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On a product-type system of difference equations of second order solvable in closed form

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Abstract
It is shown that the following system of difference equations

$$z_{n+1}=\frac{z_n^a}{w_{n-1}^b},\qquad w_{n+1}=\frac{w_n^c}{z_{n-1}^d},\quad n\in\mathbb{N}_0,$$

where $a,b,c,d\in\mathbb{Z}$, $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}$, is solvable in closed form.

MSC: Primary 39A10; 39A20

Keywords: system of difference equations; second order system; product-type system; solvable in closed form

1 Introduction

Recently there has been a great interest in studying nonlinear difference equations and systems not stemming from differential ones (see, *e.g.*, [1–30]). The old area of solving difference equations and systems has re-attracted recent attention (see, *e.g.*, [1–6, 12, 15, 19–26, 28–30]). Recent Stević's idea of transforming complicated equations and systems into simpler solvable ones, used for the first time in explaining the solvability of the equation appearing in [6] (an extension of the original result can be found in [20], see also [22]), was employed in several other papers (see, *e.g.*, [1, 2, 4, 12, 15, 19, 21, 24–26, 29, 30] and the related references therein). Another area of some recent interest, essentially initiated by Papaschinopoulos and Schinas, is studying symmetric and close to symmetric systems of difference equations (see, *e.g.*, [5, 8–10, 13, 14, 19, 21, 23, 25, 27–29]).

Stević also essentially triggered a systematic study of non-rational concrete difference equations and systems, from one side those obtained by using the translation operator (see, *e.g.*, [16] and also [11]) and from the other side those obtained by using max-type operators (see, *e.g.*, [17, 18, 27]), see also the related references cited therein. We would like to point out that for the equations and systems in [16–18, 27] only long-term behavior of their positive solutions are studied. For instance, the boundedness of positive solutions to the system

$$x_{n+1}=\max\left\{a,\frac{y_n^p}{x_{n-1}^q}\right\},\qquad y_{n+1}=\max\left\{a,\frac{x_n^p}{y_{n-1}^q}\right\},\quad n\in\mathbb{N}_0,\tag{1}$$

with $\min\{a, p, q\} > 0$, was investigated in [27]. System (1) is obviously obtained from the next product-type one

$$x_{n+1} = \frac{y_n^p}{x_{n-1}^q}, \qquad y_{n+1} = \frac{x_n^p}{y_{n-1}^q}, \quad n \in \mathbb{N}_0, \tag{2}$$

by acting with the max-type operator $m_a(t) = \max\{a, t\}$ onto the right-hand sides of both equations in (2) (see also [17] and [18] for related scalar equations). Note that for the case of positive initial values, system (2) can be solved by taking the logarithm to the both sides of both equations therein, since this transforms the system to a linear second order system of difference equations with constant coefficients, which is solvable. Note that the method does not work if initial values are not positive. Let us also mention here that positive solutions to difference equations and systems are often studied since many real-life models produce such solutions (see, *e.g.*, [7, 15, 31]). It is also interesting to note that there are max-type systems of difference equations which are solvable (see [23]). Finally, we want to note that the long-term behavior of solutions to product-type systems and those obtained from them by acting with some 'reasonable good' transformations are frequently closely related, which is another reason for studying these systems.

Hence, a natural problem is to investigate the solvability of product-type difference equations and systems with real and/or complex initial values. In [26], Stević and his collaborators started studying the problem with an approach different from the ones in [5, 21, 24, 25], but which can be regarded as a modification of some of the methods in [16–18, 27]. They showed therein that the system

$$z_{n+1} = \frac{w_n^a}{z_{n-1}^b}, \qquad w_{n+1} = \frac{z_n^c}{w_{n-1}^d}, \quad n \in \mathbb{N}_0, \tag{3}$$

where $a, b, c, d \in \mathbb{Z}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$, is solvable in closed form and presented numerous applications of obtained formulas.

In this paper we continue our investigation by studying the solvability of the following system of difference equations:

$$z_{n+1} = \frac{z_n^a}{w_{n-1}^b}, \qquad w_{n+1} = \frac{w_n^c}{z_{n-1}^d}, \quad n \in \mathbb{N}_0, \tag{4}$$

where $a, b, c, d \in \mathbb{Z}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$.

Let us mention here that although systems (3) and (4) are similar in appearance, the methods used in dealing with them are quite different.

It is easy to see that the domain of undefinable solutions [24] to system (4) is the set

$$\mathcal{U} = \left\{(z_{-1}, z_0, w_{-1}, w_0) \in \mathbb{C}^4 : z_{-1} = 0 \text{ or } z_0 = 0 \text{ or } w_{-1} = 0 \text{ or } w_0 = 0\right\}.$$

Hence, from now on we will assume that our initial values belong to the set $\mathbb{C}^4 \setminus \mathcal{U}$.

A solution $(z_n, w_n)_{n\geq -1}$ of system (4) is called *periodic* (or *eventually periodic*) with period $p \in \mathbb{N}$ if there is $n_0 \geq -1$ such that

$$(z_{n+p}, w_{n+p}) = (z_n, w_n) \quad \text{for } n \geq n_0.$$

Period p is prime if there is no $\hat{p} \in \mathbb{N}$, $\hat{p} < p$ which is a period of the solution. For $p = 1$, the solution is called *eventually constant* (see, *e.g.*, [32]). For some results on the topic, see, *e.g.*, [7, 15] and the related references therein. If it is said that a solution of system (4) is periodic with period p, it will need not mean that it is prime.

A system of difference equations of the form

$$\begin{aligned} z_n &= f(z_{n-1}, \dots, z_{n-k}, w_{n-1}, \dots, w_{n-k}) \\ w_n &= g(z_{n-1}, \dots, z_{n-k}, w_{n-1}, \dots, w_{n-k}), \quad n \in \mathbb{N}_0, \end{aligned}$$

where $k \in \mathbb{N}$, is said to be *solvable in closed form* if its general solution can be found in terms of initial values z_{-i}, w_{-i}, $i = \overline{1,k}$, delay k and index n only.

2 Main result

The main result in this paper is proved in this section.

Theorem 1 *Assume that* $a, b, c, d \in \mathbb{Z}$ *and* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then system* (4) *is solvable in closed form.*

Proof *Case* $b = 0$. In this case system (4) becomes

$$z_{n+1} = z_n^a, \qquad w_{n+1} = \frac{w_n^c}{z_{n-1}^d}, \quad n \in \mathbb{N}_0. \tag{5}$$

From the first equation in (5) we easily obtain

$$z_n = z_0^{a^n}, \quad n \in \mathbb{N}_0. \tag{6}$$

Employing (6) into the second equation in (5), we get

$$w_n = \frac{w_{n-1}^c}{z_0^{da^{n-2}}} \tag{7}$$

for $n \geq 2$.

Hence, by using (7), we have that

$$\begin{aligned} w_n &= \frac{1}{z_0^{da^{n-2}}} \left(\frac{w_{n-2}^c}{z_0^{da^{n-3}}} \right)^c = \frac{w_{n-2}^{c^2}}{z_0^{da^{n-2}+dca^{n-3}}} \\ &= \frac{1}{z_0^{da^{n-2}+dca^{n-3}}} \left(\frac{w_{n-3}^c}{z_0^{da^{n-4}}} \right)^{c^2} = \frac{w_{n-3}^{c^3}}{z_0^{da^{n-2}+dca^{n-3}+dc^2a^{n-4}}} \end{aligned}$$

for $n \geq 4$.

Assume that we have proved

$$w_n = \frac{w_{n-k}^{c^k}}{z_0^{da^{n-2}+dca^{n-3}+dc^2a^{n-4}+\cdots+dc^{k-1}a^{n-k-1}}} \tag{8}$$

for $n \geq k + 1$.

Then, by using (7) with $n \to n-k$ into (8), we get

$$w_n = \frac{1}{z_0^{da^{n-2}+dca^{n-3}+dc^2a^{n-4}+\cdots+dc^{k-1}a^{n-k-1}}}\left(\frac{w_{n-k-1}^c}{z_0^{da^{n-k-2}}}\right)^{c^k}$$
$$= \frac{w_{n-k-1}^{c^{k+1}}}{z_0^{da^{n-2}+dca^{n-3}+dc^2a^{n-4}+\cdots+dc^{k-1}a^{n-k-1}+dc^ka^{n-k-2}}} \tag{9}$$

for $n \geq k+2$.

From (7), (9) and the method of induction we see that (8) holds for every k such that $1 \leq k \leq n-1$.

By taking $k = n-1$ into (8) we get

$$w_n = \frac{w_1^{c^{n-1}}}{z_0^{da^{n-2}+dca^{n-3}+dc^2a^{n-4}+\cdots+dc^{n-2}}} \tag{10}$$

for $n \geq 2$.

Now we have two subcases to consider.

Subcase $a \neq c$. In this case from (10) we get

$$w_n = \frac{w_1^{c^{n-1}}}{z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}}}, \quad n \geq 2. \tag{11}$$

Using the next relation

$$w_1 = \frac{w_0^c}{z_{-1}^d} \tag{12}$$

in (11) we get

$$w_n = \frac{w_0^{c^n}}{z_0^{d\frac{a^{n-1}-c^{n-1}}{a-c}}z_{-1}^{dc^{n-1}}}, \quad n \in \mathbb{N}. \tag{13}$$

Subcase $a = c$. In this case from (10) we get

$$w_n = \frac{w_1^{a^{n-1}}}{z_0^{d(n-1)a^{n-2}}} \tag{14}$$

for $n \geq 2$.

Using (12) with $a = c$ into (14), we get

$$w_n = \frac{w_0^{a^n}}{z_0^{d(n-1)a^{n-2}}z_{-1}^{da^{n-1}}}, \quad n \in \mathbb{N}. \tag{15}$$

Case $d = 0$. In this case system (4) becomes

$$z_{n+1} = \frac{z_n^a}{w_{n-1}^b}, \qquad w_{n+1} = w_n^c, \quad n \in \mathbb{N}_0. \tag{16}$$

From the second equation in (16) we have that

$$w_n = w_0^{c^n}, \quad n \in \mathbb{N}_0. \tag{17}$$

Employing (17) into the first equation in (16), we get

$$z_n = \frac{z_{n-1}^a}{w_0^{bc^{n-2}}} \tag{18}$$

for $n \geq 2$.

Hence, by using (18), we have

$$\begin{aligned} z_n &= \frac{1}{w_0^{bc^{n-2}}}\left(\frac{z_{n-2}^a}{w_0^{bc^{n-3}}}\right)^a = \frac{z_{n-2}^{a^2}}{w_0^{bc^{n-2}+bac^{n-3}}} \\ &= \frac{1}{w_0^{bc^{n-2}+bac^{n-3}}}\left(\frac{z_{n-3}^a}{w_0^{bc^{n-4}}}\right)^{a^2} = \frac{z_{n-3}^{a^3}}{w_0^{bc^{n-2}+bac^{n-3}+ba^2c^{n-4}}} \end{aligned}$$

for $n \geq 4$.

Assume that we have proved

$$z_n = \frac{z_{n-k}^{a^k}}{w_0^{bc^{n-2}+bac^{n-3}+ba^2c^{n-4}+\cdots+ba^{k-1}c^{n-k-1}}} \tag{19}$$

for $n \geq k+1$.

Then, by using (18) with $n \to n-k$ into (19), we get

$$\begin{aligned} z_n &= \frac{1}{w_0^{bc^{n-2}+bac^{n-3}+ba^2c^{n-4}+\cdots+ba^{k-1}c^{n-k-1}}}\left(\frac{z_{n-k-1}^a}{w_0^{bc^{n-k-2}}}\right)^{a^k} \\ &= \frac{z_{n-k-1}^{a^{k+1}}}{w_0^{bc^{n-2}+bac^{n-3}+ba^2c^{n-4}+\cdots+ba^{k-1}c^{n-k-1}+ba^kc^{n-k-2}}} \end{aligned} \tag{20}$$

for $n \geq k+2$.

From (18), (20) and the method of induction we see that (19) holds for every k such that $1 \leq k \leq n-1$.

By taking $k = n-1$ into (19) we get

$$z_n = \frac{z_1^{a^{n-1}}}{w_0^{bc^{n-2}+bac^{n-3}+ba^2c^{n-4}+\cdots+bca^{n-3}+ba^{n-2}}} \tag{21}$$

for $n \geq 2$.

Now we have two subcases to consider.

Subcase $a \neq c$. In this case from (21) we get

$$z_n = \frac{z_1^{a^{n-1}}}{w_0^{b\frac{a^{n-1}-c^{n-1}}{a-c}}}, \quad n \geq 2. \tag{22}$$

Using the next relation

$$z_1 = \frac{z_0^a}{w_{-1}^b} \tag{23}$$

in (22) we get

$$z_n = \frac{z_0^{a^n}}{w_0^{b\frac{a^{n-1}-c^{n-1}}{a-c}} w_{-1}^{ba^{n-1}}}, \quad n \in \mathbb{N}. \tag{24}$$

Subcase $a = c$. In this case from (21) we get

$$z_n = \frac{z_1^{a^{n-1}}}{w_0^{b(n-1)a^{n-2}}} \tag{25}$$

for $n \geq 2$.

Using (23) into (25) we get

$$z_n = \frac{z_0^{a^n}}{w_0^{b(n-1)a^{n-2}} w_{-1}^{ba^{n-1}}}, \quad n \in \mathbb{N}. \tag{26}$$

Case $bd \neq 0$. First note that from the first equation in (4), for every well-defined solution, we have that

$$w_{n-1}^b = \frac{z_n^a}{z_{n+1}}, \quad n \in \mathbb{N}_0, \tag{27}$$

while from the second one it follows that

$$w_{n+1}^b = \frac{w_n^{bc}}{z_{n-1}^{bd}}, \quad n \in \mathbb{N}_0. \tag{28}$$

Using (27) into (28) we obtain

$$\frac{z_{n+2}^a}{z_{n+3}} = \frac{z_{n+1}^{ac}}{z_{n+2}^c z_{n-1}^{bd}}, \quad n \in \mathbb{N}_0,$$

which can be written as

$$z_{n+3} = z_{n+2}^{a+c} z_{n+1}^{-ac} z_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{29}$$

which is a fourth order product-type difference equation.

Note also that

$$z_1 = \frac{z_0^a}{w_{-1}^b} \quad \text{and} \quad z_2 = \frac{z_1^a}{w_0^b} = \frac{z_0^{a^2}}{w_0^b w_{-1}^{ab}}. \tag{30}$$

Let

$$a_1 = a + c, \qquad b_1 = -ac, \qquad c_1 = 0, \qquad d_1 = bd. \tag{31}$$

Then equation (29) can be written as

$$z_{n+3} = z_{n+2}^{a_1} z_{n+1}^{b_1} z_n^{c_1} z_{n-1}^{d_1}, \quad n \in \mathbb{N}_0. \tag{32}$$

From (32) with $n \to n-1$ we get

$$z_{n+2} = z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}, \quad n \in \mathbb{N}. \tag{33}$$

Employing (33) into (32) we get

$$\begin{aligned} z_{n+3} &= \left(z_{n+1}^{a_1} z_n^{b_1} z_{n-1}^{c_1} z_{n-2}^{d_1}\right)^{a_1} z_{n+1}^{b_1} z_n^{c_1} z_{n-1}^{d_1} \\ &= z_{n+1}^{a_1a_1+b_1} z_n^{a_1b_1+c_1} z_{n-1}^{a_1c_1+d_1} z_{n-2}^{a_1d_1} \\ &= z_{n+1}^{a_2} z_n^{b_2} z_{n-1}^{c_2} z_{n-2}^{d_2} \end{aligned} \tag{34}$$

for $n \in \mathbb{N}$, where

$$a_2 := a_1a_1 + b_1, \qquad b_2 := a_1b_1 + c_1, \qquad c_2 := a_1c_1 + d_1, \qquad d_2 := a_1d_1. \tag{35}$$

From (32) with $n \to n-2$ we get

$$z_{n+1} = z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} z_{n-3}^{d_1} \tag{36}$$

for $n \geq 2$.

Employing (36) into (34) we get

$$\begin{aligned} z_{n+3} &= \left(z_n^{a_1} z_{n-1}^{b_1} z_{n-2}^{c_1} z_{n-3}^{d_1}\right)^{a_2} z_n^{b_2} z_{n-1}^{c_2} z_{n-2}^{d_2} \\ &= z_n^{a_1a_2+b_2} z_{n-1}^{b_1a_2+c_2} z_{n-2}^{c_1a_2+d_2} z_{n-3}^{d_1a_2} \\ &= z_n^{a_3} z_{n-1}^{b_3} z_{n-2}^{c_3} z_{n-3}^{d_3} \end{aligned} \tag{37}$$

for $n \geq 2$, where

$$a_3 := a_1a_2 + b_2, \qquad b_3 := b_1a_2 + c_2, \qquad c_3 := c_1a_2 + d_2, \qquad d_3 := d_1a_2. \tag{38}$$

Assume that for some $2 \leq k \leq n$, we have proved that

$$z_{n+3} = z_{n+3-k}^{a_k} z_{n+2-k}^{b_k} z_{n+1-k}^{c_k} z_{n-k}^{d_k} \tag{39}$$

for $n \geq k-1$, and that

$$\begin{aligned} &a_k = a_1a_{k-1} + b_{k-1}, \qquad b_k = b_1a_{k-1} + c_{k-1}, \\ &c_k = c_1a_{k-1} + d_{k-1}, \qquad d_k = d_1a_{k-1}. \end{aligned} \tag{40}$$

Then, by using the relation

$$z_{n+3-k} = z_{n+2-k}^{a_1} z_{n+1-k}^{b_1} z_{n-k}^{c_1} z_{n-1-k}^{d_1},$$

for $n \geq k$, into (39) we obtain

$$\begin{aligned} z_{n+3} &= \left(z_{n+2-k}^{a_1} z_{n+1-k}^{b_1} z_{n-k}^{c_1} z_{n-1-k}^{d_1}\right)^{a_k} z_{n+2-k}^{b_k} z_{n+1-k}^{c_k} z_{n-k}^{d_k} \\ &= z_{n+2-k}^{a_1 a_k + b_k} z_{n+1-k}^{b_1 a_k + c_k} z_{n-k}^{c_1 a_k + d_k} z_{n-1-k}^{d_1 a_k} \\ &= z_{n+2-k}^{a_{k+1}} z_{n+1-k}^{b_{k+1}} z_{n-k}^{c_{k+1}} z_{n-1-k}^{d_{k+1}} \end{aligned} \tag{41}$$

for $n \geq k$, where

$$\begin{aligned} &a_{k+1} := a_1 a_k + b_k, \qquad b_{k+1} := b_1 a_k + c_k, \\ &c_{k+1} := c_1 a_k + d_k, \qquad d_{k+1} := d_1 a_k. \end{aligned} \tag{42}$$

This along with (34), (35) and the method of induction shows that (39) and (40) hold for every $2 \leq k \leq n+1$.

Hence, for $k = n+1$, we have

$$\begin{aligned} z_{n+3} &= z_2^{a_{n+1}} z_1^{b_{n+1}} z_0^{c_{n+1}} z_{-1}^{d_{n+1}} \\ &= \left(\frac{z_0^{a^2}}{w_0^b w_{-1}^{ab}}\right)^{a_{n+1}} \left(\frac{z_0^a}{w_{-1}^b}\right)^{b_{n+1}} z_0^{c_{n+1}} z_{-1}^{d_{n+1}} \\ &= z_0^{a^2 a_{n+1} + ab_{n+1} + c_{n+1}} z_{-1}^{d_{n+1}} w_0^{-ba_{n+1}} w_{-1}^{-aba_{n+1} - bb_{n+1}}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{43}$$

From the recurrent relations (40) we easily obtain that the sequence $(a_k)_{k\geq 5}$ satisfies the difference equation

$$a_k = a_1 a_{k-1} + b_1 a_{k-2} + c_1 a_{k-3} + d_1 a_{k-4}. \tag{44}$$

Since $b_{k-1} = a_k - a_1 a_{k-1}$ and equation (44) is linear, we have that the sequence $(b_k)_{k\in\mathbb{N}}$ is also a solution to equation (44). From this, the linearity of equation (44) and since $c_{k-1} = b_k - b_1 a_{k-1}$, we have that the sequence $(c_k)_{k\in\mathbb{N}}$ is also a solution to equation (44). Finally, since $d_k = d_1 a_{k-1}$, the linearity of equation (44) shows that $(d_k)_{k\in\mathbb{N}}$ is also a solution to the equation.

Now, we show that these four sequences can be prolonged for some negative indices of use. This enables easier getting formulas for solutions to system (4).

From (42) with $k = 0$ we get

$$a_1 = a_1 a_0 + b_0, \qquad b_1 = b_1 a_0 + c_0, \qquad c_1 = c_1 a_0 + d_0, \qquad d_1 = d_1 a_0. \tag{45}$$

Since $bd = d_1 \neq 0$, from the last equation in (45) we get $a_0 = 1$. Using this fact in the first three equalities in (45), we get $b_0 = c_0 = d_0 = 0$.

From this and by (42) with $k = -1$ we get

$$\begin{aligned} &1 = a_0 = a_1 a_{-1} + b_{-1}, \qquad 0 = b_0 = b_1 a_{-1} + c_{-1}, \\ &0 = c_0 = c_1 a_{-1} + d_{-1}, \qquad 0 = d_0 = d_1 a_{-1}. \end{aligned} \tag{46}$$

Since $d_1 \neq 0$, from the last equation in (46) we get $a_{-1} = 0$. Using this fact in other three equalities in (46), we get $b_{-1} = 1$, $c_{-1} = d_{-1} = 0$.

From this and by (42) with $k = -2$ we get

$$\begin{aligned} 0 = a_{-1} = a_1 a_{-2} + b_{-2}, \qquad 1 = b_{-1} = b_1 a_{-2} + c_{-2}, \\ 0 = c_{-1} = c_1 a_{-2} + d_{-2}, \qquad 0 = d_{-1} = d_1 a_{-2}. \end{aligned} \tag{47}$$

Since $d_1 \neq 0$, from the last equation in (47) we get $a_{-2} = 0$. Using this fact in other three equalities in (47), we get $b_{-2} = 0$, $c_{-2} = 1$ and $d_{-2} = 0$.

From this and by (42) with $k = -3$ we get

$$\begin{aligned} 0 = a_{-2} = a_1 a_{-3} + b_{-3}, \qquad 0 = b_{-2} = b_1 a_{-3} + c_{-3}, \\ 1 = c_{-2} = c_1 a_{-3} + d_{-3}, \qquad 0 = d_{-2} = d_1 a_{-3}. \end{aligned} \tag{48}$$

Since $d_1 \neq 0$, from the last equation in (48) we get $a_{-3} = 0$. Using this fact in other three equalities in (48), we get $b_{-3} = 0$, $c_{-3} = 0$ and $d_{-3} = 1$.

Hence, sequences $(a_k)_{k\ge-3}$, $(b_k)_{k\ge-3}$, $(c_k)_{k\ge-3}$ and $(d_k)_{k\ge-3}$ are solutions to linear difference equation (44) satisfying the following initial conditions:

$$\begin{aligned} &a_{-3} = 0, \qquad a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_0 = 1; \\ &b_{-3} = 0, \qquad b_{-2} = 0, \qquad b_{-1} = 1, \qquad b_0 = 0; \\ &c_{-3} = 0, \qquad c_{-2} = 1, \qquad c_{-1} = 0, \qquad c_0 = 0; \\ &d_{-3} = 1, \qquad d_{-2} = 0, \qquad d_{-1} = 0, \qquad d_0 = 0, \end{aligned} \tag{49}$$

respectively.

Since difference equation (44) is solvable, it follows that closed form formulas for $(a_k)_{k\ge-3}$, $(b_k)_{k\ge-3}$, $(c_k)_{k\ge-3}$ and $(d_k)_{k\ge-3}$ can be found. From this fact and (43) we see that equation (29) is solvable too.

From the second equation in (4), for every well-defined solution, we have that

$$z_{n-1}^d = \frac{w_n^c}{w_{n+1}}, \quad n \in \mathbb{N}_0, \tag{50}$$

while from the first one it follows that

$$z_{n+1}^d = \frac{z_n^{ad}}{w_{n-1}^{bd}}, \quad n \in \mathbb{N}_0. \tag{51}$$

Using (50) into (51) we obtain

$$\frac{w_{n+2}^c}{w_{n+3}} = \frac{w_{n+1}^{ac}}{w_{n+2}^a w_{n-1}^{bd}}, \quad n \in \mathbb{N}_0,$$

which can be written as

$$w_{n+3} = w_{n+2}^{a+c} w_{n+1}^{-ac} w_{n-1}^{bd}, \quad n \in \mathbb{N}_0, \tag{52}$$

which is nothing but difference equation (29). However, the sequence $(w_n)_{n\geq-1}$ satisfies the following initial conditions:

$$w_1 = \frac{w_0^c}{z_{-1}^d} \quad \text{and} \quad w_2 = \frac{w_1^c}{z_0^d} = \frac{w_0^{c^2}}{z_0^d z_{-1}^{cd}}. \tag{53}$$

Hence, the above presented procedure can be repeated, and it can be obtained that for $1 \leq k \leq n+1$,

$$w_{n+3} = w_{n+3-k}^{a_k} w_{n+2-k}^{b_k} w_{n+1-k}^{c_k} w_{n-k}^{d_k}, \quad n \in \mathbb{N}_0, \tag{54}$$

where $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ satisfy recurrent relations (40) with initial conditions (31).

From (54) with $k = n+1$ and by using (53) we get

$$\begin{aligned} w_{n+3} &= w_2^{a_{n+1}} w_1^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\ &= \left(\frac{w_0^{c^2}}{z_0^d z_{-1}^{cd}}\right)^{a_{n+1}} \left(\frac{w_0^c}{z_{-1}^d}\right)^{b_{n+1}} w_0^{c_{n+1}} w_{-1}^{d_{n+1}} \\ &= w_0^{c^2 a_{n+1} + c b_{n+1} + c_{n+1}} w_{-1}^{d_{n+1}} z_0^{-d a_{n+1}} z_{-1}^{-cd a_{n+1} - d b_{n+1}}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{55}$$

Also the sequences $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, $(c_k)_{k\in\mathbb{N}}$ and $(d_k)_{k\in\mathbb{N}}$ satisfy the difference equation (44) with initial conditions in (49), respectively.

As above the solvability of equation (44) shows that closed form formulas for $(a_k)_{k\geq-3}$, $(b_k)_{k\geq-3}$, $(c_k)_{k\geq-3}$ and $(d_k)_{k\geq-3}$ can be found. This fact along with (55) implies that equation (52) is solvable too. A direct calculation shows that the sequences $(z_n)_{n\geq-1}$ in (43) and $(w_n)_{n\geq-1}$ in (55) are solutions to system (4) with initial values z_{-1}, z_0, that is, w_{-1}, w_0 respectively. Hence, system (4) is also solvable in this case, finishing the proof of the theorem. □

Remark 1 Note that difference equation (44) is not only theoretically but also practically solvable since the characteristic polynomial

$$p_4(\lambda) = \lambda^4 - a_1\lambda^3 - b_1\lambda^2 - c_1\lambda - d_1 \tag{56}$$

associated to the difference equation is of fourth order, which means that we can explicitly find its roots.

Remark 2 Since we are interested in those initial values $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C}$ which uniquely define solutions to system (4), to avoid multi-valued solutions to the system, we posed the condition $a, b, c, d \in \mathbb{Z}$.

From the proof of Theorem 1 we obtain the following corollary.

Corollary 1 *Consider system* (4) *with* $a, b, c, d \in \mathbb{Z}$. *Assume that* $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. *Then the following statements are true.*

(a) *If* $b = 0$ *and* $a \neq c$, *then the general solution to system* (4) *is given by* (6) *and* (13).

(b) *If* $b = 0$ *and* $a = c$, *then the general solution to system* (4) *is given by* (6) *and* (15).
(c) *If* $d = 0$ *and* $a \neq c$, *then the general solution to system* (4) *is given by* (17) *and* (24).
(d) *If* $d = 0$ *and* $a = c$, *then the general solution to system* (4) *is given by* (17) *and* (26).
(e) *If* $bd \neq 0$, *then the general solution to system* (4) *is given by* (43) *and* (55).

Let λ_i, $i = \overline{1,4}$, be the roots of the characteristic polynomial (56) of difference equation (44). If they satisfy the condition

$$\lambda_i \neq \lambda_j \quad \text{for } i \neq j,$$

then it is known that a general solution to equation (44) has the following form:

$$u_n = \alpha_1\lambda_1^n + \alpha_2\lambda_2^n + \alpha_3\lambda_3^n + \alpha_4\lambda_4^n, \quad n \in \mathbb{N}, \tag{57}$$

where α_i, $i = \overline{1,4}$, are arbitrary constants. Since for the case $d_1 \neq 0$ the solution can be prolonged for nonpositive indices, then we may assume that formula (57) holds also for $n \geq -3$ (or $n \geq -4$ if necessary).

In order to find, in this case, a general solution to system (4) in closed form, we will need the following known lemma. We give a proof of it for the completeness and benefit of the reader.

Lemma 1 *Assume that* λ_j, $j = \overline{1,k}$, *are pairwise different zeros of the polynomial*

$$P(z) = \alpha_k z^k + \alpha_{k-1} z^{k-1} + \cdots + \alpha_1 z + \alpha_0.$$

Then

$$\sum_{j=1}^{k} \frac{\lambda_j^l}{P'(\lambda_j)} = 0$$

for $l = \overline{0, k-2}$, *and*

$$\sum_{j=1}^{k} \frac{\lambda_j^{k-1}}{P'(\lambda_j)} = \frac{1}{\alpha_k}.$$

Proof The functions

$$f_l(z) = \frac{z^l}{P(z)}, \quad l \in \mathbb{N},$$

are meromorphic on the Riemann sphere. Hence, by the residue theorem, we have that

$$\sum_{j=1}^{k} \operatorname{Res}_{z=\lambda_j} f_l(z) + \operatorname{Res}_{z=\infty} f_l(z) = 0 \tag{58}$$

for every $l \in \mathbb{N}$.

Now note that the Laurent expansion of f_l at zero is

$$f_l(z) = \frac{z^l}{\alpha_k \prod_{j=1}^k (z-\lambda_j)} = \frac{z^{l-k}}{\alpha_k \prod_{j=1}^k (1-\lambda_j/z)} = \frac{1}{\alpha_k} z^{l-k} + \sum_{s=1}^{\infty} b_{-s} z^{l-k-s}$$

for some complex numbers b_{-s}, $s \in \mathbb{N}$.

On the other hand, since λ_j, $j = \overline{1,k}$, are simple poles of f_l, we have that

$$\operatorname{Res}_{z=\lambda_j} f_l(z) = \frac{\lambda_j^l}{P'(\lambda_j)}, \quad j = \overline{1,k}.$$

From this and since $\operatorname{Res}_{z=\infty} f_l(z)$ is equal to the negative value of the coefficient at $1/z$ in the Laurent expansion, it follows that $\operatorname{Res}_{z=\infty} f_l(z) = 0$ when $l = \overline{0,k-2}$ and $\operatorname{Res}_{z=\infty} f_{k-1}(z) = -1/\alpha_k$. Using these facts in (58) the lemma follows. □

If we apply Lemma 1 to polynomial p_4 in (56), and since $p_4(t) = \prod_{l=1}^4 (t-\lambda_j)$ (note that $\alpha_4 = 1$), we have

$$\sum_{j=1}^4 \frac{\lambda_j^l}{p_4'(\lambda_j)} = 0$$

for $l = \overline{0,2}$, and

$$\sum_{j=1}^4 \frac{\lambda_j^3}{p_4'(\lambda_j)} = 1.$$

From this, since from (49) we have $a_{-3} = a_{-2} = a_{-1} = 0$ and $a_0 = 1$, and a general solution of (4) has the form in (57), we obtain

$$\begin{aligned} a_n &= \sum_{j=1}^4 \frac{\lambda_j^{n+3}}{p_4'(\lambda_j)} \\ &= \frac{\lambda_1^{n+3}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \frac{\lambda_2^{n+3}}{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)} \\ &\quad + \frac{\lambda_3^{n+3}}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)(\lambda_3-\lambda_4)} + \frac{\lambda_4^{n+3}}{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)} \end{aligned} \tag{59}$$

for $n \geq -3$.

On the other hand, from (40) we get

$$b_n = a_{n+1} - a_1 a_n, \tag{60}$$

$$c_n = b_{n+1} - b_1 a_n, \tag{61}$$

$$d_n = d_1 a_{n-1} \tag{62}$$

for $n \geq -3$.

By using (59) into (60) we get

$$b_n = \sum_{j=1}^{4} \frac{\lambda_j - a_1}{p_4'(\lambda_j)} \lambda_j^{n+3} \tag{63}$$

for $n \geq -3$.

By using (59) and (63) into (61) we get

$$c_n = \sum_{j=1}^{4} \frac{(\lambda_j - a_1)\lambda_j - b_1}{p_4'(\lambda_j)} \lambda_j^{n+3} \tag{64}$$

for $n \geq -3$.

By using (59) into (62) we get

$$d_n = \sum_{j=1}^{4} \frac{d_1}{p_4'(\lambda_j)} \lambda_j^{n+2} \tag{65}$$

for $n \geq -3$, where we have used the fact that (59) also holds for $n = -4$ (in fact, we may assume that equality (59) holds for every $n \geq -s$, for any fixed $s \in \mathbb{N}$, since due to the assumption $d_1 \neq 0$, any solution of equation (44) can be prolonged for any nonpositive value of index n).

By using (59), (63), (64) and (65) into (43) and (55), we get formulas for general solutions to system (4) in closed form.

Formulas obtained in this section can be used in describing the long-term behavior of solutions to system (4) in many cases. We will formulate and prove here only one result, just as an example. The formulations and proofs of other results, which are similar and whose proofs use standard techniques, we leave to the reader as some exercises.

Theorem 2 *Assume that $b = c = 0$ and $a, d \in \mathbb{Z}$. Then the following statements hold*:

(a) *If $a = 1$, then every solution to system* (4) *is eventually constant.*

(b) *If $a = 0$, then $z_n = w_n = 1$, $n \geq 3$.*

(c) *If $a = -1$, then every solution to system* (4) *is two-periodic.*

(d) *If $a > 1$ and $|z_0| < 1$, then $z_n \to 0$ as $n \to \infty$.*

(e) *If $a > 1$ and $|z_0| > 1$, then $|z_n| \to \infty$ as $n \to \infty$.*

(f) *If $a > 1$ and $|z_0^d| < 1$, then $|w_n| \to \infty$ as $n \to \infty$.*

(g) *If $a > 1$ and $|z_0^d| > 1$, then $w_n \to 0$ as $n \to \infty$.*

(h) *If $a < -1$ and $|z_0| < 1$, then $z_{2n} \to 0$ as $n \to \infty$ and $|z_{2n+1}| \to \infty$ as $n \to \infty$.*

(i) *If $a < -1$ and $|z_0| > 1$, then $z_{2n+1} \to 0$ as $n \to \infty$ and $|z_{2n}| \to \infty$ as $n \to \infty$.*

(j) *If $a < -1$ and $|z_0^d| > 1$, then $w_{2n} \to 0$ as $n \to \infty$ and $|w_{2n+1}| \to \infty$ as $n \to \infty$.*

(k) *If $a < -1$ and $|z_0^d| < 1$, then $w_{2n+1} \to 0$ as $n \to \infty$ and $|w_{2n}| \to \infty$ as $n \to \infty$.*

Proof (a) If we replace $a = 1$ and $c = 0$ in (6) and (7), we obtain $z_n = z_0$ and $w_{n+2} = 1/z_0^d$, $n \in \mathbb{N}_0$, from which the statement follows.

(b) By replacing $a = 0$ and $c = 0$ in (6) and (7), we get $z_n = 1$, $n \in \mathbb{N}$ and $w_n = 1$, $n \geq 3$, from which the statement follows.

(c) By replacing $a = -1$ and $c = 0$ in (6) and (7), we get $z_{2n} = z_0$, $z_{2n+1} = \frac{1}{z_0}$, $w_{2n} = 1/z_0^d$ and $w_{2n+1} = z_0^d$, $n \in \mathbb{N}$, from which the statement follows.

(d)-(k) From (6) and (7) with $c = 0$ we get

$$z_n = z_0^{a^n}, \qquad w_n = \frac{1}{z_0^{da^{n-2}}}, \quad n \geq 2. \tag{66}$$

Using the formulas in (66) all these statements easily follow. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the manuscript.

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