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Boundedness of θ-type Calderón-Zygmund operators on Hardy spaces with non-doubling measures

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Abstract

Let μ be a non-negative Radon measure on R^d which only satisfies some growth condition. In this paper, we obtain the boundedness of θ-type Calderón-Zygmund operators on the Hardy space $H^1(\mu)$.

MSC: 42B20

Keywords: non-doubling measure space; θ-type Calderón-Zygmund operator; Hardy space

1 Introduction and preliminaries

During the last few decades, the theory Calderón-Zygmund operators has played a central part of modern harmonic analysis with lots of extensive applications in the other fields of mathematics. One of the most general settings to which Calderón-Zygmund theory extends naturally is the spaces of homogeneous type in the sense of Coifman and Weiss [1]. Many results from real and harmonic analysis on Euclidean spaces have their natural extensions on these spaces (see, for example, [1–3]). A metric space (X,d) equipped with a non-negative Borel measure μ is called a space of homogeneous type if (X,d,μ) satisfies the measure doubling condition that there exists a positive constant C_μ, depending on μ, such that for any ball $B(x,r)=\{y\in X: d(x,y)<r\}$ with $x\in X$ and $r\in(0,\infty)$,

$$\mu\bigl(B(x,2r)\bigr)\le C_\mu\mu\bigl(B(x,r)\bigr). \tag{1.1}$$

This definition was introduced by Coifman and Weiss in [1]. The doubling condition (1.1) for measures plays a key role in the classical theory of Calderón-Zygmund operators. However, many results on the classical Calderón-Zygmund theory have been proved still valid if the doubling condition is replaced by some weaker conditions. In recent years, many papers focus on the analysis on R^d with non-doubling measure; see [4–8] and their references. Throughout this paper, the Euclidean space R^d is endowed with a non-negative Radon measure μ which only satisfies the following growth condition, that is, there exists $C>0$ such that

$$\mu\bigl(B(x,r)\bigr)\le Cr^n \tag{1.2}$$

for all $x \in R^d$ and $r > 0$, where $B(x,r) = \{y \in R^d : |x-y| < r\}$, n is a fixed number satisfying $0 < n \leq d$. Such a measure need not satisfy the doubling condition (1.1). In [6], Tolsa established Calderón-Zygmund theory for non-doubling measures.

The definition of θ-type Calderón-Zygmund operator was introduced by Yabuta in [9] as follows.

Definition 1.1 Let θ be a non-negative, non-decreasing function on $R^+ = (0,\infty)$ satisfying

$$\int_0^1 \frac{\theta(t)}{t}\,dt < \infty. \tag{1.3}$$

A kernel $K(\cdot,\cdot) \in L^1_{\mathrm{loc}}(X \times X \backslash \{(x,y) : x = y\})$ is called a θ-type Calderón-Zygmund kernel if the following conditions hold:

$$\bigl|K(x,y)\bigr| \leq C|x-y|^{-n} \tag{1.4}$$

and

$$\bigl|K(x,y) - K\bigl(x',y\bigr)\bigr| + \bigl|K(y,x) - K\bigl(y,x'\bigr)\bigr| \leq C\theta\biggl(\frac{|x-x'|}{|x-y|}\biggr)|x-y|^{-n}, \tag{1.5}$$

when $|x-y| \geq 2|x-x'|$.

A linear operator T is called the θ-type Calderón-Zygmund operator with kernel $K(\cdot,\cdot)$ satisfying (1.4) and (1.5) if for all $f \in L^\infty(\mu)$ with bounded support and $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{R^d} K(x,y)f(y)\,d\mu(y). \tag{1.6}$$

In [10], the authors proved that the θ-type Calderón-Zygmund operator which is bounded on $L^2(\mu)$ is also bounded from $L^\infty(\mu)$ into $RBMO(\mu)$ and from $H^{1,\infty}_{atb}(\mu)$ into $L^1(\mu)$ on the Euclidean space with non-doubling measures.

In this paper, we discuss the boundedness of the θ-type Calderón-Zygmund operator T in the Hardy space $H^1(\mu)$. In order to state our main result, we recall some necessary notations and the known results. The following grand maximal operator was introduced by Tolsa in [11].

Definition 1.2 Given $f \in L^1_{\mathrm{loc}}(\mu)$, we set

$$M_\Phi f(x) = \sup_{\varphi \sim x} \biggl|\int_{R^d} f\varphi\,d\mu\biggr|,$$

where the notation $\varphi \sim x$ means that $\varphi \in L^1(\mu) \cap C^1(R^d)$ and satisfies

(i) $\|\varphi\|_{L^1(\mu)} \leq 1$,
(ii) $0 \leq \varphi(y) \leq |y-x|^{-n}$ for all $y \in R^d$, and
(iii) $|\nabla\varphi(y)| \leq |y-x|^{-(n+1)}$ for all $y \in R^d$, where $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_d)$.

In [11], Tolsa obtained the following result.

Theorem 1.1 *A function f belongs to $H^{1,\infty}_{atb}(\mu)$ if and only if $f \in L^1(\mu)$, $\int f\,d\mu = 0$ and $M_\Phi f \in L^1(\mu)$. Moreover, in this case*

$$\|f\|_{H^{1,\infty}_{atb}(\mu)} \approx \|f\|_{L^1(\mu)} + \|M_\Phi f\|_{L^1(\mu)}.$$

In [12], the authors introduced a new atomic characterization of the Hardy space $H^1(\mu)$. Given two cubes $Q \subset R$ in R^d, set

$$K_{Q,R} = 1 + \sum_{k=1}^{N_{Q,R}} \frac{\mu(2^k Q)}{l(2^k Q)^n},$$

where $N_{Q,R}$ is the smallest positive integer k such that $l(2^k Q) \geq l(R)$; see [6] for some positive of $K_{Q,R}$. The definition of the (p,γ)-atomic block is given as follows.

Definition 1.3 Let $\rho > 1$, $1 < p \leq \infty$ and $\gamma \in N$. A function $b \in L^1_{\mathrm{loc}}(\mu)$ is called a (p,γ)-atomic block if

(1) there exists some cube R such that $\operatorname{supp}(b) \subset R$,
(2) $\int_{R^d} b\,d(\mu) = 0$,
(3) there are functions a_1, a_2 supported on cubes $Q_1, Q_2 \subset R$ and numbers $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $b = \lambda_1 a_1 + \lambda_2 a_2$, and

$$\|a_j\|_{L^p(\mu)} \leq \bigl(\mu(\rho Q_j)\bigr)^{1/p-1} (K_{Q_j,R})^{-\gamma}, \quad j = 1,2.$$

We denote $|b|_{H^{1,p}_{atb,\gamma}(\mu)} = |\lambda_1| + |\lambda_2|$. We say that $f \in H^{1,p}_{atb,\gamma}(\mu)$ if there are (p,γ)-atomic blocks b_j such that

$$f = \sum_{i=1}^{\infty} b_i, \tag{1.7}$$

with $\sum_{i=1}^{\infty} |b_i|_{H^{1,p}_{atb,\gamma}(\mu)} < \infty$ (notice that this implies that the sum in (1.7) converges in $L^1(\mu)$). The $H^{1,p}_{atb,\gamma}(\mu)$ norm of f is defined by

$$\|f\|_{H^{1,p}_{atb,\gamma}(\mu)} = \inf \sum_{i=1}^{\infty} |b_i|_{H^{1,p}_{atb,\gamma}(\mu)},$$

where the infimum is taken over all the possible decompositions of f into (p,γ)-atomic blocks.

We remark that the definition when $\gamma = 1$ was introduced by Tolsa in [6]. It was proved in [6, 12] that the definition of $H^{1,p}_{atb,\gamma}(\mu)$ is independent of the chosen constant $\rho > 1$, and for any integer $\gamma \geq 1$ and $1 < p \leq \infty$, all the atomic Hardy spaces $H^{1,p}_{atb,\gamma}(\mu)$ are just the Hardy space $H^{1,\infty}_{at}(\mu)$ with equivalent norms.

Let T^* be the transpose of T. As mentioned in [13], we have to assume that $T^*1 = 0$. Here, by $T^*1 = 0$, we mean that for any bounded function b with compact support and

$\int_{R^d} b\mu = 0$,

$$\int_{R^d} Tb(x)\, d\mu(x) = 0. \tag{1.8}$$

The main result of our paper is given as follows.

Theorem 1.2 *Let T be a θ-type Calderón-Zygmund operator defined by* (1.6) *as above, which is bounded on $L^2(\mu)$ and $T^*1 = 0$ as in* (1.8). *Then T is bounded on $H^1(\mu)$.*

Throughout this paper, C always means a positive constant independent of the main parameters involved, but it may be different in different contents.

2 Proof of our main result

The following lemma will be used in the proof of Theorem 1.2.

Lemma 2.1 *Let M_Φ be as in Definition* 1.2 *and $1 < p < \infty$. Then M_Φ is bounded on $L^p(\mu)$.*

In fact, Tolsa proved that M_Φ is bounded from $H^1(\mu)$ into $L^1(\mu)$; see Lemma 3.1 in [11]. On the other hand, it is obvious that M_Φ is bounded on $L^\infty(\mu)$ for $1 < p < \infty$. By Theorem 7.2 in [6], we obtain that M_Φ is bounded on $L^p(\mu)$ for $1 < p < \infty$.

Now we will prove Theorem 1.2.

Proof of Theorem 1.2 By the standard argument, it suffices to verify that for any atomic block b as in Definition 1.3 with $\rho = 4$, $p = \infty$ and $\gamma = 2$, Tb is in $H^1(\mu)$ with norm $C|b|_{H^{1,\infty}_{atb,2}}$. By Definition 1.3, it follows

$$\|a_j\|_{L^\infty(\mu)} \le \bigl(\mu(4Q_j)K^2_{Q_j,R}\bigr)^{-1}, \tag{2.1}$$

where $j = 1, 2$. The assumption that $T^*1 = 0$ tells us that $\int_{R^d} Tb\, d(\mu) = 0$. Recalling that T is bounded from $H^1(\mu)$ into $L^1(\mu)$ (see [6]), we obtain

$$\|Tb\|_{L^1(\mu)} \le C|b|_{H^{1,\infty}_{atb}(\mu)}.$$

By this and Theorem 1.1, we deduce that the proof of Theorem 1.2 can be reduced to proving that

$$\bigl\|M_\Phi(Tb)\bigr\|_{L^1(\mu)} \le C|b|_{H^{1,\infty}_{atb}(\mu)}. \tag{2.2}$$

We can write

$$\int_{R^d} M_\Phi(Tb)(x)\, d\mu(x) = \int_{R^d\setminus 4R} M_\Phi(Tb)(x)\, d\mu(x) + \int_{4R} M_\Phi(Tb)(x)\, d\mu(x) = I_1 + I_2.$$

Let us now estimate I_1. Let x_R be the center of the cube R. From the fact $T^*1 = 0$, we obtain

$$\begin{aligned} I_1 &= \int_{R^d\setminus 4R} \sup_{\varphi\sim x} \left| \int_{R^d} Tb(y)\bigl[\varphi(y) - \varphi(x_R)\bigr]\, d\mu(y) \right| d\mu(x) \\ &\le \int_{R^d\setminus 4R} \sup_{\varphi\sim x} \left| \int_{2R} Tb(y)\bigl[\varphi(y) - \varphi(x_R)\bigr]\, d\mu(y) \right| d\mu(x) \end{aligned}$$

$$+\int_{R^d\setminus 4R}\sup_{\varphi\sim x}\left|\int_{R^d\setminus 2R}Tb(y)\bigl[\varphi(y)-\varphi(x_R)\bigr]\,d\mu(y)\right|\,d\mu(x)$$
$$= I_{11}+I_{12}.$$

Note that for any $z\in 2R$, $x\in 2^{k+1}R\setminus 2^kR$, and $k\geq 2$, we have $|x-z|\geq l(2^{k-2}R)$.

This together with Definition 1.2 and the mean value theorem leads to

$$\bigl|\varphi(y)-\varphi(x_R)\bigr|\leq C\frac{l(R)}{l(2^{k-2}R)^{n+1}}. \tag{2.3}$$

For $j=1,2$, denote $N_{Q_j,2R}$ simply by N_j for $y\in 2R$. By (2.3), (1.4), Hölder's inequality, the boundedness of T in $L^2(\mu)$ and (2.1), we have

$$\begin{aligned}
I_{11} &= \sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\sup_{\varphi\sim x}\biggl[\int_{2R\setminus 2Q_j}\bigl|Ta_j(y)\bigr|\bigl|\varphi(y)-\varphi(x_R)\bigr|\,d\mu(y)\biggr]\,d\mu(x)\\
&\quad+\sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\sup_{\varphi\sim x}\biggl[\int_{2Q_j}\bigl|Ta_j(y)\bigr|\bigl|\varphi(y)-\varphi(x_R)\bigr|\,d\mu(y)\biggr]\,d\mu(x)\\
&\leq C\sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\frac{l(R)}{l(2^{k-2}R)^{n+1}}\sum_{l=1}^{N_j-1}\int_{2^{l+1}Q_j\setminus 2^lQ_j}\int_{Q_j}\frac{|a_j(z)|}{|y-z|^n}\,d\mu(z)\,d\mu(y)\,d\mu(x)\\
&\quad+C\sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\frac{l(R)}{l(2^{k-2}R)^{n+1}}\bigl\|(Ta_j)\chi_{2Q_j}\bigr\|_{L^1(\mu)}\,d\mu(x)\\
&\leq C\sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{\infty}2^{-k}\sum_{l=1}^{N_j-1}\frac{\mu(2^{l+1}Q_j)}{l(2^{l+1}Q_j)^n}\|a_j\|_{L^\infty(\mu)}\mu(Q_j)\\
&\quad+C\sum_{j=1}^{2}|\lambda_j|\sum_{k=2}^{\infty}2^{-k}\bigl\|(Ta_j)\chi_{2Q_j}\bigr\|_{L^2(\mu)}\mu(2Q_j)^{1/2}\\
&\leq C\sum_{j=1}^{2}|\lambda_j|K_{Q_j,R}\|a_j\|_{L^\infty(\mu)}\mu(Q_j)+C\sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^2(\mu)}\mu(2Q_j)^{1/2}\\
&\leq C\sum_{j=1}^{2}|\lambda_j|,
\end{aligned}$$

where we have used the fact that

$$K_{Q_j,2R}\leq CK_{Q_j,R}.$$

For I_{12}, we get

$$\begin{aligned}
I_{12} &= \sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\sup_{\varphi\sim x}\left|\int_{R^d\setminus 2R}Tb(y)\bigl[\varphi(y)-\varphi(x_R)\bigr]\,d\mu(y)\right|\,d\mu(x)\\
&\leq \sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}M_\Phi\bigl[|Tb|\chi_{2^{k+2}R\setminus 2^{k-1}R}\bigr](x)\,d\mu(x)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\sup_{\varphi\sim x}\left[\int_{2^{k+2}R\setminus 2^{k-1}R}\left|Tb(y)\right|\varphi(x_R)\,d\mu(y)\right]d\mu(x)\\
&+\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\sup_{\varphi\sim x}\left[\int_{R^d\setminus 2^{k+2}R}\left|Tb(y)\right|\left(\varphi(y)+\varphi(x_R)\right)d\mu(y)\right]d\mu(x)\\
&+\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\sup_{\varphi\sim x}\left[\int_{2^{k-1}R\setminus 2R}\left|Tb(y)\right|\left(\varphi(y)+\varphi(x_R)\right)d\mu(y)\right]d\mu(x)\\
={}&I_{121}+I_{122}+I_{123}+I_{124}.
\end{aligned}
$$

From Lemma 2.1, the fact that $\int_{R^d} b\,d(\mu)=0$ and (1.5), we can deduce that

$$
\begin{aligned}
I_{121}&\le\sum_{k=2}^{\infty}\mu\left(2^{k+1}R\right)^{1/2}\left\|M_\Phi\left[|Tb|\chi_{2^{k+2}R\setminus 2^{k-1}R}\right]\right\|_{L^2(\mu)}\\
&\le C\sum_{k=2}^{\infty}\mu\left(2^{k+1}R\right)^{1/2}\left(\int_{2^{k+2}R\setminus 2^{k-1}R}\left|\int_R\left(K(y,z)-K(y,x_R)\right)b(z)\,d\mu(z)\right|^2d\mu(y)\right)^{1/2}\\
&\le C\sum_{k=2}^{\infty}\mu\left(2^{k+1}R\right)^{1/2}\\
&\quad\times\left(\int_{2^{k+2}R\setminus 2^{k-1}R}\left[\int_R\theta\left(\frac{|z-x_R|}{|y-x_R|}\right)|y-x_R|^{-n}\left|b(z)\right|d\mu(z)\right]^2d\mu(y)\right)^{1/2}\\
&\le C\sum_{k=2}^{\infty}\frac{\mu(2^{k+1}R)}{l(2^kR)^n}\theta\left(2^{-k}\right)\|b\|_{L^1(\mu)}\le C\int_0^1\frac{\theta(t)}{t}\,dt\|b\|_{L^1(\mu)}\le C\sum_{j=1}^{2}|\lambda_j|,
\end{aligned}
$$

where we have used the following inequality:

$$
\int_0^1\frac{\theta(t)}{t}\ge\sum\int_{2^k}^{2^{1-k}}\frac{\theta(2^{-k})}{2^{1-k}}\ge C\sum_{k=1}^{\infty}\theta\left(2^{-k}\right),
$$

and the fact

$$
\|b\|_{L^1(\mu)}\le\sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^1(\mu)}\le C\sum_{j=1}^{2}|\lambda_j|.
$$

An argument similar to the estimate for I_{121} tells us that

$$
I_{122}\le C\sum_{j=1}^{2}|\lambda_j|.
$$

Finally, we estimate I_{123}. By the fact that $\int_{R^d} b\,d\mu=0$, Definition 1.2 and (1.5), we obtain

$$
\begin{aligned}
I_{123}\le{}&\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\sum_{l=k+2}^{\infty}\int_{2^{l+1}R\setminus 2^lR}\int_R\left|K(y,z)-K(y,x_R)\right|\left|b(z)\right|d\mu(z)\\
&\times\left[\frac{1}{|y-x|^n}+\frac{1}{|x_R-x|^n}\right]d\mu(y)\,d\mu(x)
\end{aligned}
$$

$$\begin{aligned}
&\le C\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\sum_{l=k+2}^{\infty}\int_{2^{l+1}R\setminus 2^lR}\int_R \theta\left(\frac{|z-x_R|}{|y-x_R|}\right)|y-x_R|^{-n}\left|b(z)\right|d\mu(z)\\
&\quad\times\left[\frac{1}{|y-x|^n}+\frac{1}{|x_R-x|^n}\right]d\mu(y)\,d\mu(x)\\
&\le C\sum_{k=2}^{\infty}\sum_{l=k+2}^{\infty}\theta\left(2^{-l}\right)\frac{\mu(2^{l+1}R)}{l(2^{l+1}R)^n}\frac{\mu(2^{k+1}R)}{l(2^{k+1}R)^n}\|b\|_{L^1(\mu)}\\
&\le C\sum_{j=1}^{2}|\lambda_j|.
\end{aligned}$$

An argument similar to the estimate for I_{123} indicates that

$$I_{124}\le C\sum_{j=1}^{2}|\lambda_j|.$$

Combining the estimate for I_{121}, I_{122}, I_{123} and I_{124}, we obtain the desired estimate for I_{12}. The estimates for I_{11} and I_{12} tell us that

$$I_1=\int_{R^d\setminus 4R}M_\Phi(Tb)(x)\,d\mu(x)\le C|b|_{H^{1,\infty}_{atb,2}}(\mu). \tag{2.4}$$

For I_2, by the sublinearity of M_Φ, it follows

$$I_2\le\int_{4R}M_\Phi\left[(Tb)\chi_{8R}\right](x)\,d\mu(x)+\int_{4R}M_\Phi\left[(Tb)\chi_{R^d\setminus 8R}\right](x)\,d\mu(x)=I_{21}+I_{22}.$$

From $Q_j\subset R$, Definition 1.2 and (2.1), we obtain

$$\begin{aligned}
I_{22}&\le\int_{4R}\sup_{\varphi\sim x}\left[\int_{R^d\setminus 8R}\left|Tb(y)\right|\varphi(y)\,d\mu(y)\right]d\mu(x)\\
&\le\sum_{j=1}^{2}|\lambda_j|\int_{4R}\sum_{k=2}^{\infty}\int_{2^{k+1}R\setminus 2^kR}\left|\int_{Q_j}K(y,z)a_j(z)\,d\mu(z)\right|\frac{1}{|x-y|^n}\,d\mu(y)\,d\mu(x)\\
&\le C\sum_{j=1}^{2}|\lambda_j|\sum_{k=3}^{\infty}\|a_j\|_{L^\infty(\mu)}\mu(Q_j)\frac{\mu(2^{k+1}R)}{l(2^{k-2}R)^n}\frac{\mu(4R)}{l(2^{k-2}R)^n}\\
&\le C\sum_{j=1}^{2}|\lambda_j|.
\end{aligned}$$

In order to estimate I_{21}, we write

$$\begin{aligned}
I_{21}&\le\sum_{j=1}^{2}|\lambda_j|\int_{4Q_j}M_\Phi\left[(Ta_j)\chi_{8R}\right](x)\,d\mu(x)+\sum_{j=1}^{2}|\lambda_j|\int_{4R\setminus 4Q_j}M_\Phi\left[(Ta_j)\chi_{2Q_j}\right](x)\,d\mu(x)\\
&\quad+\sum_{j=1}^{2}|\lambda_j|\int_{4R\setminus 4Q_j}M_\Phi\left[(Ta_j)\chi_{8R\setminus 2Q_j}\right](x)\,d\mu(x)\\
&\quad=I_{211}+I_{212}+I_{213}.
\end{aligned}$$

Hölder's inequality, Lemma 2.1, the boundedness of T in $L^2(\mu)$ and (2.1) lead to

$$\begin{aligned}
I_{211} &\leq \sum_{j=1}^{2} |\lambda_j| \mu(4Q_j)^{1/2} \bigl\| M_\Phi\bigl[(Ta_j)\chi_{8R}\bigr]\bigr\|_{L^2(\mu)} \\
&\leq C \sum_{j=1}^{2} |\lambda_j| \mu(4Q_j)^{1/2} \|Ta_j\|_{L^2(\mu)} \leq C \sum_{j=1}^{2} |\lambda_j| \mu(4Q_j)^{1/2} \|a_j\|_{L^2(\mu)} \\
&\leq C \sum_{j=1}^{2} |\lambda_j| \mu(4Q_j) \|a_j\|_{L^\infty(\mu)} \leq C \sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

By Definition 1.2, Hölder's inequality, the boundedness of T in $L^2(\mu)$ and (2.1), we get

$$\begin{aligned}
I_{212} &\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\backslash 2^k Q_j} \sup_{\varphi\sim x} \left| \int_{2Q_j} Ta_j(y)\varphi(y)\,d\mu(y)\right| d\mu(x) \\
&\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\backslash 2^k Q_j} \frac{1}{l(2^{k-2}Q_j)^n}\,d\mu(x) \int_{2Q_j} \bigl|Ta_j(y)\bigr|\,d\mu(y) \\
&\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \frac{\mu(2^{k+1}Q_j)}{l(2^{k-2}Q_j)^n} \|Ta_j\|_{L^2(\mu)} \mu(2Q_j)^{1/2} \\
&\leq C \sum_{j=1}^{2} |\lambda_j| K_{Q_j,R} \mu(2Q_j)^{1/2} \|a_j\|_{L^2(\mu)} \\
&\leq C \sum_{j=1}^{2} |\lambda_j|,
\end{aligned}$$

where we have used the fact that

$$K_{Q_j,4R} \leq C K_{Q_j,R}. \tag{2.5}$$

For I_{213}, we can write

$$\begin{aligned}
I_{213} &= \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\backslash 2^k Q_j} M_\Phi\bigl[(Ta_j)\chi_{8R\backslash 2Q_j}\bigr](x)\,d\mu(x) \\
&\leq \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\backslash 2^k Q_j} M_\Phi\bigl[|Ta_j|\chi_{2^{k+2}Q_j\backslash 2^{k-1}Q_j}\bigr](x)\,d\mu(x) \\
&\quad + \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\backslash 2^k Q_j} M_\Phi\bigl[|Ta_j|\chi_{\max\{2^{k+2}Q_j,8R\}\backslash 2^{k+2}Q_j}\bigr](x)\,d\mu(x) \\
&\quad + \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\backslash 2^k Q_j} M_\Phi\bigl[|Ta_j|\chi_{2^{k-1}Q_j\backslash 2Q_j}\bigr](x)\,d\mu(x) \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Lemma 2.1, (1.4) and (2.1) imply that

$$\begin{aligned}
J_1 &= \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \mu\bigl(2^{k+1}Q_j\bigr)^{1/2} \bigl\| M_\Phi\bigl[f|Ta_j|\chi_{2^{k+2}Q_j\setminus 2^{k-1}Q_j}\bigr]\bigr\|_{L^2(\mu)} \\
&\le C\sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \mu\bigl(2^{k+1}Q_j\bigr)^{1/2} \times \biggl(\int_{2^{k+2}Q_j\setminus 2^{k-1}Q_j} \biggl|\int_{Q_j} K(y,z)a_j(z)\,d\mu(z)\biggr|^2 d\mu(y)\biggr)^{1/2} \\
&\le C\sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \frac{\mu(2^{k+2}Q_j)}{l(2^{k-3}Q_j)^n} \|a_j\|_{L^\infty(\mu)}\mu(Q_j) \\
&\le C\sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

By (ii) of Definition 1.2, (1.4), (2.5) and (2.1), we have

$$\begin{aligned}
J_2 &= \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\setminus 2^kQ_j} \sup_{\varphi\sim x}\biggl[\int_{2^{k-1}Q_j\setminus 2Q_j} \bigl|Ta_j(y)\bigr|\varphi(y)\,d\mu(y)\biggr]\,d\mu(x) \\
&\le \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\setminus 2^kQ_j} \sum_{l=1}^{k-2} \int_{2^{l+1}Q_j\setminus 2^lQ_j} \biggl|\int_{Q_j} K(y,z)a_j(z)\,d\mu(z)\biggr| \frac{1}{|y-x|^n}\,d\mu(y)\,d\mu(x) \\
&\le C\sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \frac{\mu(2^{k+1}Q)}{l(2^{k+1}Q_j)^n} \sum_{l=1}^{k-2} \frac{\mu(2^{l+1}Q)}{l(2^{l+1}Q_j)^n} \|a_j\|_{L^\infty(\mu)}\mu(Q_j) \\
&\le C\sum_{j=1}^{2} |\lambda_j| (K_{Q_j,R})^2 \|a_j\|_{L^\infty(\mu)}\mu(Q_j) \\
&\le C\sum_{j=1}^{2} |\lambda_j|.
\end{aligned}$$

With the argument similar to the estimate for J_2 it follows that

$$J_3 = \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\setminus 2^kQ_j} M_\Phi\bigl[|Ta_j|\chi_{2^{k-1}Q_j\setminus 2Q_j}\bigr](x)\,d\mu(x) \le C\sum_{j=1}^{2} |\lambda_j|.$$

Thus

$$I_{213} = \sum_{j=1}^{2} |\lambda_j| \sum_{k=2}^{N_{Q_j,4R}} \int_{2^{k+1}Q_j\setminus 2^kQ_j} M_\Phi\bigl[(Ta_j)\chi_{8R\setminus 2Q_j}\bigr](x)\,d\mu(x) \le C\sum_{j=1}^{2} |\lambda_j|.$$

From the estimation of I_{21} and I_{22}, we obtain

$$I_2 = \int_{4R} M_\Phi(Tb)(x)\,d\mu(x) \le C\sum_{j=1}^{2} |\lambda_j| = C|b|_{H^{1,\infty}_{atb,2}}. \tag{2.6}$$

The estimates (2.4) and (2.6) lead to (2.2), and this completes the proof of our theorem. □

Remark 2.2 It is known that the dual space of $H^1(\mu)$ is the space $RBMO(\mu)$, which is introduced in [12]. From Theorem 1.2, the fact that $RBMO(\mu) = (H^1(\mu))^*$ and a standard dual argument, it is easy to deduce the boundedness of the transpose operator of T on the $RBMO(\mu)$ space as below.

Corollary 2.3 *Let T be the same as in Theorem* 1.2. *Then T^*, the transpose operator of T, is bounded on $RBMO(\mu)$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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