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On a Hardy-Hilbert-type inequality with parameters

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Abstract
By means of the way of weight coefficients and technique of real analysis, an extension of a Hardy-Hilbert-type inequality with parameters and a best possible constant factor is given. The equivalent forms, the operator expression with the norm, the reverses and some particular cases are also considered.

MSC: 26D15; 47A07

Keywords: Hardy-Hilbert-type inequality; weight coefficient; equivalent form; reverse; operator

1 Introduction

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$, $\|f\|_p = (\int_0^\infty f^p(x)\,dx)^{\frac{1}{p}} > 0$, $\|g\|_q > 0$. We have the following Hardy-Hilbert's integral inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q. \tag{1}$$

Assuming that $a_m, b_n \geq 0$,

$$a = \{a_m\}_{m=1}^\infty \in l^p = \left\{a; \|a\|_p = \left(\sum_{m=1}^\infty |a_m|^p\right)^{\frac{1}{p}} < \infty\right\},$$

$b = \{b_n\}_{n=1}^\infty \in l^q$, $\|a\|_p, \|b\|_q > 0$, we have the following Hardy-Hilbert's inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (*cf.* [1]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{2}$$

Hardy-Hilbert-type inequalities, specially (1) and (2), are basically important in mathematical analysis and its applications (*cf.* [1–7]).

If $\mu_i, \upsilon_j > 0$ $(i, j \in \mathbf{N})$,

$$U_m := \sum_{i=1}^m \mu_i, \qquad V_n := \sum_{j=1}^n \upsilon_j \quad (m, n \in \mathbf{N}), \tag{3}$$

then we have the following inequality (*cf.* [1], Theorem 321, p.261):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\mu_m^{1/q}\upsilon_n^{1/p}a_m b_n}{U_m+V_n}<\frac{\pi}{\sin(\pi/p)}\|a\|_p\|b\|_q. \tag{4}$$

Replacing $\mu_m^{1/q}a_m$ and $\upsilon_n^{1/p}b_n$ by a_m and b_n in (4), respectively, we obtain the following equivalent form of (4):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{U_m+V_n}<\frac{\pi}{\sin(\frac{\pi}{p})}\left(\sum_{m=1}^{\infty}\frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{5}$$

For $\mu_i=\upsilon_j=1$ ($i,j\in\mathbf{N}$), both (4) and (5) reduce to (2). We call (4) and (5) Hardy-Hilbert-type inequalities.

Note The authors of [1] (Theorem 321, p.261) did not prove that (4) is valid with the best possible constant factor.

In 1998, by introducing an independent parameter $\lambda\in(0,1]$, Yang [8] gave an extension of (1) for $p=q=2$. Following the methods of [8], Yang [5] gave some best extensions of (1) and (2) as follows.

If $\lambda_1,\lambda_2\in\mathbf{R}=(-\infty,\infty)$, $\lambda_1+\lambda_2=\lambda$, $k_\lambda(x,y)$ is a nonnegative homogeneous function of degree $-\lambda$, with $k(\lambda_1)=\int_0^\infty k_\lambda(t,1)t^{\lambda_1-1}\,dt\in\mathbf{R}_+$, $\phi(x)=x^{p(1-\lambda_1)-1}$, $\psi(x)=x^{q(1-\lambda_2)-1}$, $f(x),g(y)\geq 0$,

$$f\in L_{p,\phi}(\mathbf{R}_+)=\left\{f;\|f\|_{p,\phi}:=\left(\int_0^\infty\phi(x)\left|f(x)\right|^p\,dx\right)^{\frac{1}{p}}<\infty\right\},$$

$g\in L_{q,\psi}(\mathbf{R}_+)$, $\|f\|_{p,\phi},\|g\|_{q,\psi}>0$, then

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)f(x)g(y)\,dx\,dy<k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \tag{6}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is decreasing with respect to $x>0$ ($y>0$), then for $a_m,b_n\geq 0$,

$$a\in l_{p,\phi}=\left\{a;\|a\|_{p,\phi}:=\left(\sum_{n=1}^{\infty}\phi(n)|a_n|^p\right)^{\frac{1}{p}}<\infty\right\},$$

$b=\{b_n\}_{n=1}^\infty\in l_{q,\psi}$, $\|a\|_{p,\phi},\|b\|_{q,\psi}>0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k_\lambda(m,n)a_m b_n<k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{7}$$

where the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for $\lambda=1$, $k_1(x,y)=\frac{1}{x+y}$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, inequality (6) reduces to (1), while (7) reduces to (2). For $0<\lambda_1,\lambda_2\leq 1$, $\lambda_1+\lambda_2=\lambda$, we set $k_\lambda(x,y)=\frac{1}{(x+y)^\lambda}$. Then, by (7), it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\|a\|_{p,\phi}\|b\|_{q,\psi}, \tag{8}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible ($B(u,v)$ is the beta function). Some other results including multidimensional Hilbert-type inequalities are provided by [9–27].

In 2015, by adding a few conditions, Yang [28] gave an extension of (8) and (5) as follows:

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(U_m+V_n)^{\lambda}}\\ &\quad< B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty}\frac{U_m^{p(1-\lambda_1)-1}a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{V_n^{q(1-\lambda_2)-1}b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}},\end{aligned} \tag{9}$$

where the constant factor $B(\lambda_1,\lambda_2)$ is the best possible. For $\mu_i=\upsilon_j=1$ ($i,j\in\mathbf{N}$), (9) reduces to (8); for $\lambda=1$, $\lambda_1=\frac{1}{q}$, $\lambda_2=\frac{1}{p}$, (9) reduces to (5).

In this paper, by using the way of weight coefficients and technique of real analysis, a Hardy-Hilbert-type inequality with parameters and a best possible constant factor is given, which is with the kernel $\frac{(\min\{x,c_1y\})^{\alpha}}{(\max\{x,c_1y\})^{\lambda+\alpha}}$ similar to (9). The extended inequalities, the equivalent forms, the operator expression with the norm, the reverses and some particular cases are also considered.

2 Some lemmas

In the following, we agree on that $\mu_i,\upsilon_j>0$ ($i,j\in\mathbf{N}$), U_m and V_n are defined by (3), $p\neq 0,1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m,b_n\geq 0$ ($m,n\in\mathbf{N}$), $\|a\|_{p,\Phi_\lambda}=(\sum_{m=1}^{\infty}\Phi_\lambda(m)a_m^p)^{\frac{1}{p}}$, $\|b\|_{q,\Psi_\lambda}=(\sum_{n=1}^{\infty}\Psi_\lambda(n)b_n^q)^{\frac{1}{q}}$, where

$$\Phi_\lambda(m):=\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}},\qquad \Psi_\lambda(n):=\frac{V_n^{q(1-\lambda_2)-1}}{\upsilon_n^{q-1}}\quad(m,n\in\mathbf{N}).$$

Lemma 1 *If $g(t)$ (>0) is decreasing in $\mathbf{R}_+$ and strictly decreasing in $[n_0,\infty)\subset\mathbf{R}_+$ ($n_0\in\mathbf{N}$), satisfying $\int_0^{\infty}g(t)\,dt\in\mathbf{R}_+$, then we have*

$$\int_1^{\infty}g(t)\,dt<\sum_{n=1}^{\infty}g(n)<\int_0^{\infty}g(t)\,dt. \tag{10}$$

Proof Since, by the assumption, we have

$$\begin{aligned}&\int_n^{n+1}g(t)\,dt\leq g(n)\leq\int_{n-1}^{n}g(t)\,dt\quad(n=1,\ldots,n_0),\\ &\int_{n_0+1}^{n_0+2}g(t)\,dt<g(n_0+1)<\int_{n_0}^{n_0+1}g(t)\,dt,\end{aligned}$$

it follows that

$$0<\int_1^{n_0+2}g(t)\,dt<\sum_{n=1}^{n_0+1}g(n)<\sum_{n=1}^{n_0+1}\int_{n-1}^{n}g(t)\,dt=\int_0^{n_0+1}g(t)\,dt<\infty.$$

By the same way, we still have

$$0 < \int_{n_0+2}^{\infty} g(t)\,dt \le \sum_{n=n_0+2}^{\infty} g(n) \le \int_{n_0+1}^{\infty} g(t)\,dt < \infty.$$

Hence, making plus for the above two inequalities, we have (10). □

Example 1 For $s \in \mathbf{N}$, $0 < c_1 \le \cdots \le c_s < \infty$, $\lambda_1, \lambda_2 > -\alpha$, $\lambda_1 + \lambda_2 = \lambda$, we set

$$k_\lambda(x,y) := \prod_{k=1}^{s} \frac{(\min\{x, c_k y\})^{\frac{\alpha}{s}}}{(\max\{x, c_k y\})^{\frac{\lambda+\alpha}{s}}} \quad \bigl((x,y) \in \mathbf{R}_+^2 = \mathbf{R}_+ \times \mathbf{R}_+\bigr).$$

(a) We find

$$\begin{aligned}
k_s(\lambda_1) &:= \int_0^\infty k_\lambda(1,u) t^{\lambda_2 - 1}\,du \overset{u=1/t}{=} \int_0^\infty k_\lambda(t,1) t^{\lambda_1 - 1}\,dt \\
&= \int_0^\infty \prod_{k=1}^{s} \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1 - 1}\,dt \\
&= \int_0^{c_1} \prod_{k=1}^{s} \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}} t^{\lambda_1 - 1}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}}\,dt + \int_{c_s}^{\infty} \prod_{k=1}^{s} \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}} t^{\lambda_1 - 1}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}}\,dt \\
&\quad + \sum_{i=1}^{s-1} \int_{c_i}^{c_{i+1}} \prod_{k=1}^{s} \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}} t^{\lambda_1 - 1}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}}\,dt \\
&= \prod_{k=1}^{s} \frac{1}{c_k^{(\lambda+\alpha)/s}} \int_0^{c_1} t^{\lambda_1 + \alpha - 1}\,dt + \prod_{k=1}^{s} c_k^{\alpha/s} \int_{c_s}^{\infty} t^{-\lambda_2 - \alpha - 1}\,dt \\
&\quad + \sum_{i=1}^{s-1} \int_{c_i}^{c_{i+1}} \prod_{k=1}^{i} \frac{c_k^{\frac{\alpha}{s}}}{t^{\frac{\lambda+\alpha}{s}}} \prod_{k=i+1}^{s} \frac{t^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1 - 1}\,dt \\
&= \frac{c_1^{\lambda_1+\alpha}}{\lambda_1 + \alpha} \frac{1}{\prod_{k=1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} + \frac{1}{(\lambda_2 + \alpha) c_s^{\lambda_2 + \alpha}} \prod_{k=1}^{s} c_k^{\frac{\alpha}{s}} \\
&\quad + \sum_{i=1}^{s-1} \frac{\prod_{k=1}^{i} c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} \int_{c_i}^{c_{i+1}} t^{\lambda_1 - \frac{i\lambda}{s} + (1 - \frac{2i}{s})\alpha - 1}\,dt.
\end{aligned}$$

If $\lambda_1 - \frac{i\lambda}{s} + (1 - \frac{2i}{s})\alpha \neq 0$, then

$$\int_{c_i}^{c_{i+1}} t^{\lambda_1 - \frac{i\lambda}{s} + (1 - \frac{2i}{s})\alpha - 1}\,dt = \frac{c_{i+1}^{\lambda_1 - \frac{i\lambda}{s} + (1 - \frac{2i}{s})\alpha} - c_i^{\lambda_1 - \frac{i\lambda}{s} + (1 - \frac{2i}{s})\alpha}}{\lambda_1 - \frac{i\lambda}{s} + (1 - \frac{2i}{s})\alpha};$$

if there exists $i_0 \in \{1, \ldots, s-1\}$ such that $\lambda_1 - \frac{i_0\lambda}{s} + (1 - \frac{2i_0}{s})\alpha = 0$, then we find

$$\int_{c_{i_0}}^{c_{i_0+1}} t^{\lambda_1 - \frac{i_0\lambda}{s} + (1 - \frac{2i_0}{s})\alpha - 1}\,dt = \ln\left(\frac{c_{i_0+1}}{c_{i_0}}\right) = \lim_{i \to i_0} \int_{c_i}^{c_{i+1}} t^{\lambda_1 - \frac{i\lambda}{s} + (1 - \frac{2i}{s})\alpha - 1}\,dt,$$

and we still indicate $\ln(\frac{c_{i_0+1}}{c_{i_0}})$ by the following formal expression:

$$\frac{c_{i_0+1}^{\lambda_1-\frac{i_0\lambda}{s}+(1-\frac{2i_0}{s})\alpha} - c_{i_0}^{\lambda_1-\frac{i_0\lambda}{s}+(1-\frac{2i_0}{s})\alpha}}{\lambda_1 - \frac{i_0\lambda}{s} + (1-\frac{2i_0}{s})\alpha}.$$

Hence, we may set

$$k_s(\lambda_1) = \frac{c_1^{\lambda_1+\alpha}}{\lambda_1+\alpha}\frac{1}{\prod_{k=1}^s c_k^{\frac{\lambda+\alpha}{s}}} + \frac{1}{(\lambda_2+\alpha)c_s^{\lambda_2+\alpha}}\prod_{k=1}^s c_k^{\frac{\alpha}{s}} + \sum_{i=1}^{s-1}\left[\frac{c_{i+1}^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\alpha} - c_i^{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\alpha}}{\lambda_1-\frac{i\lambda}{s}+(1-\frac{2i}{s})\alpha}\frac{\prod_{k=1}^i c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^s c_k^{\frac{\lambda+\alpha}{s}}}\right]. \quad (11)$$

In particular, (i) for $s = 1$ (or $c_s = \cdots = c_1$), we have $k_\lambda(x,y) = \frac{(\min\{x,c_1y\})^\alpha}{(\max\{x,c_1y\})^{\lambda+\alpha}}$ and

$$k_1(\lambda_1) = \frac{\lambda+2\alpha}{(\lambda_1+\alpha)(\lambda_2+\alpha)}\frac{1}{c_1^{\lambda_2}}; \quad (12)$$

(ii) for $s = 2$, we have $k_\lambda(x,y) = \frac{(\min\{x,c_1y\}\min\{x,c_2y\})^{\alpha/2}}{(\max\{x,c_1y\}\max\{x,c_2y\})^{(\lambda+\alpha)/2}}$ and

$$k_2(\lambda_1) = \left(\frac{c_1}{c_2}\right)^{\frac{\alpha}{2}}\left[\frac{c_1^{\lambda_1-\frac{\lambda}{2}}}{(\lambda_1+\alpha)c_2^{\frac{\lambda}{2}}} + \frac{1}{(\lambda_2+\alpha)c_2^{\lambda_2}} + \frac{c_2^{\lambda_1-\frac{\lambda}{2}} - c_1^{\lambda_1-\frac{\lambda}{2}}}{(\lambda_1-\frac{\lambda}{2})c_2^{\frac{\lambda}{2}}}\right]; \quad (13)$$

(iii) for $\alpha = 0$, we have $\lambda_1, \lambda_2 > 0$, $k_\lambda(x,y) = \frac{1}{\prod_{k=1}^s(\max\{x,c_ky\})^{\frac{\lambda}{s}}}$ and

$$k_s(\lambda_1) = \widetilde{k}_s(\lambda_1) := \frac{c_1^{\lambda_1}}{\lambda_1}\frac{1}{\prod_{k=1}^s c_k^{\frac{\lambda}{s}}} + \frac{1}{\lambda_2 c_s^{\lambda_2}} + \sum_{i=1}^{s-1}\frac{c_{i+1}^{\lambda_1-\frac{i}{s}\lambda} - c_i^{\lambda_1-\frac{i}{s}\lambda}}{\lambda_1-\frac{i}{s}\lambda}\frac{1}{\prod_{k=i+1}^s c_k^{\frac{\lambda}{s}}}; \quad (14)$$

(iv) for $\alpha = -\lambda$, we have $\lambda < \lambda_1, \lambda_2 < 0$, $k_\lambda(x,y) = \frac{1}{\prod_{k=1}^s(\min\{x,c_ky\})^{\frac{\lambda}{s}}}$ and

$$k_s(\lambda_1) = \widehat{k}_s(\lambda_1) := \frac{c_1^{-\lambda_2}}{(-\lambda_2)} + \frac{1}{(-\lambda_1)c_s^{-\lambda_1}}\prod_{k=1}^s c_k^{\frac{-\lambda}{s}} + \sum_{i=1}^{s-1}\left(\frac{c_{i+1}^{\lambda_1-\frac{s-i}{s}\lambda} - c_i^{\lambda_1-\frac{s-i}{s}\lambda}}{\lambda_1-\frac{s-i}{s}\lambda}\prod_{k=1}^i c_k^{\frac{-\lambda}{s}}\right); \quad (15)$$

(v) for $\lambda = 0$, we have $\lambda_2 = -\lambda_1$, $|\lambda_1| < \alpha$ $(\alpha > 0)$,

$$k_0(x,y) = \prod_{k=1}^s\left(\frac{\min\{x,c_ky\}}{\max\{x,c_ky\}}\right)^{\frac{\alpha}{s}},$$

and

$$k_s(\lambda_1) = k_s^{(0)}(\lambda_1) := \frac{c_1^{\lambda_1+\alpha}}{a+\lambda_1}\frac{1}{\prod_{k=1}^s c_k^{\frac{\alpha}{s}}} + \frac{c_s^{\lambda_1-\alpha}}{a-\lambda_1}\prod_{k=1}^s c_k^{\frac{\alpha}{s}} + \sum_{i=1}^{s-1}\left[\frac{c_{i+1}^{\lambda_1+(1-\frac{2i}{s})\alpha} - c_i^{\lambda_1+(1-\frac{2i}{s})\alpha}}{\lambda_1+(1-\frac{2i}{s})\alpha}\frac{\prod_{k=1}^i c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^s c_k^{\frac{\alpha}{s}}}\right]. \quad (16)$$

(b) Since we find

$$k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}} = \frac{1}{y^{1-\lambda_2}}\prod_{k=1}^{s}\frac{(\min\{c_k^{-1}x,y\})^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda}{s}}(\max\{c_k^{-1}x,y\})^{\frac{\lambda+\alpha}{s}}}$$
$$= \begin{cases} \frac{1}{y^{1-\lambda_2-\alpha}}\prod_{k=1}^{s}\frac{1}{c_k^{\frac{\lambda}{s}}(c_k^{-1}x)^{\frac{\lambda+\alpha}{s}}}, & 0 < y \le c_s^{-1}x, \\ \frac{1}{y^{1+\lambda_1+\alpha-\frac{i}{s}(\lambda+2\alpha)}}\frac{\prod_{k=i+1}^{s}(c_k^{-1}x)^{\frac{\alpha}{s}}}{\prod_{k=1}^{s}c_k^{\frac{\lambda}{s}}\prod_{k=1}^{i}(c_k^{-1}x)^{\frac{\lambda+\alpha}{s}}}, & c_{i+1}^{-1}x < y \le c_i^{-1}x\ (i = 1,\ldots,s-1), \\ \frac{1}{y^{1+\lambda_1+\alpha}}\prod_{k=1}^{s}\frac{(c_k^{-1}x)^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda}{s}}(y)^{\frac{\lambda+\alpha}{s}}}, & c_1^{-1}x < y < \infty, \end{cases}$$

then for $\lambda_2 \le 1-\alpha$ ($\lambda_1 > -\alpha$), $k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}}$ is decreasing for $y > 0$ and strictly decreasing for the large enough variable y. By the same way, since

$$k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}} = \frac{1}{x^{1-\lambda_1}}\prod_{k=1}^{s}\frac{(\min\{x,c_k y\})^{\frac{\alpha}{s}}}{(\max\{x,c_k y\})^{\frac{\lambda+\alpha}{s}}}$$
$$= \begin{cases} \frac{1}{x^{1-\lambda_1-\alpha}}\prod_{k=1}^{s}\frac{1}{(c_k y)^{\frac{\lambda+\alpha}{s}}}, & 0 < x \le c_1 y, \\ \frac{1}{x^{1-\lambda_1-\alpha+\frac{i}{s}(\lambda+2\alpha)}}\frac{\prod_{k=1}^{i}(c_k y)^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s}(c_k y)^{\frac{\lambda+\alpha}{s}}}, & c_i y < x \le c_{i+1}y\ (i = 1,\ldots,s-1), \\ \frac{1}{x^{1+\lambda_2+\alpha}}\prod_{k=1}^{s}(c_k y)^{\frac{\alpha}{s}}, & c_s y < x < \infty, \end{cases}$$

then for $\lambda_1 \le 1-\alpha$ ($\lambda_2 > -\alpha$), $k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}$ is decreasing for $x > 0$ and strictly decreasing for the large enough variable x.

In view of (a) and (b), for $-\alpha < \lambda_1, \lambda_2 \le 1-\alpha$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}}$ ($k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}$) is decreasing for $y > 0$ ($x > 0$) and strictly decreasing for the large enough variable $y(x)$ satisfying $k_s(\lambda_1) \in \mathbf{R}_+$.

Lemma 2 *If* $s \in \mathbf{N}$, $0 < c_1 \le \cdots \le c_s$, $-\alpha < \lambda_1, \lambda_2 \le 1-\alpha$, $\lambda_1 + \lambda_2 = \lambda$, $k_s(\lambda_1)$ *is indicated by* (11), *define the following weight coefficients*:

$$\omega(\lambda_2,m) := \sum_{n=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_k V_n\})^{\frac{\lambda+\alpha}{s}}}\frac{U_m^{\lambda_1}\upsilon_n}{V_n^{1-\lambda_2}}, \quad m \in \mathbf{N}, \tag{17}$$

$$\varpi(\lambda_1,n) := \sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_k V_n\})^{\frac{\lambda+\alpha}{s}}}\frac{V_n^{\lambda_2}\mu_m}{U_m^{1-\lambda_1}}, \quad n \in \mathbf{N}. \tag{18}$$

Then we have the following inequalities:

$$\omega(\lambda_2,m) < k_s(\lambda_1) \quad (-\alpha < \lambda_2 \le 1-\alpha, \lambda_1 > -\alpha; m \in \mathbf{N}), \tag{19}$$

$$\varpi(\lambda_1,n) < k_s(\lambda_1) \quad (-\alpha < \lambda_1 \le 1-\alpha, \lambda_2 > -\alpha; n \in \mathbf{N}). \tag{20}$$

Proof We set $\mu(t) := \mu_m$, $t \in (m-1,m]$ ($m \in \mathbf{N}$); $\upsilon(t) := \upsilon_n$, $t \in (n-1,n]$ ($n \in \mathbf{N}$),

$$U(x) := \int_0^x \mu(t)\,dt \quad (x \ge 0), \qquad V(y) := \int_0^y \upsilon(t)\,dt \quad (y \ge 0). \tag{21}$$

Then, by (3), it follows that $U(m) = U_m$, $V(n) = V_n$ $(m, n \in \mathbf{N})$. For $x \in (m-1, m]$, $U'(x) = \mu(x) = \mu_m$ $(m \in \mathbf{N})$; for $y \in (n-1, n]$, $V'(y) = \upsilon(y) = \upsilon_n$ $(n \in \mathbf{N})$. Since $V(y)$ is strictly increasing in $(n-1, n]$, $-\alpha < \lambda_2 \le 1-\alpha$, $\lambda_1 > -\alpha$, in view of Lemma 1 and Example 1, we find

$$\begin{aligned}\omega(\lambda_2, m) &= \sum_{n=1}^{\infty} \int_{n-1}^{n} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}} V'(y)\,dy \\ &< \sum_{n=1}^{\infty} \int_{n-1}^{n} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V(y)\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V(y)\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{\lambda_1} V'(y)}{V^{1-\lambda_2}(y)}\,dy.\end{aligned}$$

Setting $t = \frac{V(y)}{U_m}$, we obtain $V'(y)\,dy = U_m\,dt$ and

$$\begin{aligned}\omega(\lambda_2, m) &< \sum_{n=1}^{\infty} \int_{\frac{V(n-1)}{U_m}}^{\frac{V(n)}{U_m}} \prod_{k=1}^{s} \frac{(\min\{1, c_k t\})^{\frac{\alpha}{s}}}{(\max\{1, c_k t\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_2-1}\,dt \\ &= \int_0^{\frac{V(\infty)}{U_m}} \prod_{k=1}^{s} \frac{(\min\{1, c_k t\})^{\frac{\alpha}{s}}}{(\max\{1, c_k t\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_2-1}\,dt \\ &\le \int_0^{\infty} \prod_{k=1}^{s} \frac{(\min\{1, c_k t\})^{\frac{\alpha}{s}}}{(\max\{1, c_k t\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_2-1}\,dt = k_s(\lambda_1).\end{aligned}$$

Since $U(x)$ is strictly increasing in $(m-1, m]$, $-\alpha < \lambda_1 \le 1-\alpha$, $\lambda_2 > -\alpha$, by the same way, we have

$$\begin{aligned}\varpi(\lambda_1, n) &= \sum_{m=1}^{\infty} \int_{m-1}^{m} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{V_n^{\lambda_2} U'(x)}{U_m^{1-\lambda_1}}\,dx \\ &< \sum_{m=1}^{\infty} \int_{m-1}^{m} \prod_{k=1}^{s} \frac{(\min\{U(x), c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U(x), c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{V_n^{\lambda_2} U'(x)}{U^{1-\lambda_1}(x)}\,dx \\ &\overset{t=U(x)/V_n}{=} \sum_{m=1}^{\infty} \int_{\frac{U(m-1)}{V_n}}^{\frac{U(m)}{V_n}} \prod_{k=1}^{s} \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1-1}\,dt \\ &= \int_0^{\frac{U(\infty)}{V_n}} \prod_{k=1}^{s} \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1-1}\,dt \le k_s(\lambda_1).\end{aligned}$$

Hence, we have (19) and (20). □

Lemma 3 *If* $s \in \mathbf{N}$, $0 < c_1 \le \cdots \le c_s$, $-\alpha < \lambda_1, \lambda_2 \le 1-\alpha$, $\lambda_1 + \lambda_2 = \lambda$, $k_s(\lambda_1)$ *is indicated by* (11), $m_0, n_0 \in \mathbf{N}$, $\mu_m \ge \mu_{m+1}$ $(m \in \{m_0, m_0+1, \ldots\})$, $\upsilon_n \ge \upsilon_{n+1}$ $(n \in \{n_0, n_0+1, \ldots\})$, $U(\infty) = V(\infty) = \infty$, *then* (i) *for* $m, n \in \mathbf{N}$, *we have*

$$k_s(\lambda_1)\bigl(1 - \theta(\lambda_2, m)\bigr) < \omega(\lambda_2, m) \quad (-\alpha < \lambda_2 \le 1-\alpha, \lambda_1 > -\alpha), \tag{22}$$

$$k_s(\lambda_1)\bigl(1 - \vartheta(\lambda_1, n)\bigr) < \varpi(\lambda_1, n) \quad (-\alpha < \lambda_1 \le 1-\alpha, \lambda_2 > -\alpha), \tag{23}$$

where

$$\theta(\lambda_2,m) := \frac{1}{k_s(\lambda_1)} \int_0^{\frac{U_{m_0}}{V_n}} \prod_{k=1}^{s} \frac{(\min\{t,c_k\})^{\frac{\alpha}{s}}}{(\max\{t,c_k\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1-1}\,dt = O\left(\frac{1}{U_m^{\lambda_2+\alpha}}\right) \in (0,1),$$

$$\vartheta(\lambda_1,n) := \frac{1}{k_s(\lambda_1)} \int_0^{\frac{U_{m_0}}{V_n}} \prod_{k=1}^{s} \frac{(\min\{t,c_k\})^{\frac{\alpha}{s}}}{(\max\{t,c_k\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1-1}\,dt = O\left(\frac{1}{V_n^{\lambda_1+\alpha}}\right) \in (0,1);$$

(ii) *for any* $b > 0$, *we have*

$$\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+b}} = \frac{1}{b}\left(\frac{1}{U_{m_0}^b} + bO(1)\right), \tag{24}$$

$$\sum_{n=1}^{\infty} \frac{\upsilon_n}{V_n^{1+b}} = \frac{1}{b}\left(\frac{1}{V_{n_0}^b} + b\widetilde{O}(1)\right). \tag{25}$$

Proof Since $\upsilon_n \geq \upsilon_{n+1}$ $(n \geq n_0)$, $-\alpha < \lambda_2 \leq 1-\alpha$, $\lambda_1 > -\alpha$ and $V(\infty) = \infty$, by Lemma 1, we have

$$\begin{aligned}
\omega(\lambda_2,m) &\geq \sum_{n=n_0}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}} \upsilon_{n+1} \\
&= \sum_{n=n_0}^{\infty} \int_n^{n+1} \prod_{k=1}^{s} \frac{(\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{\lambda_1}V'(y)}{V_n^{1-\lambda_2}}\,dy \\
&> \sum_{n=n_0}^{\infty} \int_n^{n+1} \prod_{k=1}^{s} \frac{(\{U_m,c_kV(y)\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV(y)\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{\lambda_1}V'(y)}{V^{1-\lambda_2}(y)}\,dy \\
&= \sum_{n=n_0}^{\infty} \int_{\frac{V(n)}{U_m}}^{\frac{V(n+1)}{U_m}} \prod_{k=1}^{s} \frac{(\min\{1,c_kt\})^{\frac{\alpha}{s}}}{(\max\{1,c_kt\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_2-1}\,dt \\
&= \int_{\frac{V_{n_0}}{U_m}}^{\infty} \prod_{k=1}^{s} \frac{(\min\{1,c_kt\})^{\frac{\alpha}{s}} t^{\lambda_2-1}}{(\max\{1,c_kt\})^{\frac{\lambda+\alpha}{s}}}\,dt = k_s(\lambda_1)\bigl(1-\theta(\lambda_2,m)\bigr).
\end{aligned}$$

For $U_m > c_sV_{n_0}$, we obtain $c_kt \leq c_st \leq c_s\frac{V_{n_0}}{U_m} < 1$ $(t \in (0, \frac{V_{n_0}}{U_m}]; k = 1,\ldots,s)$ and

$$\theta(\lambda_2,m) = \frac{\prod_{k=1}^{s} c_k}{k_s(\lambda_1)} \int_0^{\frac{V_{n_0}}{U_m}} t^{\lambda_2+\alpha-1}\,dt = \frac{\prod_{k=1}^{s} c_k}{(\lambda_2+\alpha)k_s(\lambda_1)} \left(\frac{V_{n_0}}{U_m}\right)^{\lambda_2+\alpha},$$

and then $\theta(\lambda_2,m) = O(\frac{1}{U_m^{\lambda_2+\alpha}})$. Hence we have (22).

By the same way, since $\mu_m \geq \mu_{m+1}$ $(m \geq m_0)$, $-\alpha < \lambda_1 \leq 1-\alpha$, $\lambda_2 > -\alpha$ and $U(\infty) = \infty$, we have

$$\begin{aligned}
\varpi(\lambda_1,n) &\geq \sum_{m=m_0}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}} \frac{V_n^{\lambda_2}\mu_{m+1}}{U_m^{1-\lambda_1}} \\
&= \sum_{m=m_0}^{\infty} \int_m^{m+1} \prod_{k=1}^{s} \frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}} \frac{V_n^{\lambda_2}U'(x)}{U_m^{1-\lambda_1}}\,dx
\end{aligned}$$

$$\begin{aligned} &> \sum_{m=m_0}^{\infty} \int_m^{m+1} \prod_{k=1}^{s} \frac{(\min\{U(x), c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U(x), c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{V_n^{\lambda_2} U'(x)}{U^{1-\lambda_1}(x)} \, dx \\ &\overset{t=U(x)/V_n}{=} \sum_{m=m_0}^{\infty} \int_{\frac{U(m)}{V_n}}^{\frac{U(m+1)}{V_n}} \prod_{k=1}^{s} \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1 - 1} \, dt \\ &= \int_{\frac{U_{m_0}}{V_n}}^{\infty} \prod_{k=1}^{s} \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}} t^{\lambda_1 - 1}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} \, dt = k_s(\lambda_1)\bigl(1 - \vartheta(\lambda_1, n)\bigr). \end{aligned}$$

For $V_n > c_1^{-1} U_{m_0}$, we obtain $t \le \frac{U_{m_0}}{V_n} < c_1 \le c_k$ $(t \in (0, \frac{U_{m_0}}{V_n}]; k = 1, \ldots, s)$ and

$$\vartheta(\lambda_1, n) = \frac{\int_0^{\frac{U_{m_0}}{V_n}} t^{\lambda_1 + \alpha - 1} \, dt}{k_s(\lambda_1) \prod_{k=1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} = \frac{(\lambda_1 + \alpha)^{-1}}{k_s(\lambda_1) \prod_{k=1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} \left(\frac{U_{m_0}}{V_n} \right)^{\lambda_1 + \alpha}.$$

Hence, we have (23).

For $b > 0$, we find

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+b}} &= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \sum_{m=m_0+1}^{\infty} \frac{\mu_m}{U_m^{1+b}} \\ &= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \sum_{m=m_0+1}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U_m^{1+b}} \, dx \\ &< \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \sum_{m=m_0+1}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U^{1+b}(x)} \, dx \\ &= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \int_{m_0}^{\infty} \frac{dU(x)}{U^{1+b}(x)} = \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \frac{1}{b U_{m_0}^b} \\ &= \frac{1}{b} \left(\frac{1}{U_{m_0}^b} + b \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} \right), \\ \sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+b}} &\ge \sum_{m=m_0}^{\infty} \frac{\mu_{m+1}}{U_m^{1+b}} = \sum_{m=m_0}^{\infty} \int_m^{m+1} \frac{U'(x)}{U_m^{1+b}} \, dx \\ &> \sum_{m=m_0}^{\infty} \int_m^{m+1} \frac{U'(x) \, dx}{U^{1+b}(x)} = \int_{m_0}^{\infty} \frac{dU(x)}{U^{1+b}(x)} = \frac{1}{b U_{m_0}^b}. \end{aligned}$$

Hence we have (24). By the same way, we still have (25). □

Note For example, $\mu_m = \frac{1}{m^\sigma}$, $\upsilon_n = \frac{1}{n^\sigma}$ $(0 \le \sigma \le 1; m, n \in \mathbf{N})$ satisfy the conditions of Lemma 3 $(m_0 = n_0 = 1)$.

3 Main results and operator expressions

Theorem 1 *If* $s \in \mathbf{N}$, $0 < c_1 \le \cdots \le c_s$, $-\alpha < \lambda_1, \lambda_2 \le 1 - \alpha$, $\lambda_1 + \lambda_2 = \lambda$, $k_s(\lambda_1)$ *is indicated by* (11), *then for* $p > 1$, $0 < \|a\|_{p,\Phi_\lambda}, \|b\|_{q,\Psi_\lambda} < \infty$, *we have the following equivalent inequalities*:

$$I := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}} a_m b_n}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} < k_s(\lambda_1) \|a\|_{p,\Phi_\lambda} \|b\|_{q,\Psi_\lambda}, \tag{26}$$

$$J := \left\{ \sum_{n=1}^{\infty} \frac{\upsilon_n}{V_n^{1-p\lambda_2}} \left[\sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}} a_m}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \right]^p \right\}^{\frac{1}{p}} < k_s(\lambda_1) \|a\|_{p,\Phi_\lambda}. \tag{27}$$

In particular, for $s = 1$ (*or* $c_s = \cdots = c_1$), *we have the following equivalent inequalities*:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\min\{U_m, c_1 V_n\})^{\alpha} a_m b_n}{(\max\{U_m, c_1 V_n\})^{\lambda+\alpha}} < k_1(\lambda_1) \|a\|_{p,\Phi_\lambda} \|b\|_{q,\Psi_\lambda}, \tag{28}$$

$$\left\{ \sum_{n=1}^{\infty} \frac{\upsilon_n}{V_n^{1-p\lambda_2}} \left[\sum_{m=1}^{\infty} \frac{(\min\{U_m, c_1 V_n\})^{\alpha} a_m}{(\max\{U_m, c_1 V_n\})^{\lambda+\alpha}} \right]^p \right\}^{\frac{1}{p}} < k_1(\lambda_1) \|a\|_{p,\Phi_\lambda}, \tag{29}$$

where $k_1(\lambda_1)$ *is indicated by* (12).

Proof By Hölder's inequality with weight (*cf.* [29]), we have

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} a_m \right]^p \\
&\quad = \left[\sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \left(\frac{U_m^{\frac{1-\lambda_1}{q}} a_m}{V_n^{\frac{1-\lambda_2}{p}} \mu_m^{\frac{1}{q}}} \right) \left(\frac{V_n^{\frac{1-\lambda_2}{p}} \mu_m^{\frac{1}{q}}}{U_m^{\frac{1-\lambda_1}{q}}} \right) \right]^p \\
&\quad \leq \sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \left(\frac{U_m^{(1-\lambda_1)p/q}}{V_n^{1-\lambda_2} \mu_m^{p/q}} a_m^p \right) \\
&\qquad \times \left[\sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{V_n^{(1-\lambda_2)(q-1)} \mu_m}{U_m^{1-\lambda_1}} \right]^{p-1} \\
&\quad = \frac{V_n^{1-p\lambda_2}}{(\varpi(\lambda_1, n))^{1-p} \upsilon_n} \sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{(1-\lambda_1)(p-1)} \upsilon_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p.
\end{aligned} \tag{30}$$

In view of (20), we find

$$\begin{aligned}
J &\leq (k_s(\lambda_1))^{\frac{1}{q}} \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{(1-\lambda_1)(p-1)} \upsilon_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}} \\
&= (k_s(\lambda_1))^{\frac{1}{q}} \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{(1-\lambda_1)(p-1)} \upsilon_n}{V_n^{1-\lambda_2} \mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}} \\
&= (k_s(\lambda_1))^{\frac{1}{q}} \left[\sum_{m=1}^{\infty} \omega(\lambda_2, m) \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right]^{\frac{1}{p}}.
\end{aligned} \tag{31}$$

Then, by (19), we have (27).

By Hölder's inequality (*cf.* [29]), we have

$$\begin{aligned}
I &= \sum_{n=1}^{\infty} \left[\frac{\upsilon_n^{\frac{1}{p}}}{V_n^{\frac{1}{p}-\lambda_2}} \sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}} a_m}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \right] \left(\frac{V_n^{\frac{1}{p}-\lambda_2}}{\upsilon_n^{\frac{1}{p}}} b_n \right) \\
&\leq J \|b\|_{q,\Psi_\lambda}.
\end{aligned} \tag{32}$$

Then, by (27), we have (26).

On the other hand, assuming that (26) is valid, we set

$$b_n := \frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}} a_m}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}}\right]^{p-1}, \quad n \in \mathbf{N}.$$

Then we find $J^p = \|b\|_{q,\Psi_\lambda}^q$. If $J = 0$, then (27) is trivially valid; if $J = \infty$, then, by (31) and (19), it is impossible. Suppose that $0 < J < \infty$. By (26), it follows that

$$\|b\|_{q,\Psi_\lambda}^q = J^p = I < k_s(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda},$$
$$\|b\|_{q,\Psi_\lambda}^{q-1} = J < k_s(\lambda_1)\|a\|_{p,\Phi_\lambda},$$

and then (27) follows, which is equivalent to (26). □

Theorem 2 *With the assumptions of Theorem* 1, *if* $m_0, n_0 \in \mathbf{N}$, $\mu_m \ge \mu_{m+1}$ ($m \in \{m_0, m_0 + 1, \ldots\}$), $\upsilon_n \ge \upsilon_{n+1}$ ($n \in \{n_0, n_0 + 1, \ldots\}$), $U(\infty) = V(\infty) = \infty$, *then the constant factor* $k_s(\lambda_1)$ *in* (26) *and* (27) *is the best possible.*

Proof For $\varepsilon \in (0, p(\lambda_1 + \alpha))$, we set $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in(-\alpha, 1-\alpha))$, $\widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ $(> -\alpha)$, and $\widetilde{a} = \{\widetilde{a}_m\}_{m=1}^\infty$, $\widetilde{b} = \{\widetilde{b}_n\}_{n=1}^\infty$,

$$\widetilde{a}_m := U_m^{\widetilde{\lambda}_1 - 1}\mu_m = U_m^{\lambda_1 - \frac{\varepsilon}{p} - 1}\mu_m, \qquad \widetilde{b}_n = V_n^{\widetilde{\lambda}_2 - \varepsilon - 1}\upsilon_n = V_n^{\lambda_2 - \frac{\varepsilon}{q} - 1}\upsilon_n. \tag{33}$$

Then, by (24), (25) and (23), we have

$$\begin{aligned}\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda} &= \left(\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}\left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}\widetilde{I} &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}}\widetilde{a}_m\widetilde{b}_n \\ &= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}}\frac{V_n^{\widetilde{\lambda}_2}\mu_m}{U_m^{1-\widetilde{\lambda}_1}}\right]\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\ &= \sum_{n=1}^{\infty}\varpi(\widetilde{\lambda}_1, n)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \ge k_s(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\bigl(1 - \vartheta(\widetilde{\lambda}_1, n)\bigr)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\ &= k_s(\widetilde{\lambda}_1)\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{\varepsilon+1}} - \sum_{n=1}^{\infty}O\left(\frac{\upsilon_n}{V_n^{\frac{\varepsilon}{q}+\lambda_1+\alpha+1}}\right)\right) \\ &= \frac{1}{\varepsilon}k_s(\widetilde{\lambda}_1)\left[\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\bigl(\widetilde{O}(1) - O(1)\bigr)\right].\end{aligned}$$

If there exists a positive constant $K \le k_s(\lambda_1)$ such that (26) is valid when replacing $k_s(\lambda_1)$ with K, then, in particular, we have $\varepsilon\widetilde{I} < \varepsilon K\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda}$, namely

$$k_s(\widetilde{\lambda}_1)\left[\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\bigl(\widetilde{O}(1) - O(1)\bigr)\right] < K\left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O(1)\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.$$

It follows that $k_s(\lambda_1) \leq K(\varepsilon \to 0^+)$. Hence, $K = k_s(\lambda_1)$ is the best possible constant factor of (26).

The constant factor $k_s(\lambda_1)$ in (27) is still the best possible. Otherwise, we would reach a contradiction by (32) that the constant factor in (26) is not the best possible. □

Remark 1 Inequality (26) is an extension of Hardy-Hilbert-type inequality (28) with parameters and a best possible constant factor.

For $p > 1$, we find $\Psi_\lambda^{1-p}(n) = \frac{\upsilon_n}{V_n^{1-p\lambda_2}}$ and define the following normed spaces:

$$l_{p,\Phi_\lambda} := \{a = \{a_m\}_{m=1}^\infty; \|a\|_{p,\Phi_\lambda} < \infty\},$$
$$l_{q,\Psi_\lambda} := \{b = \{b_n\}_{n=1}^\infty; \|b\|_{q,\Psi_\lambda} < \infty\},$$
$$l_{p,\Psi_\lambda^{1-p}} := \{c = \{c_n\}_{n=1}^\infty; \|c\|_{p,\Psi_\lambda^{1-p}} < \infty\}.$$

Assuming that $a = \{a_m\}_{m=1}^\infty \in l_{p,\Phi_\lambda}$, setting

$$c = \{c_n\}_{n=1}^\infty, \qquad c_n := \sum_{m=1}^\infty \prod_{k=1}^s \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} a_m, \quad n \in \mathbf{N},$$

we can rewrite (27) as follows:

$$\|c\|_{p,\Psi_\lambda^{1-p}} < k_s(\lambda_1)\|a\|_{p,\Phi_\lambda} < \infty,$$

namely $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 1 Define a Hardy-Hilbert-type operator $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ as follows: For any $a = \{a_m\}_{m=1}^\infty \in l_{p,\Phi_\lambda}$, there exists a unique representation $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$. Define the formal inner product of Ta and $b = \{b_n\}_{n=1}^\infty \in l_{q,\Psi_\lambda}$ as follows:

$$(Ta, b) := \sum_{n=1}^\infty \left[\sum_{m=1}^\infty \prod_{k=1}^s \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} a_m\right] b_n. \tag{34}$$

Then we can rewrite (26) and (27) as follows:

$$(Ta, b) < k_s(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{35}$$
$$\|Ta\|_{p,\Psi_\lambda^{1-p}} < k_s(\lambda_1)\|a\|_{p,\Phi_\lambda}. \tag{36}$$

Define the norm of operator T as follows:

$$\|T\| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}} \frac{\|Ta\|_{p,\Psi_\lambda^{1-p}}}{\|a\|_{p,\Phi_\lambda}}. \tag{37}$$

Then, by (36), we find $\|T\| \leq k_s(\lambda_1)$. Since by Theorem 2 the constant factor in (36) is the best possible, we have

$$\|T\| = k_s(\lambda_1). \tag{38}$$

4 Some reverses

In the following, we also set

$$\widetilde{\Phi}_\lambda(m) := \big(1-\theta(\lambda_2,m)\big)\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}},$$
$$\widetilde{\Psi}_\lambda(n) := \big(1-\vartheta(\lambda_1,n)\big)\frac{V_n^{q(1-\lambda_2)-1}}{\upsilon_n^{q-1}} \quad (m,n\in\mathbf{N}).$$

For $0<p<1$ or $p<0$, we still use the formal symbols of $\|a\|_{p,\Phi_\lambda}$, $\|b\|_{q,\Psi_\lambda}$, $\|a\|_{p,\widetilde{\Phi}_\lambda}$ and $\|b\|_{q,\widetilde{\Psi}_\lambda}$.

Theorem 3 *If* $s\in\mathbf{N}$, $0<c_1\le\cdots\le c_s$, $-\alpha<\lambda_1,\lambda_2\le 1-\alpha$, $\lambda_1+\lambda_2=\lambda$, $k_s(\lambda_1)$ *is indicated by* (11), $m_0,n_0\in\mathbf{N}$, $\mu_m\ge\mu_{m+1}$ $(m\in\{m_0,m_0+1,\ldots\})$, $\upsilon_n\ge\upsilon_{n+1}$ $(n\in\{n_0,n_0+1,\ldots\})$, $U(\infty)=V(\infty)=\infty$, *then for* $0<p<1$, $0<\|a\|_{p,\Phi_\lambda},\|b\|_{q,\Psi_\lambda}<\infty$, *we have the following equivalent inequalities with the best possible constant factor* $k_s(\lambda_1)$:

$$I=\sum_{n=1}^\infty\sum_{m=1}^\infty\prod_{k=1}^s\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}a_mb_n}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}$$
$$>k_s(\lambda_1)\|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{39}$$
$$J=\left\{\sum_{n=1}^\infty\frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^\infty\prod_{k=1}^s\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}a_m}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\right]^p\right\}^{\frac{1}{p}}$$
$$>k_s(\lambda_1)\|a\|_{p,\widetilde{\Phi}_\lambda}. \tag{40}$$

Proof By the reverse Hölder's inequality (*cf.* [29]) and (20), we have the reverses of (30), (31) and (32). Then, by (22), we have (40). By (40) and the reverse of (32), we have (39).

On the other hand, assuming that (39) is valid, we set b_n as in Theorem 1. Then we find $J^p=\|b\|_{q,\Psi_\lambda}^q$. If $J=\infty$, then (40) is trivially valid; if $J=0$, then, by reverse of (31) and (22), it is impossible. Suppose that $0<J<\infty$. By (39), it follows that

$$\|b\|_{q,\Psi_\lambda}^q=J^p=I>k_s(\lambda_1)\|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\Psi_\lambda},$$
$$\|b\|_{q,\Psi_\lambda}^{q-1}=J>k_s(\lambda_1)\|a\|_{p,\widetilde{\Phi}_\lambda},$$

and then (40) follows, which is equivalent to (39).

For $\varepsilon\in(0,p(\lambda_1+\alpha))$, we set $\widetilde{\lambda}_1$, $\widetilde{\lambda}_2$, $\widetilde{a}_m$ and $\widetilde{b}_n$ as (33). Then, by (24), (25) and (20), we find

$$\begin{aligned}\|a\|_{p,\widetilde{\Phi}_\lambda}\|b\|_{q,\Psi_\lambda}&=\left[\sum_{m=1}^\infty\big(1-\theta(\lambda_2,m)\big)\frac{\mu_m}{U_m^{1+\varepsilon}}\right]^{\frac{1}{p}}\left(\sum_{n=1}^\infty\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}}\\&=\left(\sum_{m=1}^\infty\frac{\mu_m}{U_m^{1+\varepsilon}}-\sum_{m=1}^\infty O\left(\frac{\mu_m}{U_m^{1+\lambda_2+\alpha+\varepsilon}}\right)\right)^{\frac{1}{p}}\left(\sum_{n=1}^\infty\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}}\\&=\frac{1}{\varepsilon}\left[\frac{1}{U_{m_0}^\varepsilon}+\varepsilon\big(O(1)-O_1(1)\big)\right]^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^\varepsilon}+\varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}},\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\widetilde{a}_m\widetilde{b}_n \\
&= \sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\frac{V_n^{\widetilde{\lambda}_2}\mu_m}{U_m^{1-\widetilde{\lambda}_1}}\right]\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \sum_{n=1}^{\infty}\varpi(\widetilde{\lambda}_1,n)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \le k_s(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \frac{1}{\varepsilon}k_s(\widetilde{\lambda}_1)\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\widetilde{O}(1)\right).
\end{aligned}$$

If there exists a constant $K \ge k_s(\lambda_1)$ such that (39) is valid when replacing $k_s(\lambda_1)$ with K, then, in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{a}\|_{p,\widetilde{\Phi}_\lambda}\|\widetilde{b}\|_{q,\Psi_\lambda}$, namely

$$k_s(\widetilde{\lambda}_1)\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\widetilde{O}(1)\right) > K\left[\frac{1}{U_{m_0}^{\varepsilon}}+\varepsilon\big(O(1)-O_1(1)\big)\right]^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\widetilde{O}(1)\right)^{\frac{1}{q}}.$$

It follows that $k_s(\lambda_1) \ge K$ ($\varepsilon \to 0^+$). Hence, $K = k_s(\lambda_1)$ is the best possible constant factor of (39).

The constant factor $k_s(\lambda_1)$ in (40) is still the best possible. Otherwise, we would reach a contradiction by the reverse of (32) that the constant factor in (39) is not the best possible. □

Theorem 4 *With the assumptions of Theorem* 3, *if* $p < 0$, *then we have the following equivalent inequalities with the best possible constant factor* $k_s(\lambda_1)$:

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}a_mb_n}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}} > k_s(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\widetilde{\Psi}_\lambda}, \tag{41}$$

$$\begin{aligned}
J_1 &:= \left\{\sum_{n=1}^{\infty}\frac{V_n^{p\lambda_2-1}\upsilon_n}{(1-\vartheta(\lambda_1,n))^{p-1}}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}a_m}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\right]^p\right\}^{\frac{1}{p}} \\
&> k_s(\lambda_1)\|a\|_{p,\Phi_\lambda}.
\end{aligned} \tag{42}$$

Proof By the reverse Hölder's inequality with weight (*cf.* [29]), since $p < 0$, by (23), we have

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}a_m}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\right]^p \\
&\quad= \left[\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\left(\frac{U_m^{(1-\lambda_1)/q}}{V_n^{(1-\lambda_2)/p}\mu_m^{1/q}}a_m\right)\left(\frac{V_n^{(1-\lambda_2)/p}\mu_m^{1/q}}{U_m^{(1-\lambda_1)/q}}\right)\right]^p \\
&\quad\le \sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\frac{U_m^{(1-\lambda_1)p/q}}{V_n^{1-\lambda_2}\mu_m^{p/q}}a_m^p \\
&\qquad\times\left[\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\frac{V_n^{(1-\lambda_2)(q-1)}\mu_m}{U_m^{1-\lambda_1}}\right]^{p-1} \\
&\quad= \frac{V_n^{1-p\lambda_2}}{(\varpi(\lambda_1,n))^{1-p}}\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\frac{U_m^{(1-\lambda_1)(p-1)}}{V_n^{1-\lambda_2}\mu_m^{p-1}}a_m^p
\end{aligned}$$

$$\leq \frac{(k_s(\lambda_1))^{p-1} V_n^{1-p\lambda_2}}{(1-\vartheta(\lambda_1,n))^{1-p}\upsilon_n} \sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p,$$

$$J_1 \geq \big(k_s(\lambda_1)\big)^{\frac{1}{q}} \left\{ \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p \right\}^{\frac{1}{p}}$$

$$= \big(k_s(\lambda_1)\big)^{\frac{1}{q}} \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}}}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p \right\}^{\frac{1}{p}}$$

$$= \big(k_s(\lambda_1)\big)^{\frac{1}{q}} \left\{ \sum_{m=1}^{\infty} \omega(\lambda_2, m) \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p \right\}^{\frac{1}{p}}. \tag{43}$$

Then, by (19), we have (44).

By the reverse Hölder's inequality (*cf.* [29]), we have

$$I = \sum_{n=1}^{\infty} \frac{V_n^{\lambda_2-\frac{1}{p}} \upsilon_n^{1/p}}{(1-\vartheta(\lambda_1,n))^{1/q}} \left[\sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}} a_m}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \right] \left[\big(1-\vartheta(\lambda_1,n)\big)^{\frac{1}{q}} \frac{V_n^{\frac{1}{p}-\lambda_2}}{\upsilon_n^{1/p}} b_n \right]$$
$$\geq J_1 \|b\|_{q,\widetilde{\Psi}_\lambda}. \tag{44}$$

Then, by (42), we have (41).

On the other hand, assuming that (41) is valid, we set b_n as follows:

$$b_n := \frac{V_n^{p\lambda_2-1}\upsilon_n}{(1-\vartheta(\lambda_1,n))^{p-1}} \left[\sum_{m=1}^{\infty} \prod_{k=1}^{s} \frac{(\min\{U_m, c_k V_n\})^{\frac{\alpha}{s}} a_m}{(\max\{U_m, c_k V_n\})^{\frac{\lambda+\alpha}{s}}} \right]^{p-1}, \quad n \in \mathbf{N}.$$

Then we find $J_1^p = \|b\|_{q,\widetilde{\Psi}_\lambda}^q$. If $J_1 = \infty$, then (42) is trivially valid; if $J_1 = 0$, then by (43) and (19) it is impossible. Suppose that $0 < J_1 < \infty$. By (41), it follows that

$$\|b\|_{q,\widetilde{\Psi}_\lambda}^q = J_1^p = I > k_s(\lambda_1) \|a\|_{p,\Phi_\lambda} \|b\|_{q,\widetilde{\Psi}_\lambda},$$
$$\|b\|_{q,\widetilde{\Psi}_\lambda}^{q-1} = J_1 > k_s(\lambda_1) \|a\|_{p,\Phi_\lambda},$$

and then (42) follows, which is equivalent to (41).

For $\varepsilon \in (0, q(\lambda_2+\alpha))$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}$ $(> -\alpha)$, $\widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$ $(\in(-\alpha, 1-\alpha))$, and

$$\widetilde{a}_m := U_m^{\widetilde{\lambda}_1-1-\varepsilon}\mu_m = U_m^{\lambda_1-\frac{\varepsilon}{p}-1}\mu_m, \qquad \widetilde{b}_n = V_n^{\widetilde{\lambda}_2-1}\upsilon_n = V_n^{\lambda_2-\frac{\varepsilon}{q}-1}\upsilon_n.$$

Then, by (24), (25) and (19), we have

$$\|\widetilde{a}\|_{p,\Phi_\lambda} \|\widetilde{b}\|_{q,\widetilde{\Psi}_\lambda} = \left(\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+\varepsilon}} \right)^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} \big(1-\vartheta(\lambda_1,n)\big) \frac{\upsilon_n}{V_n^{1+\varepsilon}} \right]^{\frac{1}{q}}$$
$$= \left(\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+\varepsilon}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{\upsilon_n}{V_n^{1+\varepsilon}} - \sum_{n=1}^{\infty} O\left(\frac{\upsilon_n}{V_n^{1+\lambda_1+\alpha+\varepsilon}} \right) \right)^{\frac{1}{q}}$$
$$= \frac{1}{\varepsilon} \left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O(1) \right)^{\frac{1}{p}} \left[\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon \big(\widetilde{O}(1) - O_1(1) \big) \right]^{\frac{1}{q}},$$

$$\begin{aligned}\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\widetilde{a}_m\widetilde{b}_n \\ &= \sum_{m=1}^{\infty}\left[\sum_{n=1}^{\infty}\prod_{k=1}^{s}\frac{(\min\{U_m,c_kV_n\})^{\frac{\alpha}{s}}}{(\max\{U_m,c_kV_n\})^{\frac{\lambda+\alpha}{s}}}\frac{U_m^{\widetilde{\lambda}_1}\upsilon_n}{V_n^{1-\widetilde{\lambda}_2}}\right]\frac{\mu_m}{U_m^{1+\varepsilon}} \\ &= \sum_{m=1}^{\infty}\omega(\widetilde{\lambda}_2,m)\frac{\mu_m}{U_m^{1+\varepsilon}} \leq k_s(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}} \\ &= \frac{1}{\varepsilon}k_s(\widetilde{\lambda}_1)\left(\frac{1}{U_{m_0}^{\varepsilon}}+\varepsilon O(1)\right).\end{aligned}$$

If there exists a constant $K \geq k_s(\lambda_1)$ such that (41) is valid when replacing $k_s(\lambda_1)$ with K, then, in particular, we have $\varepsilon\widetilde{I} > \varepsilon K\|\widetilde{a}\|_{p,\Phi_\lambda}\|\widetilde{b}\|_{q,\widetilde{\Psi}_\lambda}$, namely

$$\begin{aligned}&k_s(\widetilde{\lambda}_1)\left(\frac{1}{U_{m_0}^{\varepsilon}}+\varepsilon O(1)\right) \\ &\quad > K\left(\frac{1}{U_{m_0}^{\varepsilon}}+\varepsilon O(1)\right)^{\frac{1}{p}}\left[\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon\big(\widetilde{O}(1)-O_1(1)\big)\right]^{\frac{1}{q}}.\end{aligned}$$

It follows that $k_s(\lambda_1) \geq K$ ($\varepsilon \to 0^+$). Hence, $K = k_s(\lambda_1)$ is the best possible constant factor of (41).

The constant factor $k_s(\lambda_1)$ in (42) is still the best possible. Otherwise, we would reach a contradiction by (44) that the constant factor in (41) is not the best possible. □

Remark 2 (i) For $\alpha = 0$, $0 < \lambda_1, \lambda_2 \leq 1$ in (26) and (27), we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\prod_{k=1}^{s}(\max\{U_m,c_kV_n\})^{\frac{\lambda}{s}}} < \widetilde{k}_s(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{45}$$

$$\left\{\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(\max\{U_m,c_kV_n\})^{\frac{\lambda}{s}}}\right]^p\right\}^{\frac{1}{p}} < \widetilde{k}_s(\lambda_1)\|a\|_{p,\Phi_\lambda}, \tag{46}$$

where $\widetilde{k}_s(\lambda_1)$ is indicated by (14);

(ii) for $\alpha = -\lambda$, $-1 \leq \lambda_1, \lambda_2 < 0$ in (26) and (27), we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_mb_n}{\prod_{k=1}^{s}(\min\{U_m,c_kV_n\})^{\frac{\lambda}{s}}} < \widehat{k}_s(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{47}$$

$$\left\{\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty}\frac{a_m}{\prod_{k=1}^{s}(\min\{U_m,c_kV_n\})^{\frac{\lambda}{s}}}\right]^p\right\}^{\frac{1}{p}} < \widehat{k}_s(\lambda_1)\|a\|_{p,\Phi_\lambda}, \tag{48}$$

where $\widehat{k}_s(\lambda_1)$ is indicated by (15);

(iii) for $\lambda = 0$, $\lambda_2 = -\lambda_1$, in (26) and (27), we have the following equivalent inequalities:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\prod_{k=1}^{s}\left(\frac{\min\{U_m,c_kV_n\}}{\max\{U_m,c_kV_n\}}\right)^{\frac{\alpha}{s}}a_mb_n < k_s^{(0)}(\lambda_1)\|a\|_{p,\Phi_\lambda}\|b\|_{q,\Psi_\lambda}, \tag{49}$$

$$\left\{\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+p\lambda_1}}\left[\sum_{m=1}^{\infty}\prod_{k=1}^{s}\left(\frac{\min\{U_m,c_kV_n\}}{\max\{U_m,c_kV_n\}}\right)^{\frac{\alpha}{s}}a_m\right]^p\right\}^{\frac{1}{p}} < k_s^{(0)}(\lambda_1)\|a\|_{p,\Phi_\lambda}, \tag{50}$$

where $k_s^{(0)}(\lambda_1)$ is indicated by (16) ($|\lambda_1| < \alpha$, $0 < \alpha \le \frac{1}{2}$; $|\lambda_1| < 1-\alpha$, $\frac{1}{2} < \alpha \le 1$).

By Theorem 2, the constant factors in the above inequalities are all the best possible. We still can obtain some particular reverse inequalities with the best possible constant factors by Theorem 3 and Theorem 4.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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References

1. Hardy, GH, Littlewood, JE, Pólya, G: Inequalities. Cambridge University Press, Cambridge (1934)
2. Mitrinović, DS, Pečarić, JE, Fink, AM: Inequalities Involving Functions and Their Integrals and Derivatives. Kluwer Academic, Boston (1991)
3. Yang, BC: Hilbert-Type Integral Inequalities. Bentham Science Publishers Ltd., Sharjah (2009)
4. Yang, BC: Discrete Hilbert-Type Inequalities. Bentham Science Publishers Ltd., Sharjah (2011)
5. Yang, BC: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijing (2009)
6. Yang, BC: Two Types of Multiple Half-Discrete Hilbert-Type Inequalities. Lambert Academic Publishing, Saarbrücken (2012)
7. Persson, LE, et al.: Commutators of Hardy operators in vanishing Morrey spaces. In: 9th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences (ICNPAA 2012). AIP Conference Proceedings, vol. 1493, pp. 859-866 (2012). doi:10.1063/1.4765588
8. Yang, BC: On Hilbert's integral inequality. J. Math. Anal. Appl. **220**, 778-785 (1998)
9. Yang, BC, Brnetić, I, Krnić, M, Pečarić, JE: Generalization of Hilbert and Hardy-Hilbert integral inequalities. Math. Inequal. Appl. **8**(2), 259-272 (2005)
10. Krnić, M, Pečarić, JE: Hilbert's inequalities and their reverses. Publ. Math. (Debr.) **67**(3-4), 315-331 (2005)
11. Yang, BC, Rassias, TM: On the way of weight coefficient and research for Hilbert-type inequalities. Math. Inequal. Appl. **6**(4), 625-658 (2003)
12. Yang, BC, Rassias, TM: On a Hilbert-type integral inequality in the subinterval and its operator expression. Banach J. Math. Anal. **4**(2), 100-110 (2010)
13. Azar, L: On some extensions of Hardy-Hilbert's inequality and applications. J. Inequal. Appl. **2009**, Article ID 546829 (2009)
14. Arpad, B, Choonghong, O: Best constant for certain multilinear integral operator. J. Inequal. Appl. **2006**, Article ID 28582 (2006)
15. Kuang, JC, Debnath, L: On Hilbert's type inequalities on the weighted Orlicz spaces. Pac. J. Appl. Math. **1**(1), 95-103 (2007)
16. Zhong, WY: The Hilbert-type integral inequality with a homogeneous kernel of Lambda-degree. J. Inequal. Appl. **2008**, Article ID 917392 (2008)
17. Hong, Y: On Hardy-Hilbert integral inequalities with some parameters. J. Inequal. Pure Appl. Math. **6**(4), 92 (2005)
18. Zhong, WY, Yang, BC: On multiple Hardy-Hilbert's integral inequality with kernel. J. Inequal. Appl. **2007**, Article ID 27962 (2007). doi:10.1155/2007/27962
19. Yang, BC, Krnić, M: On the norm of a multi-dimensional Hilbert-type operator. Sarajevo J. Math. **7**(20), 223-243 (2011)
20. Krnić, M, Pečarić, JE, Vuković, P: On some higher-dimensional Hilbert's and Hardy-Hilbert's type integral inequalities with parameters. Math. Inequal. Appl. **11**, 701-716 (2008)
21. Krnić, M, Vuković, P: On a multidimensional version of the Hilbert-type inequality. Anal. Math. **38**, 291-303 (2012)
22. Rassias, TM, Yang, BC: On half-discrete Hilbert's inequality. Appl. Math. Comput. **220**, 75-93 (2013)
23. Rassias, TM, Yang, BC: A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function. Appl. Math. Comput. **225**, 263-277 (2013)
24. Rassias, TM, Yang, BC: On a multidimensional half - discrete Hilbert - type inequality related to the hyperbolic cotangent function. Appl. Math. Comput. **242**, 800-813 (2014)

25. Rassias, TM, Yang, BC: On a multidimensional Hilbert-type integral inequality associated to the gamma function. Appl. Math. Comput. **249**, 408-418 (2014)
26. Li, YJ, He, B: On inequalities of Hilbert's type. Bull. Aust. Math. Soc. **76**(1), 1-13 (2007)
27. Yang, BC: On a more accurate multidimensional Hilbert-type inequality with parameters. Math. Inequal. Appl. **18**(2), 429-441 (2015)
28. Yang, BC: An extension of a Hardy-Hilbert-type inequality. J. Guangdong Univ. Educ. **35**(3), 1-8 (2015)
29. Kuang, JC: Applied Inequalities. Shangdong Science Technic Press, Jinan (2004)