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Operator ideal of Norlund-type sequence spaces

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Abstract
Let E be a Norlund sequence space which is invariant under the doubling operator

$$D: x = (x_0, x_1, x_2, \ldots) \mapsto y = (x_0, x_0, x_1, x_1, x_2, x_2, \ldots).$$

Using the approximation numbers $(\alpha_n(T))_{n=0}^{\infty}$ of operators from a Banach space X into a Banach space Y, we give the sufficient (not necessary) conditions on E such that the components

$$U_E^{\mathrm{app}}(X,Y) := \{T \in L(X,Y) : (\alpha_n(T))_{n=0}^{\infty} \in E\}$$

form an operator ideal, the finite rank operators are dense in the complete space of operators $U_E^{\mathrm{app}}(X,Y)$ which is a longstanding open problem. Finally we give an answer for Rhoades (Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 59(3-4):238-241, 1975) about the linearity of E-type spaces $(U_E^{\mathrm{app}}(X,Y))$, and we conclude under a few conditions that every compact operator would be approximated by finite rank operators. Our results agree with those in (J. Inequal. Appl., 2013, doi:10.1186/1029-242x-2013-186) for the space $\mathrm{ces}((p_n))$, where (p_n) is a sequence of positive reals.

MSC: Primary 47Bxx; secondary 47B10

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1 Introduction

As an aftereffect of the enormous applications in geometry of Banach spaces, spectral hypothesis, hypothesis of eigenvalue dispersions and fixed point hypothesis and so on, the hypothesis of operator ideal goals possesses an uncommon essentialness in useful examination. A large portion of the administrator goals in the family of Banach spaces or normed spaces in straight practical examination are characterized by diverse scalar grouping spaces. All through the paper

$$L(X,Y) = \{T : X \to Y; T \text{ is bounded and linear; } X \text{ and } Y \text{ are Banach spaces}\}.$$

And $\mathbb{N} = \{0, 1, 2, \ldots\}$, by w we denote the space of all real sequences and θ is the zero vector of E. In [1], by using the approximation numbers and p-absolutely summable sequences of real numbers ℓ^p $(0 < p < \infty)$, Pietsch formed the operator ideals. In [2], Faried and Bakery considered the space $\operatorname{ces}((p_n))$, where (p_n) is a sequence of positive reals and ℓ_M, when $M(t) = t^p$ $(0 < p < \infty)$, which match in special with ℓ^p. Bakery [3, 4] took some different mean of Cesaro type spaces involving Lacunary sequence $\operatorname{Ces}(\theta, p)$ defined in [5] to form an operator ideal with its approximation numbers.

2 Definitions and preliminaries

Definition 2.1 An s-function is a map allocating to each operator $T \in L(X, Y)$ a non-negative scalar sequence $(s_n(T))_{n=0}^{\infty}$, and the number $s_n(T)$ is called the nth s-number of T assuming that the following conditions are verified:

(a) $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for $T \in L(X, Y)$;
(b) $s_n(S_1 + S_2) \leq s_n(S_1) + \|S_2\|$ for all $S_1, S_2 \in L(X, Y)$;
(c) ideal property: $s_n(RVT) \leq \|R\| s_n(V) \|T\|$ for all $T \in L(X_0, X)$, $V \in L(X, Y)$ and $R \in L(Y, Y_0)$, where X_0 and Y_0 are arbitrary Banach spaces;
(d) if $G \in L(X, Y)$ and $\lambda \in \mathbb{R}$, we obtain $s_n(\lambda G) = |\lambda| s_n(G)$;
(e) rank property: If $\operatorname{rank}(T) \leq n$, then $s_n(T) = 0$ for each $T \in L(X, Y)$;
(f) norming property: $s_{r \geq n}(I_n) = 0$ or $s_{r<n}(I_n) = 1$, where I_n represents the unit operator on the n-dimensional Hilbert space ℓ_2^n.

There a few cases would define s-numbers, we specify close approximation numbers $\alpha_n(S)$, Gelfand numbers $c_n(S)$, Kolmogorov numbers $d_n(S)$ and Tichomirov numbers $d_n^*(S)$. The sum of these numbers fulfills the next condition:

(g) Additivity: $s_{n+m}(T_1 + T_2) \leq s_n(T_1) + s_m(T_2)$ for all $T_1, T_2 \in L(X, Y)$.

Definition 2.2 Let L be the class of all bounded linear operators between any arbitrary Banach spaces. A subclass U of L is called an operator ideal if each element $U(X, Y) = U \cap L(X, Y)$ fulfills the accompanying conditions:

(i) $I_K \in U$, where K represents a Banach space of one dimension.
(ii) The space $U(X, Y)$ is a linear space over $\mathbb{R}$.
(iii) If $T \in L(X_0, X)$, $V \in U(X, Y)$ and $R \in L(Y, Y_0)$, then $RVT \in U(X_0, Y_0)$. See [6] and [7].

Definition 2.3 An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is convex, positive, non-decreasing, continuous with $M(0) = 0$ and $\lim_{x \to \infty} M(x) = \infty$.

See [8] and [9]. In the event that convexity of M is supplanted by $M(x + y) \leq M(x) + M(y)$, it is known as a modulus function, presented by Nakano [10].

Definition 2.4 An Orlicz function M is said to fulfill Δ_2-condition for all estimations of $x \geq 0$ if there exists a steady $k > 0$ such that $M(2x) \leq kM(x)$. The Δ_2-condition is compared to $M(lx) \leq klM(x)$ for all estimations of x and for $l > 1$.

Lindentrauss and Tzafriri [11] used the idea of an Orlicz function to define the following sequence spaces:

$$\ell_M = \{x \in \omega : \exists \lambda > 0 \text{ with } \rho(\lambda x) < \infty\}, \quad \text{where } \rho(x) = \sum_{k=1}^{\infty} M(|x_k|),$$

which is a Banach space with the Luxemburg norm defined by

$$\|x\| = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \le 1\right\}.$$

The space ℓ_M is directly related to the space ℓ^p, which is an Orlicz sequence space with $M(x) = x^p$ for $1 \le p < \infty$.

For $\rho(x) = \sum_{n=0}^{\infty}(\frac{\sum_{k=0}^{n} q_k|x_k|}{\sum_{k=0}^{n} q_k})^{p_n}$, the Norlund sequence spaces considered by Wang [12] are defined as

$$\operatorname{ces}\bigl((p_n),(q_n)\bigr) = \bigl\{x = (x_k) \in \omega : \exists\lambda > 0 \text{ with } \rho(\lambda x) < \infty\bigr\},$$

where (p_n) and (q_n) are sequences of positive reals, $p_n \ge 1$ for all $n \in \mathbb{N}$. A Norlund sequence space is a Banach space with the Luxemburg norm defined by

$$\|x\| = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \le 1\right\}.$$

If (p_n) is bounded, we might essentially compose

$$\operatorname{ces}\bigl((p_n),(q_n)\bigr) = \left\{x = (x_k) \in \omega : \sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|x_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n} < \infty\right\}.$$

Remark 2.5

(1) Taking $q_n = 1$ for all $n \in \mathbb{N}$, then $\operatorname{ces}((p_n),(q_n))$ is reduced to $\operatorname{ces}((p_n))$ studied by Sanhan and Suantai [13].

(2) Taking $q_n = 1$ and $p_n = p$ for all $n \in \mathbb{N}$, then $\operatorname{ces}((p_n),(q_n))$ is reduced to ces_p studied by many authors (see [14, 15] and [16]).

In order to give full knowledge to the reader, we add the article [17] on the paranormed Nörlund sequence spaces and the textbook [18] containing five chapters on the sequence spaces.

Definition 2.6 Let E be a linear space of sequences, then E is called a (sss) if:

(1) For $n \in \mathbb{N}$, $e_n \in E$;

(2) E is solid, i.e., assuming $x = (x_n) \in w$, $y = (y_n) \in E$ and $|x_n| \le |y_n|$ for all $n \in \mathbb{N}$, then $x \in E$;

(3) $(x_{[\frac{n}{2}]})_{n=0}^{\infty} \in E$, where $[\frac{n}{2}]$ indicates the integral part of $\frac{n}{2}$, whenever $(x_n)_{n=0}^{\infty} \in E$.

Example 2.7 ℓ^p is a (sss) if $p \in (0,\infty)$.

Example 2.8 Let M be an Orlicz function with Δ_2-condition, then ℓ_M is a (sss).

Example 2.9 ces_p is a (sss) if $p \in (1,\infty)$.

Example 2.10 Pick an increasing sequence (p_n) with $\lim_{n\to\infty}\inf p_n > 1$ and $\lim_{n\to\infty}\sup p_n < \infty$, hence $\operatorname{ces}((p_n))$ will be a (sss).

Definition 2.11 A subclass of the (sss) is called a pre-modular (sss) assuming that we have a map $\rho : E \to [0,\infty[$ with the following:

(i) For $x \in E$, $x = \theta \Leftrightarrow \rho(x) = 0$ with $\rho(x) \geq 0$;
(ii) For each $x \in E$ and scalar λ, we get a real number $L \geq 1$ for which $\rho(\lambda x) \leq L|\lambda|\rho(x)$;
(iii) $\rho(x+y) \leq K(\rho(x) + \rho(y))$ for each $x, y \in E$ holds for a few numbers $K \geq 1$;
(iv) For $n \in \mathbb{N}$, $|x_n| \leq |y_n|$, we obtain $\rho((x_n)) \leq \rho((y_n))$;
(v) The inequality $\rho((x_n)) \leq \rho((x_{[\frac{n}{2}]})) \leq K_0 \rho((x_n))$ holds for some numbers $K_0 \geq 1$;
(vi) $\overline{F} = E_\rho$, where F is the space of any finite sequences;
(vii) There is steady $\xi > 0$ such that $\rho(\lambda, 0, 0, 0, \ldots) \geq \xi|\lambda|\rho(1, 0, 0, 0, \ldots)$ for any $\lambda \in \mathbb{R}$.

Condition (ii) is equivalent to $\rho(x)$ is continuous at θ. The pre-modular ρ characterizes a metric topology in E, and the linear space E enriched with this topology will be indicated further by E_ρ. Moreover, the condition (i) of Definition 2.6 and the condition (vi) of Definition 2.11 explain that $(e_n)_{n\in\mathbb{N}}$ is a Schauder basis of E_ρ.

Example 2.12 ℓ^p is a pre-modular (sss) if $p \in (0,\infty)$.

Example 2.13 Suppose that M is an Orlicz function with Δ_2-condition, hence ℓ_M is a pre-modular (sss).

Example 2.14 ces_p is a pre-modular (sss) if $p \in (1,\infty)$.

Example 2.15 $\operatorname{ces}((p_n))$ is a pre-modular (sss) if (p_n) is an increasing sequence, $\lim_{n\to\infty} \inf p_n > 1$ and $\lim_{n\to\infty} \sup p_n < \infty$.

Definition 2.16

$$U_E^{\text{app}} := \{U_E^{\text{app}}(V,W); V \text{ and } W \text{ are Banach spaces}\}, \quad \text{and its components}$$
$$U_E^{\text{app}}(V,W) := \{S \in L(V,W) : (\alpha_n(S))_{n=0}^{\infty} \in E\}.$$

Theorem 2.17 *If E is a (sss), then U_E^{app} is an operator ideal.*

Now and in what follows, (p_n) and (q_n) are assumed to be bounded sequences of positive reals. We define $e_n = \{0, 0, \ldots, 1, 0, 0, \ldots\}$, where 1 appears at the nth place for all $n \in \mathbb{N}$, and the given inequality will be used in the sequel: $|a_n + b_n|^{p_n} \leq H(|a_n|^{p_n} + |b_n|^{p_n})$, where $H = \max\{1, 2^{h-1}\}$, $h = \sup_n p_n$ and $p_n \geq 1$ for all $n \in \mathbb{N}$. See [19].

3 Main results

3.1 Linear problem

We study here the operator ideals generated by the approximation numbers and Norlund sequence spaces such that the class of all bounded linear operators between any arbitrary Banach spaces with nth approximation numbers of the bounded linear operators in these sequence spaces form an operator ideal.

Theorem 3.1 $\operatorname{ces}((p_n),(q_n))$ *is a (sss) if the following conditions are satisfied*:

(b1) *The sequence (p_n) is increasing*;
(b2) *The sequence (q_n) with $\sum_{n=0}^{\infty}(\sum_{k=0}^{n} q_k)^{-p_n} < \infty$*;

(b3) (q_n) *is either monotone decreasing or monotone increasing, and there exists a constant* $C \geq 1$ *such that* $q_{2n+1} \leq Cq_n$.

Proof (1-i) Suppose $x, y \in \operatorname{ces}((p_n),(q_n))$. Since

$$\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|x_k+y_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n} \leq H\left[\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|x_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|y_k|}{\sum_{k=0}^{n} |q_k|}\right)^{p_n}\right]<\infty,$$

$x+y \in \operatorname{ces}((p_n),(q_n))$.

(1-ii) Let $x \in \operatorname{ces}((p_n),(q_n))$ and $\lambda \in \mathbb{R}$. Since (p_n) is bounded, then we have

$$\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|\lambda x_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n} \leq \sup_n |\lambda|^{p_n}\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|x_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n}<\infty,$$

that is, $\lambda x \in \operatorname{ces}((p_n),(q_n))$. Hence, by (1-i) and (1-ii), $\operatorname{ces}((p_n),(q_n))$ is a linear space. Let us show that $\{e_m\}_{m\in\mathbb{N}} \subseteq \operatorname{ces}((p_n),(q_n))$. Since (p_n) is bounded and $\sum_{n=0}^{\infty}(\sum_{k=0}^{n} q_k)^{-p_n}<\infty$, we get

$$\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|e_m(k)|}{\sum_{k=0}^{n} q_k}\right)^{p_n} = \sum_{n=m}^{\infty}\left(\frac{q_m}{\sum_{k=0}^{n} q_k}\right)^{p_n} \leq \sup_n (q_m)^{p_n}\sum_{n=m}^{\infty}\left(\sum_{k=0}^{n} q_k\right)^{-p_n}<\infty.$$

Hence $e_m \in \operatorname{ces}((p_n),(q_n))$ for all $m \in \mathbb{N}$.

(2) Let $y \in \operatorname{ces}((p_n),(q_n))$ and $|x_n| \leq |y_n|$ for every $n \in \mathbb{N}$. Since $q_n > 0$ for every $n \in \mathbb{N}$,

$$\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|x_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n} \leq \sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|y_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n}<\infty,$$

we get $x \in \operatorname{ces}((p_n),(q_n))$.

(3) Let $(x_n) \in \operatorname{ces}((p_n),(q_n))$, (p_n) be an increasing sequence and (q_n) be increasing with a constant $C \geq 1$ such that $q_{2n+1} \leq Cq_n$. Then we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k|x_{[\frac{k}{2}]}|}{\sum_{k=0}^{n} q_k}\right)^{p_n}\\
&\quad=\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{2n} q_k|x_{[\frac{k}{2}]}|}{\sum_{k=0}^{2n} q_k}\right)^{p_{2n}}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{2n+1} q_k|x_{[\frac{k}{2}]}|}{\sum_{k=0}^{2n+1} q_k}\right)^{p_{2n+1}}\\
&\quad\leq\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{2n} q_k|x_{[\frac{k}{2}]}|}{\sum_{k=0}^{2n} q_k}\right)^{p_n}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{2n+1} q_k|x_{[\frac{k}{2}]}|}{\sum_{k=0}^{2n+1} q_k}\right)^{p_n}\\
&\quad=\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n}(q_{2k}+q_{2k+1})|x_k|+q_{2n}|x_n|}{\sum_{k=0}^{2n} q_k}\right)^{p_n}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n}(q_{2k}+q_{2k+1})|x_k|}{\sum_{k=0}^{2n+1} q_k}\right)^{p_n}\\
&\quad\leq\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n}(q_{2k}+q_{2k+1})|x_k|+q_{2n}|x_n|}{\sum_{k=0}^{n} q_k}\right)^{p_n}+\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n}(q_{2k}+q_{2k+1})|x_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n}\\
&\quad\leq 2^{h-1}\left[\sum_{n=0}^{\infty}\left(\frac{2C\sum_{k=0}^{n} q_k|x_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n}+\sum_{n=0}^{\infty}\left(\frac{C\sum_{k=0}^{n} q_k|x_k|}{\sum_{k=0}^{n} q_k}\right)^{p_n}\right]
\end{aligned}$$

$$+\sum_{n=0}^{\infty}\left(\frac{2C\sum_{k=0}^{n}q_k|x_k|}{\sum_{k=0}^{n}q_k}\right)^{p_n}$$

$$\leq 2^{h-1}\left(2^h+1\right)C^h\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n}q_k|x_k|}{\sum_{k=0}^{n}q_k}\right)^{p_n}+2^hC^h\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n}q_k|x_k|}{\sum_{k=0}^{n}q_k}\right)^{p_n}$$

$$\leq\left(2^{2h-1}+2^{h-1}+2^h\right)C^h\sum_{n=0}^{\infty}\left(\frac{\sum_{k=0}^{n}q_k|x_k|}{\sum_{k=0}^{n}q_k}\right)^{p_n}<\infty,$$

then $(x_{[\frac{n}{2}]})\in \operatorname{ces}((p_n),(q_n))$. □

By Theorem 2.17, we derive the following corollaries.

Corollary 3.2 $U^{\text{app}}_{\operatorname{ces}((p_n),(q_n))}$ *is an ideal operator if conditions* (b1), (b2) *and* (b3) *are satisfied.*

Corollary 3.3 $U^{\text{app}}_{\operatorname{ces}((p_n))}$ *is an ideal operator if* (p_n) *is increasing,* $\lim_{n\to\infty}\inf p_n>1$ *and* $\lim_{n\to\infty}\sup p_n<\infty$.

Corollary 3.4 *If* $1<p<\infty$, *then* $U^{\text{app}}_{\operatorname{ces}_p}$ *is an operator ideal.*

3.2 Topological problem

For a Norlund sequence space E, the ideal of the finite rang operators in the class of Banach spaces is dense in $U^{\text{app}}_E(X,Y)$, which gives a negative answer to Rhoades [20] open problem about the linearity of E-type spaces $(U^{\text{app}}_E(X,Y))$.

Theorem 3.5 $U^{\text{app}}_{\operatorname{ces}((p_n),(q_n))}=\overline{F(X,Y)}$, *assuming conditions* (b1), (b2) *and* (b3) *are fulfilled, and the converse is not true, in general.*

Proof Firstly, we substantiate that each finite operator $T\in F(X,Y)$ belongs to $U^{\text{app}}_{\operatorname{ces}((p_n),(q_n))}(X,Y)$. Since $e_m\in\operatorname{ces}((p_n),(q_n))$ for each $m\in\mathbb{N}$ and $\operatorname{ces}((p_n),(q_n))$ is a linear space, for each finite operator $T\in F(X,Y)$, i.e., we obtain that $(\alpha_n(T))_{n=0}^{\infty}$ holds main finitely a significant unique number in relation to zero. Currently we substantiate that $U^{\text{app}}_{\operatorname{ces}((p_n),(q_n))}\subseteq\overline{F(X,Y)}$. Let (q_n) be a monotonic increasing sequence such that there exists a constant $C\geq 1$ for which $q_{2n+1}\leq Cq_n$, then we have for $n\geq s$ that

$$q_{2s+n}\leq q_{2s+2n+1}\leq Cq_{s+n}\leq Cq_{2n}\leq Cq_{2n+1}\leq C^2q_n. \tag{1}$$

By taking $T\in U^{\text{app}}_{\operatorname{ces}((p_n),(q_n))}(X,Y)$, we obtain $(\alpha_n(T))_{n=0}^{\infty}\in\operatorname{ces}((p_n),(q_n))$, and while $\rho((\alpha_n(T))_{n=0}^{\infty})<\infty$, let $\varepsilon\in(0,1)$, at that point there exists $s\in\mathbb{N}-\{0\}$ such-and-such $\rho((\alpha_n(T))_{n=s}^{\infty})<\frac{\varepsilon}{2^{h+2}\delta C}$, where $\delta=\max\{1,\sum_{n=s}^{\infty}(\sum_{k=0}^{n}q_k)^{-p_n}\}$. As $\alpha_n(T)$ is decreasing for all $n\in\mathbb{N}$, we obtain

$$\begin{aligned}\sum_{n=s+1}^{2s}\left(\frac{\sum_{k=0}^{n}q_k\alpha_{2s}(T)}{\sum_{k=0}^{n}q_k}\right)^{p_n}&\leq\sum_{n=s+1}^{2s}\left(\frac{\sum_{k=0}^{n}q_k\alpha_k(T)}{\sum_{k=0}^{n}q_k}\right)^{p_n}\\&\leq\sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{n}q_k\alpha_k(T)}{\sum_{k=0}^{n}q_k}\right)^{p_n}<\frac{\varepsilon}{2^{h+2}\delta C}.\end{aligned}\tag{2}$$

Hence, there exists $A \in F_{2s}(X,Y)$ such that $\operatorname{rank} A \leq 2s$ and

$$\sum_{n=2s+1}^{3s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} \leq \sum_{n=s+1}^{2s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} < \frac{\varepsilon}{2^{h+2}\delta C}, \tag{3}$$

since the sequence (p_n) is bounded, then on considering

$$\sup_{n\geq s\in\mathbb{N}}\left(\sum_{k=0}^{2s-1} q_k\|T-A\|\right)^{p_n} < \frac{\varepsilon}{2^h\delta}, \tag{4}$$

and from the definition of approximation numbers, there exists $N \in \mathbb{N} - \{0\}$, A_N with $\operatorname{rank}(A_N) \leq N$ and $\|T-A\| \leq 2\alpha_n(T)$. Since $\lim_{n\to\infty}\alpha_n(T) = 0$, then $\lim_{N\to\infty}\|T-A_N\| = 0$, hence we put

$$\sum_{n=0}^{s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} < \frac{\varepsilon}{2^{h+3}\delta C}. \tag{5}$$

Since (p_n) is increasing and $(\frac{1}{\sum_{k=0}^{n} q_k})$ is decreasing for each $n \in \mathbb{N}$, we have by using (1), (2), (3), (4) and (5) that

$$\begin{aligned}
&d(A,T)\\
&\quad= \rho\big(\alpha_n(T-A)\big)_{n=0}^{\infty}\\
&\quad= \sum_{n=0}^{3s-1}\left(\frac{\sum_{k=0}^{n} q_k\alpha_k(T-A)}{\sum_{k=0}^{n} q_k}\right)^{p_n} + \sum_{n=3s}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k\alpha_k(T-A)}{\sum_{k=0}^{n} q_k}\right)^{p_n}\\
&\quad\leq \sum_{n=0}^{3s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{n+2s} q_k\alpha_k(T-A)}{\sum_{k=0}^{n+2s} q_k}\right)^{p_{n+2s}}\\
&\quad\leq \sum_{n=0}^{3s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{n+2s} q_k\alpha_k(T-A)}{\sum_{k=0}^{n+2s} q_k}\right)^{p_n}\\
&\quad\leq \sum_{n=0}^{3s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{n+2s} q_k\alpha_k(T-A)}{\sum_{k=0}^{n} q_k}\right)^{p_n}\\
&\quad\leq 3\sum_{n=0}^{s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{2s-1} q_k\alpha_k(T-A) + \sum_{k=2s}^{n+2s} q_k\alpha_k(T-A)}{\sum_{k=0}^{n} q_k}\right)^{p_n}\\
&\quad\leq 3\sum_{n=0}^{s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n}\\
&\qquad+ 2^{h-1}\left[\sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{2s-1} q_k\alpha_k(T-A)}{\sum_{k=0}^{n} q_k}\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{\sum_{k=2s}^{n+2s} q_k\alpha_k(T-A)}{\sum_{k=0}^{n} q_k}\right)^{p_n}\right]\\
&\quad\leq 3\sum_{n=0}^{s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n}\\
&\qquad+ 2^{h-1}\left[\sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{2s-1} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} + \sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{n} q_{k+2s}\alpha_{k+2s}(T-A)}{\sum_{k=0}^{n} q_k}\right)^{p_n}\right]
\end{aligned}$$

$$\begin{aligned}
&\leq 3\sum_{n=0}^{s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} \\
&\quad + 2^{h-1}\sup_{n=s}^{\infty}\left(\sum_{k=0}^{2s-1} q_k\|T-A\|\right)^{p_n}\sum_{n=s}^{\infty}\left(\sum_{k=0}^{n} q_k\right)^{-p_n} + 2^{h-1}\sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{n} C^2 q_k\alpha_k(T)}{\sum_{k=0}^{n} q_k}\right)^{p_n} \\
&\leq 3\sum_{n=0}^{s}\left(\frac{\sum_{k=0}^{n} q_k\|T-A\|}{\sum_{k=0}^{n} q_k}\right)^{p_n} \\
&\quad + 2^{h-1}\sup_{n=s}^{\infty}\left(\sum_{k=0}^{2s-1} q_k\|T-A\|\right)^{p_n}\sum_{n=s}^{\infty}\left(\sum_{k=0}^{n} q_k\right)^{-p_n} + 2^{h-1}C^{2h}\sum_{n=s}^{\infty}\left(\frac{\sum_{k=0}^{n} q_k\alpha_k(T)}{\sum_{k=0}^{n} q_k}\right)^{p_n} \\
&< \varepsilon.
\end{aligned}$$

Since $I_3 \in U^{\mathrm{app}}_{\mathrm{ces}((1),(1))}$ but condition (b2) is not satisfied, which gives a counter example for the converse part. This completes the proof. From Theorem 3.5, we can say that if (b1), (b2) and (b3) are satisfied, then every compact operator would be approximated by finite rank operators and the converse does not hold, in general. □

Corollary 3.6 $U^{\mathrm{app}}_{\mathrm{ces}((p_n))} = \overline{F(X,Y)}$ *if* (p_n) *is increasing,*

$$\lim_{n\to\infty}\inf p_n > 1 \quad \textit{and} \quad \lim_{n\to\infty}\sup p_n < \infty.$$

Corollary 3.7 $U^{\mathrm{app}}_{\mathrm{ces}_p} = \overline{F(X,Y)}$, *where* $1 < p < \infty$.

3.3 Completeness of the ideal components

For a Norlund sequence space E, the components of ideal $U^{\mathrm{app}}_E(X,Y)$ are complete.

Theorem 3.8 $\mathrm{ces}((p_n),(q_n))$ *is a pre-modular (sss) if conditions* (b1), (b2) *and* (b3) *are satisfied.*

Proof We define a functional ρ on $\mathrm{ces}((p_n),(q_n))$ as $\rho(x) = \sum_{n=0}^{\infty}(\frac{\sum_{k=0}^{n} q_k|x_k|}{\sum_{k=0}^{n} q_k})^{p_n}$.

(i) Evidently, $\rho(x) \geq 0$ and $\rho(x) = 0 \Leftrightarrow x = \theta$.

(ii) There exists steady $L = \max\{1, \sup_n |\lambda|^{p_n}\} \geq 1$ such that $\rho(\lambda x) \leq L\lambda\rho(x)$ for all $x \in \mathrm{ces}((p_n),(q_n))$ and for each real λ.

(iii) For some numbers $K = \max\{1, 2^{h-1}\} \geq 1$, we have the inequality $\rho(x+y) \leq K(\rho(x) + \rho(y))$ for all $x, y \in \mathrm{ces}((p_n),(q_n))$.

(iv) Clearly, (2) of Theorem 3.1 holds.

(v) It is obtained from (3) of Theorem 3.1 that $K_0 \geq (2^{2h-1} + 2^{h-1} + 2^h) \geq 1$.

(vi) It is clear that $\overline{F} = \mathrm{ces}((p_n),(q_n))$.

(vii) There exists a steady $0 < \xi \leq |\lambda|^{p_0-1}$ such that $\rho(\lambda, 0, 0, 0, \ldots) \geq \xi|\lambda|\rho(1, 0, 0, 0, \ldots)$ for all $\lambda \in \mathbb{R}$. □

Theorem 3.9 $U^{\mathrm{app}}_{E_\rho}(X,Y)$ *is complete if* X *and* Y *are Banach spaces and* E_ρ *is a pre-modular* (*sss*).

Proof Let (T_m) be a Cauchy sequence in $U^{\mathrm{app}}_{E_\rho}(X,Y)$. Then, by utilizing part (vii) of Definition 2.11 and since $L(X,Y) \supseteq U^{\mathrm{app}}_{E_\rho}(X,Y)$, we get

$$\begin{aligned}\rho\big(\big(\alpha_n(T_i - T_j)\big)_{n=0}^{\infty}\big) &\geq \rho\big(\alpha_0(T_i - T_j), 0, 0, 0, \ldots\big)\\ &= \rho\big(\|T_i - T_j\|, 0, 0, 0, \ldots\big) \geq \xi\|T_i - T_j\|\rho(1, 0, 0, 0, \ldots),\end{aligned}$$

then $(T_m)_{m\in\mathbb{N}}$ is plus a Cauchy sequence in $L(X,Y)$. While the space $L(X,Y)$ is a Banach space, so there exists $T \in L(X,Y)$ such that $\lim_{m\to\infty}\|T_m - T\| = 0$ and while $(\alpha_n(T_m))_{n=0}^{\infty} \in E$ for each $m \in \mathbb{N}$. Therefore, using parts (iii) and (iv) of Definition 2.11 and ρ is continuous at θ, we obtain

$$\begin{aligned}\rho\big(\big(\alpha_n(T)\big)_{n=0}^{\infty}\big) &= \rho\big(\big(\alpha_n(T - T_m + T_m)\big)_{n=0}^{\infty}\big)\\ &\leq K\rho\big(\big(\alpha_{[\frac{n}{2}]}(T - T_m)\big)_{n=0}^{\infty}\big) + K\rho\big(\big(\alpha_{[\frac{n}{2}]}(T_m)_{n=0}^{\infty}\big)\big)\\ &\leq K\rho\big(\big(\|T_m - T\|\big)_{n=0}^{\infty}\big) + K\rho\big(\big(\alpha_n(T_m)_{n=0}^{\infty}\big)\big) < \varepsilon,\end{aligned}$$

we have $(\alpha_n(T))_{n=0}^{\infty} \in E$, then $T \in U^{\mathrm{app}}_{E_\rho}(X,Y)$. □

Corollary 3.10 *On the off chance that X and Y are Banach spaces and conditions* (b1), (b2) *and* (b3) *are fulfilled, then $U^{\mathrm{app}}_{\mathrm{ces}((p_n),(q_n))}(X,Y)$ is complete.*

Corollary 3.11 *If X and Y are Banach spaces and (p_n) is an increasing sequence, $\lim_{n\to\infty}\inf p_n > 1$ and $\lim_{n\to\infty}\sup p_n < \infty$, then $U^{\mathrm{app}}_{\mathrm{ces}((p_n))}(X,Y)$ is complete.*

Corollary 3.12 *If X and Y are Banach spaces and $1 < p < \infty$, then $U^{\mathrm{app}}_{\mathrm{ces}_p}(X,Y)$ is complete.*

Competing interests
The author declares that they have no competing interests.

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References
1. Pietsch, A: Operator Ideals. North-Holland, Amsterdam (1980)
2. Faried, N, Bakery, AA: Mappings of type Orlicz and generalized Cesáro sequence space. J. Inequal. Appl. (2013). doi:10.1186/1029-242X-2013-186
3. Bakery, AA: Mappings of type generalized de La Vallée Poussin's mean. J. Inequal. Appl. (2013). doi:10.1186/1029-242X-2013-518
4. Bakery, AA: Operator ideal of Cesaro type sequence spaces involving lacunary sequence. Abstr. Appl. Anal. **2014**, Article ID 419560 (2014). doi:10.1155/2014/419560
5. Lim, KP: Matrix transformation on certain sequence spaces. Tamkang J. Math. **8**(2), 213-220 (1977)
6. Kalton, NJ: Spaces of compact operators. Math. Ann. **208**, 267-278 (1974)
7. Lima, Å, Oja, E: Ideals of finite rank operators, intersection properties of balls, and the approximation property. Stud. Math. **133**, 175-186 (1999)
8. Krasnoselskii, MA, Rutickii, YB: Convex Functions and Orlicz Spaces. Gorningen, Netherlands (1961)
9. Orlicz, W: Über Raume (L^M) Bull. Int. Acad Polon. Sci. A, 93-107 (1936)
10. Nakano, H: Concave modulars. J. Math. Soc. Jpn. **5**, 29-49 (1953)
11. Lindenstrauss, J, Tzafriri, L: On Orlicz sequence spaces. Isr. J. Math. **10**, 379-390 (1971)
12. Wang, CS: On Norlund sequence spaces. Tamkang J. Math. **9**(2), 269-274 (1978)
13. Sanhan, W, Suantai, S: On *k*-nearly uniformly convex property in generalized Cesàro sequence space. Int. J. Math. Math. Sci. **57**, 3599-3607 (2003)
14. Lee, PY: Cesáro sequence spaces. Math. Chron. **13**, 29-45 (1984)
15. Lui, YQ, Wu, BE, Lee, PY: Method of Sequence Spaces. Guangdong of Science and Technology Press, Guangzhou (1996) (in Chinese)

16. Shiue, JS: On the Cesàro sequence spaces. Tamkang J. Math. **1**(1), 19-25 (1970)
17. Yeşilkayagil, M, Başar, F: On the paranormed Nörlund sequence space of non-absolute type. Abstr. Appl. Anal. **2014**, Article ID 858704 (2014)
18. Başar, F: Summability Theory and Its Applications. Bentham Science Publishers, Istanbul (2012)
19. Altay, B, Başar, F: Generalization of the sequence space $\ell(p)$ derived by weighted means. J. Math. Anal. Appl. **330**(1), 147-185 (2007)
20. Rhoades, BE: Operators of $A - p$ type. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **59**(3-4), 238-241 (1975)