_j(u) - u^T w_j] \\
 & > \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{1}{\bar{\gamma}_i(x, u)} [\bar{H}_i(u, s) - s^T \nabla_s \bar{H}_i(u, s)] + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{1}{\delta_j(x, u)} [K_j(u, r) - r^T \nabla_r K_j(u, r)] \\
 & + \sum_{i=1}^k \frac{\lambda_i}{\tau} \frac{\bar{\rho}_i \bar{d}_i(x, u)}{\bar{\alpha}_i(x, u)} + \sum_{j=1}^m \frac{\mu_j}{\tau} \frac{\eta_j c_j(x, u)}{\xi_j(x, u)}.
 \end{aligned}$$

This, together with (7)-(8) and hypotheses (iii)-(iv), shows

$$\sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) + \sum_{i=1}^m \frac{\mu_j}{\tau \sigma(x, u)} [h_j(x) + x^T w_j] > 0.$$

Since x is a feasible solution of (MFP), it follows that

$$\sum_{i=1}^k \frac{\lambda_i}{\tau \bar{\alpha}_i(x, u)} \left(\frac{f_i(x) + x^T z_i}{g_i(x) - x^T v_i} - \frac{f_i(u) + u^T z_i}{g_i(u) - u^T v_i} \right) > 0.$$

This contradicts the hypothesis (v). Hence the proof is completed. □

5 Conclusions

In this article, we consider a class of fractional programming problem having k -objectives in which each numerator and denominator of the objective function is nondifferentiable in terms of the support function of a compact convex set. The important property that the ratio of higher-order $(C, \alpha, \gamma, \rho, d)$ -convex function is also a higher-order $(\bar{C}, \bar{\alpha}, \bar{\gamma}, \bar{\rho}, \bar{d})$ -convex function is obtained. We also derive sufficient optimality conditions for an efficient solution for this problem. Furthermore, a higher-order Mond-Weir type dual is formulated and appropriate duality relations are obtained under higher-order $(C, \alpha, \gamma, \rho, d)$ -convexity assumptions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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