Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

The moment of maximum normed randomly weighted sums of martingale differences

Mei Yao[1,2,3*] and Lu Lin[1,3]

*Correspondence:
ymwalzn@163.com
[1]Shandong University Qilu Securities Institute for Financial Studies, Shandong University, Jinan, China
[2]School of Mathematics, Hefei University of Technology, Hefei, China
Full list of author information is available at the end of the article

Abstract

By using some inequalities and properties of martingale differences, we investigate the moment of maximum normed randomly weighted sums of martingale differences under some weakly conditions. A sufficient condition to the moment of this stochastic sequence with maximum norm is presented in this paper.

MSC: 60F15; 60F25

Keywords: random weighted; maximum normed; martingale differences

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence, independent and identically distributed with $EX_1 = 0$. Denote $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$. For $0 < r < 2$ and $p > 0$, it is well known that

$$\begin{cases} E|X_1|^r < \infty, & \text{if } p < r, \\ E[|X_1|^r \log(1 + |X_1|)] < \infty, & \text{if } p = r, \\ E|X_1|^p < \infty, & \text{if } p > r, \end{cases} \tag{1.1}$$

$$E\left(\sup_{n\geq 1}\left|\frac{X_n}{n^{1/r}}\right|^p\right) < \infty, \tag{1.2}$$

$$E\left(\sup_{n\geq 1}\left|\frac{S_n}{n^{1/r}}\right|^p\right) < \infty \tag{1.3}$$

are all equivalent. Marcinkiewicz and Zygmund [1] obtain (1.1) ⇒ (1.3) for the case $p \geq r = 1$, Burkerholder [2] gets (1.3) ⇒ (1.1) for the case $p = r = 1$, Gut [3] proves that (1.1)-(1.3) are equivalent in the case $p \geq r$, and Choi and Sung [4] show that (1.1)-(1.3) are equivalent in the case $p < r$. For $0 < r < 2$ and $p > 0$, Chen and Gan [5] prove that (1.1)-(1.3) are equivalent under the dependent case such as ρ-mixing random variables.

By using the method of dominated by a nonnegative random variable, we investigate the randomly weighted sums of martingale differences under some weakly conditions. A sufficient condition to (1.2) and (1.3) is presented. To a certain extent, we generalize the result of Chen and Gan [5] for ρ-mixing random variables to the case of randomly weighted sums of martingale differences. For the details, please see our main result of Theorem 2.1 in Section 2.

Recall that the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X if

$$\sup_{n\geq 1} P\bigl(|X_n| > t\bigr) \leq CP(X > t) \quad \text{for some positive constant } C \text{ and all } t \geq 0$$

(see Adler and Rosalsky [6] and Adler *et al.* [7]). A bound on tail probabilities for quadratic forms in independent random variables is seen by using the following condition. There exist $C > 0$ and $\gamma > 0$ such that for all $n \geq 1$ and all $x \geq 0$, we have $P(|X_n| \geq x) \leq C \int_x^\infty e^{-\gamma t^2}\,dt$. For details, see Hanson and Wright [8] and Wright [9].

Meanwhile, the definition of martingale differences can be found in many books such as Stout [10], Hall and Heyde [11], Shiryaev [12], Gut [13], and so on. There are many results of martingale differences. For example, Ghosal and Chandra [14] gave the complete convergence of martingale arrays; Stoica [15, 16] investigated the Baum-Katz-Nagaev-type results for martingale differences; Wang *et al.* [17] also studied the complete moment convergence for martingale differences; Yang *et al.* [18] obtained the complete convergence for the moving average process of martingale differences; Yang *et al.* [19] investigated the complete moment convergence for randomly weighted sums of martingale differences, *etc.*

On the other hand, randomly weighted sums have been an attractive research topic in the literature of applied probability. For example, Thanh and Yin [20] studied the almost sure and complete convergence of randomly weighted sums of independent random elements in Banach spaces; Thanh *et al.* [21] investigated the convergence analysis of double-indexed and randomly weighted sums of mixing processes and gave its application to state observers of linear-time-invariant systems; Kevei and Mason [22] and Hormann and Swan [23] studied the asymptotic properties of randomly weighted sums and self-normalized sums; Cabrera *et al.* [24] and Shen *et al.* [25] investigated the conditional convergence for randomly weighted sums; Gao and Wang [26] and Tang and Yuan [27] investigated the randomly weighted sums of random variables and have given an application to ruin theory and capital allocation; Chen [28] obtained some asymptotically results of randomly weighted sums of dependent random variables with dominated variation, and so on.

Throughout the paper, $I(A)$ is the indicator function of set A and $C, C_1, C_2, \ldots$ denote some positive constants not depending on n. The following lemmas are our basic techniques to prove our results.

Lemma 1.1 (*cf.* Hall and Heyde (Theorem 2.11 in [11])) *If $\{X_i, \mathscr{F}_i, 1 \leq i \leq n\}$ is a martingale difference and $p > 0$, then there exists a constant C depending only on p such that*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq C\left\{E\left(\sum_{i=1}^{n} E\bigl(X_i^2|\mathscr{F}_{i-1}\bigr)\right)^{p/2} + E\Bigl(\max_{1\leq i\leq n}|X_i|^p\Bigr)\right\}, \quad n \geq 1.$$

Lemma 1.2 (*cf.* Adler and Rosalsky (Lemma 1 in [6]) and Adler *et al.* (Lemma 3 in [7])) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables, which is stochastically dominated by a nonnegative random variable X. Then, for any $\alpha > 0$ and $b > 0$, the following two statements hold*:

$E[|X_n|^\alpha I(|X_n| \leq b)] \leq C_1\{E[X^\alpha I(X \leq b)] + b^\alpha P(X > b)\}$,

$E[|X_n|^\alpha I(|X_n| > b)] \leq C_2 E[X^\alpha I(X > b)]$, *where C_1 and C_2 are positive constants not depending on n. Consequently, for all $n \geq 1$, one has $E|X_n|^\alpha \leq C_3 EX^\alpha$, where C_3 is a positive constant not depending on n.*

2 The main result and its proof

Theorem 2.1 *Let $0 < r < 2$, $0 < p < 2$, and $\{X_n, \mathscr{F}_n, n \geq 1\}$ be a martingale difference sequence, which is stochastically dominated by a nonnegative random variable X such that*

$$\begin{cases} \text{for } p < r, & \begin{cases} C_1 EX < \infty, & \text{if } 0 < r < 1, \\ C_2 E[X \log(1+X)] < \infty, & \text{if } r = 1, \\ C_3 EX^r < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p = r, & \begin{cases} C_4 EX < \infty, & \text{if } 0 < r < 1, \\ C_5 E[X \log^2(1+X)] < \infty, & \text{if } r = 1, \\ C_6 E[X^r \log(1+X)] < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p > r, & \begin{cases} C_7 EX < \infty, & \text{if } 0 < p < 1, \\ C_8 E[X \log(1+X)] < \infty, & \text{if } p = 1, \\ C_9 EX^p < \infty, & \text{if } p > 1. \end{cases} \end{cases} \tag{2.1}$$

Assume that $\{A_n, n \geq 1\}$ is an independent sequence of random variables, which is also independent of the sequence $\{X_n, n \geq 1\}$. In addition, it is assumed that

$$\sum_{i=1}^n EA_i^2 = O(n). \tag{2.2}$$

Let $S_n = \sum_{i=1}^n A_i X_i$, $n \geq 1$. Then one has the result (1.3), *which implies the result* (1.2).

Remark 2.1 In Theorem 2.1, $\{A_n X_n, \mathscr{F}_n, n \geq 1\}$ may be not a martingale difference, since A_n is not required to be measurable with respect to $\mathscr{F}_{n-1}$. We use the property of independence and the method of martingales to study the moments of maximum normed (1.2) and (1.3) and give a sufficient condition (2.1) for them.

Proof It can be argued that

$$\begin{aligned} E\left(\sup_{n\geq 1}\left|\frac{S_n}{n^{1/r}}\right|^p\right) &= \int_0^\infty P\left(\sup_{n\geq 1}\left|\frac{S_n}{n^{1/r}}\right| > t^{1/p}\right) dt \\ &\leq 2^{p/r} + \int_{2^{p/r}}^\infty \sum_{k=1}^\infty P\left(\max_{2^{k-1}\leq n<2^k}\left|\frac{S_n}{n^{1/r}}\right| > t^{1/p}\right) dt \\ &\leq 2^{p/r} + \int_{2^{p/r}}^\infty \sum_{k=1}^\infty P\left(\max_{1\leq n\leq 2^k}|S_n| > 2^{(k-1)/r} t^{1/p}\right) dt \quad \left(\text{let } s = 2^{(k-1)p/r} t\right) \\ &= 2^{p/r} + 2^{p/r} \sum_{k=1}^\infty 2^{-kp/r} \int_{2^{kp/r}}^\infty P\left(\max_{1\leq n\leq 2^k}|S_n| > s^{1/p}\right) ds. \end{aligned} \tag{2.3}$$

Let $\mathscr{G}_0 = \{\emptyset, \Omega\}$, $\mathscr{G}_n = \sigma(X_1, \ldots, X_n)$, $n \geq 1$, and $X_{si} = X_i I(|X_i| \leq s^{1/p})$, where $s^{1/p} > 0$. It can be argued that

$$A_i X_i = A_i X_i I\left(|X_i| > s^{1/p}\right) + \left[A_i X_{si} - E(A_i X_{si}|\mathscr{G}_{i-1})\right] + E(A_i X_{si}|\mathscr{G}_{i-1}), \quad 1 \leq i \leq n.$$

Then

$$
\begin{aligned}
&\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\Big(\max_{1\le n\le 2^k} |S_n| > s^{1/p}\Big)\, ds \\
&\quad\le \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\left(\max_{1\le n\le 2^k} \left|\sum_{i=1}^{n} A_i X_i I\big(|X_i| > s^{1/p}\big)\right| > s^{1/p}/2\right) ds \\
&\qquad + \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\left(\max_{1\le n\le 2^k} \left|\sum_{i=1}^{n} \big[A_i X_{si} - E(A_i X_{si}|\mathscr{G}_{i-1})\big]\right| > s^{1/p}/4\right) ds \\
&\qquad + \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} P\left(\max_{1\le n\le 2^k} \left|\sum_{i=1}^{n} E(A_i X_{si}|\mathscr{G}_{i-1})\right| > s^{1/p}/4\right) ds \\
&\quad=: I_1 + I_2 + I_3.
\end{aligned}
\tag{2.4}
$$

Combining Hölder's inequality with (2.2), we have

$$
\sum_{i=1}^{n} E|A_i| \le \left(\sum_{i=1}^{n} EA_i^2\right)^{1/2} \left(\sum_{i=1}^{n} 1\right)^{1-1/2} = O(n).
\tag{2.5}
$$

In view of $\{A_n, n\ge 1\}$ is independent of the sequence $\{X_n, n\ge 1\}$; by Markov's inequality, (2.5), and Lemma 1.2, we get

$$
\begin{aligned}
I_1 &\le 2\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\left(\max_{1\le n\le 2^k} \left|\sum_{i=1}^{n} A_i X_i I\big(|X_i| > s^{1/p}\big)\right|\right) ds \\
&\le 2\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} \left(\sum_{i=1}^{2^k} E|A_i| E|X_i| I\big(|X_i| > s^{1/p}\big)\right) ds \\
&\le C_1 \sum_{k=1}^{\infty} 2^{k-kp/r} \sum_{m=k}^{\infty} \int_{2^{mp/r}}^{2^{(m+1)p/r}} s^{-1/p} E\big[XI\big(X > s^{1/p}\big)\big]\, ds \\
&\le C_2 \sum_{k=1}^{\infty} 2^{k-kp/r} \sum_{m=k}^{\infty} 2^{mp/r-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \\
&= C_2 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \sum_{k=1}^{m} 2^{k-kp/r} \\
&\le \begin{cases} C_3 \sum_{m=1}^{\infty} 2^{m-m/r} E[XI(X > 2^{m/r})], & \text{if } p < r, \\ C_4 \sum_{m=1}^{\infty} m 2^{m-m/r} E[XI(X > 2^{m/r})], & \text{if } p = r, \\ C_5 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[XI(X > 2^{m/r})], & \text{if } p > r. \end{cases}
\end{aligned}
\tag{2.6}
$$

For the case $p < r$, if $0 < r < 1$, then

$$
\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \\
&\quad= \sum_{m=1}^{\infty} 2^{m-m/r} \sum_{k=m}^{\infty} E\big[XI\big(2^{k/r} < Y \le 2^{(k+1)/r}\big)\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \sum_{m=1}^{k} 2^{m(1-1/r)} \\
&\le C_1 \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \le C_1 EX.
\end{aligned}
$$

If $r = 1$, then

$$
\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \\
&\quad = \sum_{m=1}^{\infty} E\big[XI\big(X > 2^{m}\big)\big] = \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} E\big[XI\big(2^{k} < X \le 2^{k+1}\big)\big] \\
&\quad = \sum_{k=1}^{\infty} E\big[XI\big(2^{k} < X \le 2^{k+1}\big)\big] \sum_{m=1}^{k} 1 = \sum_{k=1}^{\infty} kE\big[XI\big(2^{k} < X \le 2^{k+1}\big)\big] \\
&\quad \le C_2 \sum_{k=1}^{\infty} E\big[X \log(1+X) I\big(2^{k} < X \le 2^{k+1}\big)\big] \\
&\quad \le C_2 E\big[X \log(1+X)\big].
\end{aligned}
$$

Otherwise, for $r > 1$, one has

$$
\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \\
&\quad = \sum_{m=1}^{\infty} 2^{m-m/r} \sum_{k=m}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \\
&\quad = \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \sum_{m=1}^{k} 2^{m-m/r} \\
&\quad \le C_1 \sum_{k=1}^{\infty} 2^{k-k/r} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \\
&\quad \le C_3 \sum_{k=1}^{\infty} E\big[X^{r} I\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \le C_3 EX^{r}.
\end{aligned}
$$

Similarly, for the case $p = r$, if $0 < r < 1$, then

$$
\begin{aligned}
&\sum_{m=1}^{\infty} m 2^{m-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \\
&\quad = \sum_{m=1}^{\infty} m 2^{m-m/r} \sum_{k=m}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \\
&\quad = \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \sum_{m=1}^{k} m 2^{m(1-1/r)} \\
&\quad \le C_4 \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \le C_4 EX.
\end{aligned}
$$

If $r = 1$, then

$$\begin{aligned}
&\sum_{m=1}^{\infty} m2^{m-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \\
&\quad = \sum_{m=1}^{\infty} m \sum_{k=m}^{\infty} E\bigl[XI\bigl(2^k < X \le 2^{k+1}\bigr)\bigr] \\
&\quad = \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^k < X \le 2^{k+1}\bigr)\bigr] \sum_{m=1}^{k} m \le C_5 \sum_{k=1}^{\infty} k^2 E\bigl[XI\bigl(2^k < X \le 2^{k+1}\bigr)\bigr] \\
&\quad \le C_5 \sum_{k=1}^{\infty} E\bigl[X \log^2(1+X) I\bigl(2^k < X \le 2^{k+1}\bigr)\bigr] \le C_5 E\bigl[X \log^2(1+X)\bigr].
\end{aligned}$$

Otherwise, for $r > 1$, it follows that

$$\begin{aligned}
&\sum_{m=1}^{\infty} m2^{m-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \\
&\quad = \sum_{m=1}^{\infty} m2^{m-m/r} \sum_{k=m}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \\
&\quad \le \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] k \sum_{m=1}^{k} 2^{m-m/r} \\
&\quad \le C_6 \sum_{k=1}^{\infty} k2^{k-k/r} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \le C_7 E\bigl[X^r \log(1+X)\bigr].
\end{aligned}$$

On the other hand, for the case $p > r$, if $0 < p < 1$, then

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{mp/r-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \\
&\quad = \sum_{m=1}^{\infty} 2^{m(p-1)/r} \sum_{k=m}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \\
&\quad = \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \sum_{m=1}^{k} 2^{m(p-1)/r} \\
&\quad \le C_8 \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \le C_8 EX.
\end{aligned}$$

If $p = 1$, then

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{mp/r-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \\
&\quad = \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \\
&\quad = \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \sum_{m=1}^{k} 1
\end{aligned}$$

$$= \sum_{k=1}^{\infty} kE\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big]$$
$$\le C_9 EX \log(1+X).$$

For $p > 1$, one has

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{mp/r-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \\
&\quad = \sum_{m=1}^{\infty} 2^{m(p-1)/r} \sum_{k=m}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \\
&\quad = \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \sum_{m=1}^{k} 2^{m(p-1)/r} \\
&\quad \le C_{10} \sum_{k=1}^{\infty} 2^{k(p-1)/r} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \\
&\quad \le C_{10} \sum_{k=1}^{\infty} E\big[X^p I\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \le C_{10} EX^p.
\end{aligned}$$

Consequently, in view of (2.6), the conditions of Theorem 2.1, and the inequalities above, we have

$$\begin{aligned}
I_1 &\le \begin{cases} C_1 \sum_{m=1}^{\infty} 2^{m-m/r} E[XI(X > 2^{m/r})], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m2^{m-m/r} E[XI(X > 2^{m/r})], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[XI(X > 2^{m/r})], & \text{if } p > r \end{cases} \\
&\le \begin{cases} \text{for } p < r, & \begin{cases} C_4 EX < \infty, & \text{if } 0 < r < 1, \\ C_5 E[X \log(1+X)] < \infty, & \text{if } r = 1, \\ C_6 EX^r < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p = r, & \begin{cases} C_7 EX < \infty, & \text{if } 0 < r < 1, \\ C_8 E[X \log^2(1+X)] < \infty, & \text{if } r = 1, \\ C_9 E[X^r \log(1+X)] < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p > r, & \begin{cases} C_{10} EX < \infty, & \text{if } 0 < p < 1, \\ C_{11} E[X \log(1+X)] < \infty, & \text{if } p = 1, \\ C_{12} EX^p < \infty, & \text{if } p > 1. \end{cases} \end{cases}
\end{aligned} \tag{2.7}$$

It can be checked that for the fixed real numbers $a_1, \ldots, a_n$,

$$\big\{a_i X_{si} - E(a_i X_{si} | \mathscr{G}_{i-1}), \mathscr{G}_i, 1 \le i \le n\big\}$$

is also a martingale difference. So one has by Lemma 1.1

$$\begin{aligned}
&E\left\{\max_{1\le n\le 2^k} \sum_{i=1}^{n} \big[A_i X_{si} - E(A_i X_{si} | \mathscr{G}_{i-1})\big]\right\}^2 \\
&\quad = E\left\{E\left(\max_{1\le n\le 2^k} \sum_{i=1}^{n} \big[a_i X_{si} - E(a_i X_{si} | \mathscr{G}_{i-1})\big]\right)^2 \middle| A_1 = a_1, \ldots, A_n = a_n\right\}
\end{aligned}$$

$$
\begin{aligned}
&\le C_1 E\left\{\sum_{i=1}^{2^k} E\bigl(a_i^2 X_{si}^2\bigr)\Big| A_1 = a_1, \ldots, A_n = a_n\right\} \\
&= C_1 \sum_{i=1}^{2^k} EA_i^2 EX_{si}^2,
\end{aligned}
\tag{2.8}
$$

by using the fact that $\{A_1, \ldots, A_n\}$ is independent of $\{X_{s1}, \ldots, X_{sn}\}$. Consequently, by Markov's inequality, (2.2), (2.8), and Lemma 1.2, one can check that

$$
\begin{aligned}
I_2 &\le C_1 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-2/p} E\left\{\max_{1\le n\le 2^k} \sum_{i=1}^{n} \bigl[A_i X_{si} - E(A_i X_{si}|\mathscr{G}_{i-1})\bigr]\right\}^2 ds \\
&\le C_2 \sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-2/p} \left(\sum_{i=1}^{2^k} EA_i^2 EX_{si}^2\right) ds \\
&\le C_3 \sum_{k=1}^{\infty} 2^{-kp/r+k} \int_{2^{kp/r}}^{\infty} s^{-2/p} E\bigl[X^2 I\bigl(X \le s^{1/p}\bigr)\bigr] ds \\
&\quad + C_4 \sum_{k=1}^{\infty} 2^{-kp/r+k} \int_{2^{kp/r}}^{\infty} P\bigl(X > s^{1/p}\bigr) ds \\
&=: C_3 I_{21} + C_4 I_{22}.
\end{aligned}
\tag{2.9}
$$

For I_{21}, it follows that

$$
\begin{aligned}
I_{21} &= \sum_{k=1}^{\infty} 2^{-kp/r+k} \sum_{m=k}^{\infty} \int_{2^{mp/r}}^{2^{(m+1)p/r}} s^{-2/p} E\bigl[X^2 I\bigl(X \le s^{1/p}\bigr)\bigr] ds \\
&\le \sum_{k=1}^{\infty} 2^{-kp/r+k} \sum_{m=k}^{\infty} 2^{mp/r-2m/r} E\bigl[X^2 I\bigl(X \le 2^{(m+1)/r}\bigr)\bigr] \\
&= \sum_{m=1}^{\infty} 2^{m(p-2)/r} E\bigl[X^2 I\bigl(X \le 2^{(m+1)/r}\bigr)\bigr] \sum_{k=1}^{m} 2^{k(1-p/r)}.
\end{aligned}
$$

If $p < r$, one has by $r < 2$

$$
\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m(p-2)/r} E\bigl[X^2 I\bigl(X \le 2^{(m+1)/r}\bigr)\bigr] \sum_{k=1}^{m} 2^{k(1-p/r)} \\
&\quad\le C_1 \sum_{m=1}^{\infty} 2^{m(r-2)/r} E\bigl[X^2 I\bigl(X \le 2^{(m+1)/r}\bigr)\bigr] \\
&\quad= C_1 \sum_{m=1}^{\infty} 2^{m(r-2)/r} E\bigl[X^2 I\bigl(X \le 2^{1/r}\bigr)\bigr] \\
&\qquad + C_1 \sum_{m=1}^{\infty} 2^{m(r-2)/r} \sum_{i=1}^{m} E\bigl[X^2 I\bigl(2^{i/r} < X \le 2^{(i+1)/r}\bigr)\bigr] \\
&\quad\le C_2 + C_1 \sum_{i=1}^{\infty} E\bigl[X^2 I\bigl(2^{i/r} < X \le 2^{(i+1)/r}\bigr)\bigr] \sum_{m=i}^{\infty} 2^{m(r-2)/r}
\end{aligned}
$$

$$
\begin{aligned}
&\le C_2 + C_3 2^{(2-r)/r} \sum_{i=1}^{\infty} 2^{(i+1)(r-2)/r} E\bigl[X^2 I\bigl(2^{i/r} < X \le 2^{(i+1)/r}\bigr)\bigr] \\
&\le C_4 + C_5 EX^r.
\end{aligned}
$$

For the case $p = r$, by $r < 2$, one has

$$
\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m(p-2)/r} E\bigl[X^2 I\bigl(X \le 2^{(m+1)/r}\bigr)\bigr] \sum_{k=1}^{m} 2^{k(1-p/r)} \\
&\quad \le C_1 \sum_{m=1}^{\infty} m 2^{m(r-2)/r} E\bigl[X^2 I\bigl(X \le 2^{(m+1)/r}\bigr)\bigr] \\
&\quad = C_1 \sum_{m=1}^{\infty} m 2^{m(r-2)/r} E\bigl[X^2 I\bigl(X \le 2^{1/r}\bigr)\bigr] \\
&\qquad + C_1 \sum_{m=1}^{\infty} m 2^{m(r-2)/r} \sum_{i=1}^{m} E\bigl[X^2 I\bigl(2^{i/r} < X \le 2^{(i+1)/r}\bigr)\bigr] \\
&\quad \le C_2 + C_1 \sum_{i=1}^{\infty} E\bigl[X^2 I\bigl(2^{i/r} < X \le 2^{(i+1)/r}\bigr)\bigr] \sum_{m=i}^{\infty} m 2^{m(r-2)/r} \\
&\quad \le C_2 + C_3 2^{(2-r)/r} \sum_{i=1}^{\infty} i 2^{(i+1)(r-2)/r} E\bigl[X^2 I\bigl(2^{i/r} < X \le 2^{(i+1)/r}\bigr)\bigr] \\
&\quad \le C_4 + C_5 E\bigl[X^r \log(1+X)\bigr].
\end{aligned}
$$

For the case $p > r$, it can be checked by $p < 2$ that

$$
\sum_{m=1}^{\infty} 2^{m(p-2)/r} E\bigl[X^2 I\bigl(Y \le 2^{(m+1)/r}\bigr)\bigr] \le CEX^p.
$$

Consequently, it follows that

$$
\begin{aligned}
I_{21} &\le \begin{cases} C_1 \sum_{m=1}^{\infty} 2^{m(r-2)/r} E[X^2 I(X \le 2^{(m+1)/r})], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m 2^{m(r-2)/r} E[X^2 I(X \le 2^{(m+1)/r})], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{m(p-2)/r} E[X^2 I(Y \le 2^{(m+1)/r})], & \text{if } p > r \end{cases} \\
&\le \begin{cases} C_4 + C_5 EX^r < \infty, & \text{if } p < r, \\ C_6 + C_7 E[X^r \log(1+X)] < \infty, & \text{if } p = r, \\ C_8 EX^p < \infty, & \text{if } p > r. \end{cases}
\end{aligned}
\tag{2.10}
$$

On the other hand, similar to the proofs of (2.6) and (2.7), we obtain

$$
\begin{aligned}
I_{22} &\le \sum_{k=1}^{\infty} 2^{k-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\bigl[XI\bigl(X > s^{1/p}\bigr)\bigr]\,ds \\
&\le C_2 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \sum_{k=1}^{m} 2^{k-kp/r} \\
&\le \begin{cases} C_3 \sum_{m=1}^{\infty} 2^{m-m/r} E[XI(X > 2^{m/r})], & \text{if } p < r, \\ C_4 \sum_{m=1}^{\infty} m 2^{m-m/r} E[XI(X > 2^{m/r})], & \text{if } p = r, \\ C_5 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[XI(X > 2^{m/r})], & \text{if } p > r. \end{cases}
\end{aligned}
\tag{2.11}
$$

For the case $p < r$, if $0 < r < 1$, then

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \\
&\quad = \sum_{m=1}^{\infty} 2^{m-m/r} \sum_{k=m}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \\
&\quad = \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \sum_{m=1}^{k} 2^{m(1-1/r)} \\
&\quad \le C_1 \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \le C_1 EX.
\end{aligned}$$

If $r = 1$, then

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \\
&\quad = \sum_{m=1}^{\infty} E\bigl[XI\bigl(X > 2^{m}\bigr)\bigr] = \sum_{m=1}^{\infty} \sum_{k=m}^{\infty} E\bigl[XI\bigl(2^{k} < X \le 2^{k+1}\bigr)\bigr] \\
&\quad = \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^{k} < X \le 2^{k+1}\bigr)\bigr] \sum_{m=1}^{k} 1 \le \sum_{k=1}^{\infty} kE\bigl[XI\bigl(2^{k} < X \le 2^{k+1}\bigr)\bigr] \\
&\quad \le C_2 \sum_{k=1}^{\infty} E\bigl[X \log(1+X) I\bigl(2^{k} < X \le 2^{k+1}\bigr)\bigr] \le C_2 E\bigl[X \log(1+X)\bigr].
\end{aligned}$$

Otherwise, for $r > 1$, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty} 2^{m-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \\
&\quad = \sum_{m=1}^{\infty} 2^{m-m/r} \sum_{k=m}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \\
&\quad = \sum_{k=1}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \sum_{m=1}^{k} 2^{m-m/r} \\
&\quad \le C_3 \sum_{k=1}^{\infty} 2^{k-k/r} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \\
&\quad \le C_3 \sum_{k=1}^{\infty} E\bigl[X^r I\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr] \le C_3 EX^r.
\end{aligned}$$

Similarly, for the case $p = r$, if $0 < r < 1$, then

$$\begin{aligned}
&\sum_{m=1}^{\infty} m2^{m-m/r} E\bigl[XI\bigl(X > 2^{m/r}\bigr)\bigr] \\
&\quad = \sum_{m=1}^{\infty} m2^{m-m/r} \sum_{k=m}^{\infty} E\bigl[XI\bigl(2^{k/r} < X \le 2^{(k+1)/r}\bigr)\bigr]
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \sum_{m=1}^{k} m2^{m(1-1/r)} \\
&\le C_4 \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \le C_4 EX.
\end{aligned}
$$

For $r = 1$, it follows that

$$
\begin{aligned}
&\sum_{m=1}^{\infty} m2^{m-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \\
&\quad = \sum_{m=1}^{\infty} m \sum_{k=m}^{\infty} E\big[XI\big(2^{k} < X \le 2^{k+1}\big)\big] \\
&\quad = \sum_{k=1}^{\infty} E\big[XI\big(2^{k} < X \le 2^{k+1}\big)\big] \sum_{m=1}^{k} m \le C_5 \sum_{k=1}^{\infty} k^2 E\big[XI\big(2^{k} < X \le 2^{k+1}\big)\big] \\
&\quad \le C_5 \sum_{k=1}^{\infty} E\big[X \log^2(1+X) I\big(2^{k} < X \le 2^{k+1}\big)\big] \le C_5 E\big[X \log^2(1+X)\big].
\end{aligned}
$$

Otherwise, for $r > 1$, it follows that

$$
\begin{aligned}
&\sum_{m=1}^{\infty} m2^{m-m/r} E\big[XI\big(X > 2^{m/r}\big)\big] \\
&\quad = \sum_{m=1}^{\infty} m2^{m-m/r} \sum_{k=m}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \\
&\quad \le \sum_{k=1}^{\infty} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \sum_{m=1}^{k} m2^{m-m/r} \\
&\quad \le C_6 \sum_{k=1}^{\infty} k2^{k-k/r} E\big[XI\big(2^{k/r} < X \le 2^{(k+1)/r}\big)\big] \le C_7 E\big[X^r \log(1+X)\big].
\end{aligned}
$$

Therefore, by (2.7) for case $p > r$, (2.11), and the inequalities above, we obtain

$$
\begin{aligned}
I_{22} &\le \begin{cases} C_1 \sum_{m=1}^{\infty} 2^{m-m/r} E[XI(X > 2^{m/r})], & \text{if } p < r, \\ C_2 \sum_{m=1}^{\infty} m2^{m-m/r} E[XI(X > 2^{m/r})], & \text{if } p = r, \\ C_3 \sum_{m=1}^{\infty} 2^{mp/r-m/r} E[XI(X > 2^{m/r})], & \text{if } p > r \end{cases} \\
&\le \begin{cases} \text{for } p < r, & \begin{cases} C_4 EX < \infty, & \text{if } 0 < r < 1, \\ C_5 E[X \log(1+X)] < \infty, & \text{if } r = 1, \\ C_6 EX^r < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p = r, & \begin{cases} C_7 EX < \infty, & \text{if } 0 < r < 1, \\ C_8 E[X \log^2(1+X)] < \infty, & \text{if } r = 1, \\ C_9 E[X^r \log(1+X)] < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p > r, & \begin{cases} C_{10} EX < \infty, & \text{if } 0 < p < 1, \\ C_{11} E[X \log(1+X)] < \infty, & \text{if } p = 1, \\ C_{12} EX^p < \infty, & \text{if } p > 1. \end{cases} \end{cases}
\end{aligned}
\tag{2.12}
$$

Obviously, it can be seen that $\{X_n, \mathscr{G}_n, n \geq 1\}$ is also a martingale difference, since $\{X_n, \mathscr{F}_n, n \geq 1\}$ is a martingale difference. Combining with the fact that $\{A_n, n \geq 1\}$ is independent of $\{X_n, n \geq 1\}$, we have

$$\begin{aligned} E(A_iX_i|\mathscr{G}_{i-1}) &= E\big[E(A_iX_i|\mathscr{G}_i)|\mathscr{G}_{i-1}\big] = E\big[X_iE(A_i|\mathscr{G}_i)|\mathscr{G}_{i-1}\big] \\ &= EA_iE[X_i|\mathscr{G}_{i-1}] = 0, \quad \text{a.s.}, 1 \leq i \leq n. \end{aligned}$$

In view of the proofs of (2.6), (2.7), and the inequality above, we obtain by Markov's inequality and (2.5)

$$\begin{aligned} I_3 &\leq 4\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\left(\max_{1\leq n\leq 2^k}\left|\sum_{i=1}^{n} E(A_iX_{si}|\mathscr{G}_{i-1})\right|\right) ds \\ &= 4\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\left(\max_{1\leq n\leq 2^k}\left|\sum_{i=1}^{n} E\big(A_iX_iI\big(|X_i| \leq s^{1/p}\big)|\mathscr{G}_{i-1}\big)\right|\right) ds \\ &= 4\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\left(\max_{1\leq n\leq 2^k}\left|\sum_{i=1}^{n} E\big(A_iX_iI\big(|X_i| > s^{1/p}\big)|\mathscr{G}_{i-1}\big)\right|\right) ds \\ &\leq 4\sum_{k=1}^{\infty} 2^{-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} \sum_{i=1}^{2^k} E|A_i|E|X_i|I\big(|X_i| > s^{1/p}\big)\, ds \\ &\leq C_1\sum_{k=1}^{\infty} 2^{k-kp/r} \int_{2^{kp/r}}^{\infty} s^{-1/p} E\big[XI\big(X > s^{1/p}\big)\big]\, ds \\ &\leq \begin{cases} \text{for } p < r, & \begin{cases} C_1EX < \infty, & \text{if } 0 < r < 1, \\ C_2E[X\log(1+X)] < \infty, & \text{if } r = 1, \\ C_3EX^r < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p = r, & \begin{cases} C_4EX < \infty, & \text{if } 0 < r < 1, \\ C_5E[X\log^2(1+X)] < \infty, & \text{if } r = 1, \\ C_6E[X^r\log(1+X)] < \infty, & \text{if } r > 1, \end{cases} \\ \text{for } p > r, & \begin{cases} C_7EX < \infty, & \text{if } 0 < p < 1, \\ C_8E[X\log(1+X)] < \infty, & \text{if } p = 1, \\ C_9EX^p < \infty, & \text{if } p > 1. \end{cases} \end{cases} \end{aligned} \tag{2.13}$$

Consequently, in view of (2.1), (2.3), (2.4), (2.7)-(2.10), (2.12), and (2.13), one has (1.3). By (1.3), it is easy to get (1.2). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the manuscript.

Author details
[1] Shandong University Qilu Securities Institute for Financial Studies, Shandong University, Jinan, China. [2] School of Mathematics, Hefei University of Technology, Hefei, China. [3] School of Mathematics, Shandong University, Jinan, China.

Acknowledgements
The authors are deeply grateful to the editor and two anonymous referees, whose insightful comments and suggestions have contributed substantially to the improvement of this paper. This work is supported by the National Natural Science Foundation of China (11171188), National Social Science Fund of China (14ATJ005), the Humanities and Social Science

Planning Foundation of Ministry of Education of China (14YJCZH155) and the Fundamental Scientific Research Funds for the Central Universities (2015HGZX0018).

Received: 12 April 2015 Accepted: 13 August 2015 Published online: 28 August 2015

References

1. Marcinkiewicz, J, Zygmund, A: Sur les fonctions independantes. Fundam. Math. **29**, 60-90 (1937)
2. Burkerholder, DL: Successive conditional expectations of an integrable function. Ann. Math. Stat. **33**(3), 887-893 (1962)
3. Gut, A: Moments of the maximum of normed partial sums of random variables with multidimensional indices. Z. Wahrscheinlichkeitstheor. Verw. Geb. **64**, 205-220 (1979)
4. Choi, BD, Sung, SH: On moment conditions for the supremum of normed sums. Stoch. Process. Appl. **26**, 99-106 (1987)
5. Chen, PY, Gan, SX: On moments of the maximum of normed partial sums of ρ-mixing random variables. Stat. Probab. Lett. **78**(10), 1215-1221 (2008)
6. Adler, A, Rosalsky, A: Some general strong laws for weighted sums of stochastically dominated random variables. Stoch. Anal. Appl. **5**(1), 1-16 (1987)
7. Adler, A, Rosalsky, A, Taylor, RL: Strong laws of large numbers for weighted sums of random elements in normed linear spaces. Int. J. Math. Math. Sci. **12**(3), 507-530 (1989)
8. Hanson, DL, Wright, FT: A bound on tail probabilities for quadratic forms in independent random variables. Ann. Math. Stat. **42**(3), 1079-1083 (1971)
9. Wright, FT: A bound on tail probabilities for quadratic forms in independent random variables whose distributions are not necessarily symmetric. Ann. Probab. **1**(6), 1068-1070 (1973)
10. Stout, WF: Almost Sure Convergence. Academic Press, New York (1974)
11. Hall, P, Heyde, CC: Martingale Limit Theory and Its Application. Academic Press, New York (1980)
12. Shiryaev, AN: Probability, 2nd edn. Springer, New York (1996)
13. Gut, A: Probability: A Graduate Course. Springer, Berlin (2005)
14. Ghosal, S, Chandra, TK: Complete convergence of martingale arrays. J. Theor. Probab. **11**(3), 621-631 (1998)
15. Stoica, G: Baum-Katz-Nagaev type results for martingales. J. Math. Anal. Appl. **336**(2), 1489-1492 (2007)
16. Stoica, G: A note on the rate of convergence in the strong law of large numbers for martingales. J. Math. Anal. Appl. **381**(2), 910-913 (2011)
17. Wang, XJ, Hu, SH, Yang, WZ, Wang, XH: Convergence rates in the strong law of large numbers for martingale difference sequences. Abstr. Appl. Anal. **2012**, Article ID 572493 (2012)
18. Yang, WZ, Hu, SH, Wang, XJ: Complete convergence for moving average process of martingale differences. Discrete Dyn. Nat. Soc. **2012**, Article ID 128492 (2012)
19. Yang, WZ, Wang, YW, Wang, XH, Hu, SH: Complete moment convergence for randomly weighted sums of martingale differences. J. Inequal. Appl. **2013**, 396 (2013)
20. Thanh, LV, Yin, G: Almost sure and complete convergence of randomly weighted sums of independent random elements in Banach spaces. Taiwan. J. Math. **15**(4), 1759-1781 (2011)
21. Thanh, LV, Yin, G, Wang, LY: State observers with random sampling times and convergence analysis of double-indexed and randomly weighted sums of mixing processes. SIAM J. Control Optim. **49**(1), 106-124 (2011)
22. Kevei, P, Mason, DM: The asymptotic distribution of randomly weighted sums and self-normalized sums. Electron. J. Probab. **17**, 46 (2012)
23. Hormann, S, Swan, Y: A note on the normal approximation error for randomly weighted self-normalized sums. Period. Math. Hung. **67**(2), 143-154 (2013)
24. Cabrera, MO, Rosalsky, A, Volodin, A: Some theorems on conditional mean convergence and conditional almost sure convergence for randomly weighted sums of dependent random variables. Test **21**(2), 369-385 (2012)
25. Shen, AT, Wu, RC, Chen, Y, Zhou, Y: Conditional convergence for randomly weighted sums of random variables based on conditional residual *h*-integrability. J. Inequal. Appl. **2013**, 122 (2013)
26. Gao, QW, Wang, YB: Randomly weighted sums with dominated varying-tailed increments and application to risk theory. J. Korean Stat. Soc. **39**(3), 305-314 (2010)
27. Tang, QH, Yuan, ZY: Randomly weighted sums of subexponential random variables with application to capital allocation. Extremes **17**(3), 467-493 (2014)
28. Chen, DY: Randomly weighted sums of dependent random variables with dominated variation. J. Math. Anal. Appl. **420**(2), 1617-1633 (2014)