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Degenerate poly-Bernoulli polynomials with umbral calculus viewpoint

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Abstract
In this paper, we consider the degenerate poly-Bernoulli polynomials. We present several explicit formulas and recurrence relations for these polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

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Keywords: degenerate poly-Bernoulli polynomials; umbral calculus

1 Introduction

The *degenerate Bernoulli polynomials* $\beta_n(\lambda, x)$ ($\lambda \neq 0$) were introduced by Carlitz [1] and rediscovered by Ustinov [2] under the name *Korobov polynomials of the second kind*. They are given by the generating function

$$\frac{t}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0} \beta_n(\lambda, x)\frac{t^n}{n!}.$$

When $x = 0$, $\beta_n(\lambda) = \beta_n(\lambda, 0)$ are called the *degenerate Bernoulli numbers* (see [3]). We observe that $\lim_{\lambda\to 0} \beta_n(\lambda, x) = B_n(x)$, where $B_n(x)$ is the nth *ordinary Bernoulli polynomial* (see the references).

The *poly-Bernoulli polynomials* $PB_n^{(k)}(x)$ are defined by

$$\frac{Li_k(1-e^{-t})}{e^t-1}e^{xt} = \sum_{n\geq 0} PB_n^{(k)}(x)\frac{t^n}{n!},$$

where $Li_k(x)$ ($k \in \mathbb{Z}$) is the classical *polylogarithm function* given by $Li_k(x) = \sum_{n\geq 1} \frac{x^n}{n^k}$ (see [4–6]).

For $0 \neq \lambda \in \mathbb{C}$ and $k \in \mathbb{Z}$, the *degenerate poly-Bernoulli polynomials* $P\beta_n^{(k)}(\lambda, x)$ are defined by Kim and Kim to be

$$\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{x/\lambda} = \sum_{n\geq 0} P\beta_n^{(k)}(\lambda, x)\frac{t^n}{n!} \quad \text{(see [5])}. \tag{1.1}$$

When $x = 0$, $P\beta_n^{(k)}(\lambda) = P\beta_n^{(k)}(\lambda, 0)$ are called *degenerate poly-Bernoulli numbers*. We observe that $\lim_{\lambda\to 0} P\beta_n^{(k)}(\lambda, x) = PB_n^{(k)}(x)$.

The goal of this paper is to use umbral calculus to obtain several new and interesting identities of degenerate poly-Bernoulli polynomials. To do that we recall the umbral calculus as given in [7, 8]. We denote the algebra of polynomials in a single variable x over $\mathbb{C}$ by Π and the vector space of all linear functionals on Π by Π^*. The action of a linear functional L on a polynomial $p(x)$ is denoted by $\langle L \mid p(x)\rangle$. We define the vector space structure on Π^* by $\langle cL + c'L' \mid p(x)\rangle = c\langle L \mid p(x)\rangle + c'\langle L' \mid p(x)\rangle$, where $c, c' \in \mathbb{C}$. We define the algebra of formal power series in a single variable t to be

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \;\middle|\; a_k \in \mathbb{C} \right\}. \tag{1.2}$$

A power series $f(t) \in \mathcal{H}$ defines a linear functional on Π by setting

$$\langle f(t) \mid x^n \rangle = a_n, \quad \text{for all } n \geq 0 \text{ (see [6, 8–10])}. \tag{1.3}$$

By (1.2) and (1.3), we have

$$\langle t^k \mid x^n \rangle = n!\delta_{n,k}, \quad \text{for all } n, k \geq 0, \tag{1.4}$$

where $\delta_{n,k}$ is the Kronecker symbol. Let $f_L(t) = \sum_{n\geq 0} \langle L \mid x^n\rangle \frac{t^n}{n!}$. From (1.4), we have $\langle f_L(t) \mid x^n\rangle = \langle L \mid x^n\rangle$. So, the map $L \mapsto f_L(t)$ is a vector space isomorphism from Π^* onto $\mathcal{H}$. Thus, $\mathcal{H}$ is thought of as set of both formal power series and linear functionals. We call $\mathcal{H}$ the *umbral algebra*. The *umbral calculus* is the study of umbral algebra.

The *order* $O(f(t))$ of the non-zero power series $f(t) \in \mathcal{H}$ is the smallest integer k for which the coefficient of t^k does not vanish. Suppose that $f(t), g(t) \in \mathcal{H}$ such that $O(f(t)) = 1$ and $O(g(t)) = 0$, then there exists a unique sequence $s_n(x)$ of polynomials such that

$$\langle g(t)\bigl(f(t)\bigr)^k \mid s_n(x)\rangle = n!\delta_{n,k}, \tag{1.5}$$

where $n, k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer* sequence for $(g(t), f(t))$, which is denoted by $s_n(x) \sim (g(t), f(t))$ (see [7, 8]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have $\langle e^{yt} \mid p(x)\rangle = p(y)$, $\langle f(t)g(t) \mid p(x)\rangle = \langle g(t) \mid f(t)p(x)\rangle$, and

$$f(t) = \sum_{n\geq 0} \langle f(t) \mid x^n\rangle \frac{t^n}{n!}, \qquad p(x) = \sum_{n\geq 0} \langle t^n \mid p(x)\rangle \frac{x^n}{n!} \tag{1.6}$$

(see [7, 8]). From (1.6), we obtain $\langle t^k \mid p(x)\rangle = p^{(k)}(0)$ and $\langle 1 \mid p^{(k)}(x)\rangle = p^{(k)}(0)$, where $p^{(k)}(0)$ denotes the kth derivative of $p(x)$ with respect to x at $x = 0$. So, we get $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k} p(x)$, for all $k \geq 0$. Let $s_n(x) \sim (g(t), f(t))$, then we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y) \frac{t^n}{n!}, \tag{1.7}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [7, 8]). For $s_n(x) \sim (g(t), f(t))$ and $r_n(x) \sim (h(t), \ell(t))$, let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, then we have

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}\left(\ell\left(\bar{f}(t)\right)\right)^k \,\Big|\, x^n\right\rangle \tag{1.8}$$

(see [7, 8]).

From (1.1), we see that $P\beta_n^{(k)}(\lambda,x)$ is the Sheffer sequence for the pair

$$\left(g(t),f(t)\right) = \left(\frac{e^t-1}{Li_k(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right). \tag{1.9}$$

In this paper, we will use umbral calculus in order to derive some properties, explicit formulas, recurrence relations, and identities as regards the degenerate poly-Bernoulli polynomials. Also, we establish a connection between our polynomials and several known families of polynomials.

2 Explicit formulas

In this section we present several explicit formulas for the degenerate poly-Bernoulli polynomials, namely $P\beta_n^{(k)}(\lambda,x)$. To do so, we recall that Stirling numbers $S_1(n,k)$ of the first kind can be defined by means of exponential generating functions as $\sum_{\ell\geq j} S_1(\ell,j)\frac{t^\ell}{\ell!} = \frac{1}{j!}\log^j(1+t)$ and can be defined by means of ordinary generating functions as

$$(x)_n = \sum_{m=0}^{n} S_1(n,m)x^m \sim \left(1, e^t-1\right), \tag{2.1}$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ with $(x)_0 = 1$. For $\lambda \neq 0$, we define $(x\mid\lambda)_n = \lambda^n(x/\lambda)_n$. Sometimes, for simplicity, we denote the function $\frac{e^t-1}{Li_k(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}$ by $G_k(t)$.

First, we express the degenerate poly-Bernoulli polynomials in terms of degenerate poly-Bernoulli numbers.

Theorem 2.1 *For all* $n \geq 0$,

$$P\beta_n^{(k)}(\lambda,x) = \sum_{j=0}^{n}\sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)\lambda^{\ell-j}P\beta_{n-\ell}^{(k)}(\lambda)x^j.$$

Proof By (1.5), for $s_n(x) \sim (g(t),f(t))$ we have $s_n(x) = \sum_{j=0}^{n}\frac{1}{j!}\langle g(\bar{f}(t))^{-1}\bar{f}(t)^j \mid x^n\rangle x^j$. Thus, in the case of degenerate poly-Bernoulli polynomials (see (1.9)), we have

$$\begin{aligned}
&\frac{1}{j!}\left\langle g\left(\bar{f}(t)\right)^{-1}\bar{f}(t)^j \mid x^n\right\rangle \\
&= \frac{1}{j!}\left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^j \,\Big|\, x^n\right\rangle \\
&= \lambda^{-j}\left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1} \,\Big|\, \frac{\log^j(1+\lambda t)}{j!}x^n\right\rangle \\
&= \lambda^{-j}\left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1} \,\Big|\, \sum_{\ell\geq j} S_1(\ell,j)\frac{\lambda^\ell t^\ell}{\ell!}x^n\right\rangle \\
&= \sum_{\ell=j}^{n}\binom{n}{\ell}S_1(\ell,j)\lambda^{\ell-j}\left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1} \,\Big|\, x^{n-\ell}\right\rangle
\end{aligned}$$

$$= \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell,j)\lambda^{\ell-j}\left\langle \sum_{m\geq 0} P\beta_m^{(k)}(\lambda)\frac{t^m}{m!} \,\Big|\, x^{n-\ell}\right\rangle$$

$$= \sum_{\ell=j}^{n} \binom{n}{\ell} S_1(\ell,j)\lambda^{\ell-j} P\beta_{n-\ell}^{(k)}(\lambda),$$

which completes the proof. □

Note that Stirling numbers $S_2(n,k)$ of the second kind can be defined by the exponential generating functions as

$$\sum_{n\geq k} S_2(n,k)\frac{x^n}{n!} = \frac{(e^t-1)^k}{k!}. \tag{2.2}$$

Theorem 2.2 *For all* $n \geq 0$,

$$P\beta_n^{(k)}(\lambda,x) = \sum_{j=0}^{n}\left(\sum_{m=j}^{n}\sum_{\ell=0}^{m-j}\binom{m}{j} S_1(n,m)S_2(m-j,\ell)\lambda^{n-\ell-j}P\beta_\ell^{(k)}(\lambda)\right)x^j.$$

Proof By (2.1), we have $(x \mid \lambda)_n = \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}x^m \sim (1, \frac{1}{\lambda}(e^{\lambda t}-1))$, and by (1.9), we have

$$G_k(t)P\beta_n^{(k)}(\lambda,x) \sim \left(1, \frac{1}{\lambda}\left(e^{\lambda t}-1\right)\right), \tag{2.3}$$

which implies $G_k(t)P\beta_n^{(k)}(\lambda,x) = \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}x^m$. Thus,

$$\begin{aligned}
P\beta_n^{(k)}(\lambda,x) &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}\frac{Li_k(1-e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}{e^t-1}x^m \\
&= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}\frac{Li_k(1-e^{-\nu})}{(1+\lambda\nu)^{1/\lambda}-1}\Big|_{\nu=\frac{1}{\lambda}(e^{\lambda t}-1)}x^m \\
&= \sum_{m=0}^{n}\sum_{\ell\geq 0} S_1(n,m)\lambda^{n-m}P\beta_\ell^{(k)}(\lambda)\frac{(\frac{1}{\lambda}(e^{\lambda t}-1))^\ell}{\ell!}x^m \\
&= \sum_{m=0}^{n}\sum_{\ell=0}^{m} S_1(n,m)\lambda^{n-m-\ell}P\beta_\ell^{(k)}(\lambda)\sum_{j\geq\ell} S_2(j,\ell)\frac{\lambda^j t^j}{j!}x^m \\
&= \sum_{m=0}^{n}\sum_{\ell=0}^{m}\sum_{j=\ell}^{m}\binom{m}{j} S_1(n,m)S_2(j,\ell)\lambda^{n-m-\ell+j}P\beta_\ell^{(k)}(\lambda)x^{m-j} \\
&= \sum_{m=0}^{n}\sum_{\ell=0}^{m}\sum_{j=0}^{m-\ell}\binom{m}{j} S_1(n,m)S_2(m-j,\ell)\lambda^{n-\ell-j}P\beta_\ell^{(k)}(\lambda)x^j \\
&= \sum_{j=0}^{n}\left(\sum_{m=j}^{n}\sum_{\ell=0}^{m-j}\binom{m}{j} S_1(n,m)S_2(m-j,\ell)\lambda^{n-\ell-j}P\beta_\ell^{(k)}(\lambda)\right)x^j,
\end{aligned} \tag{2.4}$$

which completes the proof. □

Theorem 2.3 *For all* $n \geq 1$,

$$P\beta_n^{(k)}(\lambda, x) = \sum_{j=0}^{n} \left(\sum_{\ell=0}^{n-j} \sum_{m=0}^{n-j-\ell} \binom{n-1}{\ell} \binom{n-\ell}{j} \lambda^{n-m-j} S_2(n-j-\ell, m) B_\ell^{(n)} P\beta_m^{(k)}(\lambda) \right) x^j.$$

Proof Note that $x^n \sim (1, t)$. Thus, by (2.3) and transfer formula, we have

$$\begin{aligned} G_k(t) P\beta_n^{(k)}(\lambda, x) &= x \left(\frac{\lambda t}{e^{\lambda t} - 1} \right)^n x^{-1} x^n = x \left(\frac{\lambda t}{e^{\lambda t} - 1} \right)^n x^{n-1} \\ &= x \sum_{\ell \geq 0} B_\ell^{(n)} \frac{\lambda^\ell t^\ell}{\ell!} x^{n-1} = x \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda^\ell B_\ell^{(n)} x^{n-1-\ell} \\ &= \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda^\ell B_\ell^{(n)} x^{n-\ell}. \end{aligned}$$

Therefore, $P\beta_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} \lambda^\ell B_\ell^{(n)} G_k(t)^{-1} x^{n-\ell}$, which, by (2.4), completes the proof. □

Theorem 2.4 *For all* $n \geq 0$,

$$P\beta_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n} \left(\sum_{m=0}^{\ell} (-1)^{m+\ell} \binom{n}{\ell} \frac{(m+1)!}{(m+1)^k (\ell+1)} S_2(\ell+1, m+1) \right) \beta_{n-\ell}(\lambda, x).$$

Proof By (2.3), we have

$$\begin{aligned} P\beta_n^{(k)}(\lambda, y) &= \left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda} - 1} (1+\lambda t)^{y/\lambda} \,\middle|\, x^n \right\rangle \\ &= \left\langle \frac{Li_k(1-e^{-t})}{t} \,\middle|\, \frac{t}{(1+\lambda t)^{1/\lambda} - 1} (1+\lambda t)^{y/\lambda} x^n \right\rangle \\ &= \left\langle \frac{Li_k(1-e^{-t})}{t} \,\middle|\, \sum_{\ell \geq 0} \beta_\ell(\lambda, y) \frac{t^\ell}{\ell!} x^n \right\rangle \\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} \beta_\ell(\lambda, y) \left\langle \frac{1}{t} \sum_{m \geq 1} \frac{(1-e^{-t})^m}{m^k} \,\middle|\, x^{n-\ell} \right\rangle \\ &= \sum_{\ell=0}^{n} \sum_{m=1}^{n-\ell+1} \binom{n}{\ell} \beta_\ell(\lambda, y) \left\langle \frac{(-1)^m (e^{-t}-1)^m}{m^k t} \,\middle|\, x^{n-\ell} \right\rangle. \end{aligned} \tag{2.5}$$

Thus, by (2.2), we obtain

$$\begin{aligned} P\beta_n^{(k)}(\lambda, y) &= \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \beta_\ell(\lambda, y) \left\langle \frac{(-1)^{m+1}(m+1)!}{(m+1)^k} \sum_{j=m+1}^{n-\ell+1} S_2(j, m+1) \frac{(-1)^j}{j!} t^{j-1} \,\middle|\, x^{n-\ell} \right\rangle \\ &= \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \beta_\ell(\lambda, y) \frac{(-1)^{m+1}(m+1)!}{(m+1)^k} S_2(n-\ell+1, m+1) \frac{(-1)^{n-\ell+1}(n-\ell)!}{(n-\ell+1)!} \\ &= \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} (-1)^{n+m-\ell} \binom{n}{\ell} \frac{(m+1)!}{(m+1)^k (n-\ell+1)} S_2(n-\ell+1, m+1) \beta_\ell(\lambda, y), \end{aligned}$$

which completes the proof. □

Note that the above theorem has been obtained in Theorem 2.2 in [5].

Theorem 2.5 *For all* $n \geq 0$,

$$P\beta_n^{(k)}(\lambda, x) = \frac{1}{n+1} \sum_{\ell=0}^{n} \sum_{m=0}^{\ell} \binom{n+1}{n-\ell, m, \ell-m+1} P\beta_m^{(k)} \beta_{n-\ell}(\lambda, x),$$

where $\binom{a}{b_1, b_2, b_3} = \frac{a!}{b_1! b_2! b_3!}$ *is the multinomial coefficient.*

Proof By (2.5), we have

$$\begin{aligned} P\beta_n^{(k)}(\lambda, y) &= \sum_{\ell=0}^{n} \binom{n}{\ell} \beta_\ell(\lambda, y) \left\langle \frac{e^t - 1}{t} \,\Big|\, \frac{Li_k(1 - e^{-t})}{e^t - 1} x^{n-\ell} \right\rangle \\ &= \sum_{\ell=0}^{n} \binom{n}{\ell} \beta_\ell(\lambda, y) \left\langle \frac{e^t - 1}{t} \,\Big|\, \sum_{m \geq 0} P\beta_m^{(k)} \frac{t^m}{m!} x^{n-\ell} \right\rangle \\ &= \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{m} \beta_\ell(\lambda, y) P\beta_m^{(k)} \left\langle \frac{e^t - 1}{t} \,\Big|\, x^{n-\ell-m} \right\rangle. \end{aligned}$$

Note that $\langle \frac{e^t-1}{t} \mid x^{n-\ell-m} \rangle = \int_0^1 u^{n-\ell-m} \, du = \frac{1}{n-\ell-m+1}$. Thus,

$$\begin{aligned} P\beta_n^{(k)}(\lambda, y) &= \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} \frac{1}{n-\ell-m+1} \binom{n}{\ell} \binom{n-\ell}{m} P\beta_m^{(k)} \beta_\ell(\lambda, y) \\ &= \sum_{\ell=0}^{n} \sum_{m=0}^{\ell} \frac{1}{\ell-m+1} \binom{n}{\ell} \binom{\ell}{m} P\beta_m^{(k)} \beta_{n-\ell}(\lambda, y) \\ &= \frac{1}{n+1} \sum_{\ell=0}^{n} \sum_{m=0}^{\ell} \binom{n+1}{n-\ell, m, \ell-m+1} P\beta_m^{(k)} \beta_{n-\ell}(\lambda, y), \end{aligned}$$

which completes the proof. □

Note that $Li_2(1 - e^{-t}) = \int_0^t \frac{y}{e^y-1} \, dy = \sum_{j\geq 0} B_j \frac{1}{j!} \int_0^t y^j \, dy = \sum_{j\geq 0} \frac{B_j t^{j+1}}{j!(j+1)}$. For general $k \geq 2$, the function $Li_k(1 - e^{-t})$ has the integral representation

$$Li_k\left(1 - e^{-t}\right) = \underbrace{\int_0^t \frac{1}{e^y - 1} \int_0^y \frac{1}{e^y - 1} \int_0^y \cdots \frac{1}{e^y - 1}}_{(k-2) \text{ times}} \int_0^y \frac{y}{e^y - 1} \, dy \cdots dy \, dy \, dy,$$

which, by induction on k, implies

$$Li_k\left(1 - e^{-t}\right) = \sum_{j_1 \geq 0} \cdots \sum_{j_{k-1} \geq 0} t^{j_1 + \cdots + j_{k-1} + 1} \prod_{i=1}^{k-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)}. \tag{2.6}$$

Theorem 2.6 *For all* $n \geq 0$ *and* $k \geq 2$,

$$P\beta_n^{(k)}(\lambda, x) = \sum_{\ell=0}^{n} (n)_\ell \beta_{n-\ell}(\lambda, x) \left(\sum_{j_1 + \cdots + j_{k-1} = \ell} \prod_{i=1}^{k-1} \frac{B_{j_i}}{j_i!(j_1 + \cdots + j_i + 1)} \right).$$

Proof By (2.5), we have

$$P\beta_n^{(k)}(\lambda,x)=\sum_{\ell=0}^{n}\binom{n}{\ell}\beta_\ell(\lambda,x)\left\langle\frac{Li_k(1-e^{-t})}{t}\;\middle|\;x^{n-\ell}\right\rangle.$$

Thus, by (2.6), we obtain

$$P\beta_n^{(k)}(\lambda,x)=\sum_{\ell=0}^{n}\frac{n!}{\ell!}\beta_\ell(\lambda,x)\left(\sum_{j_1+\cdots+j_{k-1}=n-\ell}\prod_{i=1}^{k-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}\right),$$

which completes the proof. □

Note that here we compute $A=\langle Li_k(1-e^{-t})\mid x^{n+1}\rangle$ in several different ways. As for the first way, we have

$$\begin{aligned}A&=\left\langle\int_0^t\frac{d}{ds}Li_k\bigl(1-e^{-s}\bigr)\,ds\;\middle|\;x^{n+1}\right\rangle=\left\langle\int_0^t\frac{e^{-s}Li_{k-1}(1-e^{-s})}{1-e^{-s}}\,ds\;\middle|\;x^{n+1}\right\rangle\\&=\left\langle\int_0^t\frac{Li_{k-1}(1-e^{-s})}{e^s-1}\,ds\;\middle|\;x^{n+1}\right\rangle=\sum_{m\geq0}\frac{PB_m^{(k-1)}}{m!}\left\langle\int_0^t s^m\,ds\;\middle|\;x^{n+1}\right\rangle\\&=\sum_{m\geq0}\frac{PB_m^{(k-1)}}{(m+1)!}\bigl\langle t^{m+1}\mid x^{n+1}\bigr\rangle=PB_n^{(k-1)}.\end{aligned}$$

As for the second way, we have

$$\begin{aligned}A&=\left\langle\frac{(e^t-1)Li_k(1-e^{-t})}{e^t-1}\;\middle|\;x^{n+1}\right\rangle=\left\langle\frac{Li_k(1-e^{-t})}{e^t-1}\;\middle|\;\bigl(e^t-1\bigr)x^{n+1}\right\rangle\\&=\left\langle\frac{Li_k(1-e^{-t})}{e^t-1}\;\middle|\;(x+1)^{n+1}-x^{n+1}\right\rangle=\sum_{m=0}^{n}\binom{n+1}{m}\left\langle\frac{Li_k(1-e^{-t})}{e^t-1}\;\middle|\;x^m\right\rangle\\&=\sum_{m=0}^{n}\binom{n+1}{m}PB_m^{(k)}.\end{aligned}$$

As for the third way, by (2.6), we have

$$A=(n+1)!\sum_{j_1+\cdots+j_{k-1}=n}\prod_{i=1}^{k-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.$$

Hence, we can state the following result.

Theorem 2.7 *For all* $n\geq0$,

$$PB_n^{(k-1)}=\sum_{m=0}^{n}\binom{n+1}{m}PB_m^{(k)}=(n+1)!\sum_{j_1+\cdots+j_{k-1}=n}\prod_{i=1}^{k-1}\frac{B_{j_i}}{j_i!(j_1+\cdots+j_i+1)}.$$

3 Recurrences

In this section, we present several recurrences for the degenerate poly-Bernoulli polynomials, namely $P\beta_n^{(k)}(\lambda,x)$. Note that, by (1.9) and the fact that $(x \mid \lambda)_n \sim (1, \frac{e^{\lambda t}-1}{\lambda})$, we obtain the following identity.

Proposition 3.1 *For all* $n \geq 0$, $P\beta_n^{(k)}(\lambda, x+y) = \sum_{j=0}^{n} \binom{n}{j} P\beta_j^{(k)}(\lambda,x)(y \mid \lambda)_{n-j}$.

It is well known that if $s_n(x) \sim (g(t), f(t))$, then we have $f(t)s_n(x) = ns_{n-1}(x)$. Thus, by (1.9), we obtain $\frac{e^{\lambda t}-1}{\lambda} P\beta_n^{(k)}(\lambda,x) = nP\beta_{n-1}^{(k)}(\lambda,x)$, which implies the following result.

Proposition 3.2 *For all* $n \geq 0$, $P\beta_n^{(k)}(\lambda, x+\lambda) = P\beta_n^{(k)}(\lambda,x) + n\lambda P\beta_{n-1}^{(k)}(\lambda,x)$.

Theorem 3.3 *For all* $n \geq 0$,

$$\begin{aligned} &P\beta_{n+1}^{(k)}(\lambda,x) - xP\beta_n^{(k)}(\lambda,x-\lambda) \\ &\quad = \sum_{i=0}^{m+1} \sum_{\ell=0}^{m+1-i} \binom{m+1}{i} \lambda^{m+1-i-\ell} S_2(m+1-i,\ell)\bigl(PB_\ell^{(k)} B_i(x) - P\beta_\ell^{(k)}(\lambda) B_i(x+1-\lambda)\bigr). \end{aligned}$$

Proof By applying the fact that $s_{n+1}(x) = (x - \frac{g'(t)}{g(t)})\frac{1}{f'(t)} s_n(x)$ for all $s_n(x) \sim (g(t), f(t))$ and (1.9), we obtain

$$P\beta_{n+1}^{(k)}(\lambda,x) = \left(x - \frac{g'(t)}{g(t)}\right) e^{-\lambda t} P\beta_n^{(k)}(\lambda,x) = xP\beta_n^{(k)}(\lambda,x-\lambda) - e^{-\lambda t}\frac{g'(t)}{g(t)} P\beta_n^{(k)}(\lambda,x),$$

where

$$\begin{aligned} \frac{g'(t)}{g(t)} &= \bigl(\log\bigl(e^t - 1\bigr) - \log Li_k\bigl(1 - e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}\bigr)\bigr)' \\ &= \frac{e^t}{e^t - 1} - \frac{1}{Li_k(1 - e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})} \frac{Li_{k-1}(1 - e^{-\frac{1}{\lambda}(e^{\lambda t}-1)})}{1 - e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}} e^{\lambda t} e^{-\frac{1}{\lambda}(e^{\lambda t}-1)}. \end{aligned}$$

Thus, the expression $A = e^{-\lambda t}\frac{g'(t)}{g(t)} P\beta_n^{(k)}(\lambda,x)$ is given by

$$\frac{1}{t}\left(\frac{te^{(1-\lambda)t}}{e^t - 1} G_k(t)^{-1} - \frac{t}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)} - 1} G_{k-1}(t)^{-1}\right) G_k(t) P\beta_n^{(k)}(\lambda,x).$$

Note that, by (1.9), we have $G_k(x) P\beta_n^{(k)}(\lambda,x) = \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}x^m$. Therefore,

$$\begin{aligned} A &= \sum_{m=0}^{n} S_1(n,m)\lambda^{n-m} \frac{1}{t}\left(\frac{te^{(1-\lambda)t}}{e^t - 1} G_k(t)^{-1} - \frac{t}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)} - 1} G_{k-1}(t)^{-1}\right) x^m \\ &= \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1} \lambda^{n-m} \left(\frac{te^{(1-\lambda)t}}{e^t - 1} G_k(t)^{-1} - \frac{t}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)} - 1} G_{k-1}(t)^{-1}\right) x^{m+1}. \end{aligned} \tag{3.1}$$

We remark that the expression in the parentheses in (3.1) has order at least one. Now, let us simplify (3.1):

$$
\begin{aligned}
&\frac{te^{(1-\lambda)t}}{e^t-1}G_k(t)^{-1}x^{m+1}\\
&\quad=\frac{te^{(1-\lambda)t}}{e^t-1}\frac{Li_k(1-e^{-s})}{(1+\lambda s)^{1/\lambda}-1}\bigg|_{s=\frac{e^{\lambda t}-1}{\lambda}}x^{m+1}\\
&\quad=\frac{te^{(1-\lambda)t}}{e^t-1}\sum_{\ell=0}^{m+1}P\beta_\ell^{(k)}(\lambda)\frac{(\frac{e^{\lambda t}-1}{\lambda})^\ell}{\ell!}x^{m+1}\\
&\quad=\frac{te^{(1-\lambda)t}}{e^t-1}\sum_{\ell=0}^{m+1}\sum_{i=\ell}^{m+1}\binom{m+1}{i}\lambda^{i-\ell}S_2(i,\ell)P\beta_\ell^{(k)}(\lambda)x^{m+1-i}\\
&\quad=\sum_{i=0}^{m+1}\sum_{\ell=0}^{m+1-i}\binom{m+1}{i}\lambda^{m+1-i-\ell}S_2(m+1-i,\ell)P\beta_\ell^{(k)}(\lambda)\frac{te^{(1-\lambda)t}}{e^t-1}x^i\\
&\quad=\sum_{i=0}^{m+1}\sum_{\ell=0}^{m+1-i}\binom{m+1}{i}\lambda^{m+1-i-\ell}S_2(m+1-i,\ell)P\beta_\ell^{(k)}(\lambda)B_i(x+1-\lambda)
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
&\frac{t}{e^{\frac{1}{\lambda}(e^{\lambda t}-1)}-1}G_{k-1}(t)^{-1}x^{m+1}\\
&\quad=\frac{t}{e^t-1}\frac{Li_{k-1}(1-e^{-s})}{e^s-1}\bigg|_{s=\frac{e^{\lambda t}-1}{\lambda}}x^{m+1}\\
&\quad=\frac{t}{e^t-1}\sum_{\ell=0}^{m+1}PB_\ell^{(k)}\frac{(\frac{e^{\lambda t}-1}{\lambda})^\ell}{\ell!}x^{m+1}\\
&\quad=\frac{t}{e^t-1}\sum_{\ell=0}^{m+1}\sum_{i=\ell}^{m+1}\binom{m+1}{i}\lambda^{i-\ell}S_2(i,\ell)PB_\ell^{(k)}x^{m+1-i}\\
&\quad=\sum_{i=0}^{m+1}\sum_{\ell=0}^{m+1-i}\binom{m+1}{i}\lambda^{m+1-i-\ell}S_2(m+1-i,\ell)PB_\ell^{(k)}\frac{t}{e^t-1}x^i\\
&\quad=\sum_{i=0}^{m+1}\sum_{\ell=0}^{m+1-i}\binom{m+1}{i}\lambda^{m+1-i-\ell}S_2(m+1-i,\ell)PB_\ell^{(k)}B_i(x).
\end{aligned}
\tag{3.3}
$$

Hence, by (3.1)-(3.3), we complete the proof. □

In the next result we express $\frac{d}{dx}P\beta_n^{(k)}(\lambda,x)$ in terms of $P\beta_n^{(k)}(\lambda,x)$.

Proposition 3.4 *For all* $n\geq 0$, $\frac{d}{dx}P\beta_n^{(k)}(\lambda,x)=n!\sum_{\ell=0}^{n-1}\frac{(-\lambda)^{n-\ell-1}}{\ell!(n-\ell)}P\beta_\ell^{(k)}(\lambda,x)$.

Proof Note that $\frac{d}{dx}s_n(x)=\sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle\bar{f}(t)\mid x^{n-\ell}\rangle s_\ell(x)$ for all $s_n(x)\sim(g(t),f(t))$. Thus, by (1.9), we have

$$
\begin{aligned}
\langle\bar{f}(t)\mid x^{n-\ell}\rangle&=\left\langle\frac{1}{\lambda}\log(1+\lambda t)\;\Big|\;x^{n-\ell}\right\rangle=\frac{1}{\lambda}\sum_{m\geq 1}(-1)^{m-1}\lambda^m(m-1)!\left\langle\frac{x^m}{m!}\;\Big|\;x^{n-\ell}\right\rangle\\
&=(-\lambda)^{n-\ell-1}(n-\ell-1)!,
\end{aligned}
$$

which completes the proof. □

Theorem 3.5 *For all $n \geq 1$,*

$$\begin{aligned} &P\beta_n^{(k)}(\lambda, x) - xP\beta_{n-1}^{(k)}(\lambda, x-\lambda) \\ &\quad = \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}\bigl(P\beta_m^{(k-1)}(\lambda, x)B_{n-m} - P\beta_m^{(k)}(\lambda, x+1-\lambda)\beta_{n-m}(\lambda)\bigr). \end{aligned}$$

Proof By (1.9), we have

$$\begin{aligned} P\beta_n^{(k)}(\lambda, y) &= \left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{y/\lambda} \,\middle|\, x^n \right\rangle \\ &= \left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}\frac{d}{dt}(1+\lambda t)^{y/\lambda} \,\middle|\, x^{n-1} \right\rangle \qquad (3.4) \\ &\quad + \left\langle \frac{d}{dt}\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{y/\lambda} \,\middle|\, x^{n-1} \right\rangle. \qquad (3.5) \end{aligned}$$

The term in (3.4) is given by

$$y\left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{(y-\lambda)/\lambda} \,\middle|\, x^{n-1} \right\rangle = yP\beta_{n-1}^{(k)}(\lambda, y-\lambda). \qquad (3.6)$$

For the term in (3.5), we observe that $\frac{d}{dt}\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1} = \frac{1}{t}(A-B)$, where

$$A = \frac{t}{e^t-1}\frac{Li_{k-1}(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}, \qquad B = \frac{t}{(1+\lambda t)^{1/\lambda}-1}\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{1/\lambda-1}.$$

Note that the expression $A - B$ has order of at least 1. Now, we are ready to compute the term in (3.5). By (1.9), we have

$$\begin{aligned} &\left\langle \frac{d}{dt}\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}(1+\lambda t)^{y/\lambda} \,\middle|\, x^{n-1} \right\rangle \\ &\quad = \left\langle \frac{1}{t}(A-B)(1+\lambda t)^{y/\lambda} \,\middle|\, x^{n-1} \right\rangle \\ &\quad = \frac{1}{n}\bigl\langle A(1+\lambda t)^{y/\lambda} \mid x^n \bigr\rangle - \frac{1}{n}\bigl\langle B(1+\lambda t)^{y/\lambda} \mid x^n \bigr\rangle \\ &\quad = \frac{1}{n}\left\langle \frac{t}{e^t-1} \,\middle|\, \sum_{m\geq 0} P\beta_m^{(k-1)}(\lambda, y)\frac{t^m}{m!}x^n \right\rangle \\ &\qquad - \frac{1}{n}\left\langle \frac{t}{(1+\lambda t)^{1/\lambda}-1} \,\middle|\, \sum_{m\geq 0} P\beta_m^{(k)}(\lambda, y+1-\lambda)\frac{t^m}{m!}x^n \right\rangle \\ &\quad = \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}P\beta_m^{(k-1)}(\lambda, y)\left\langle \frac{t}{e^t-1} \,\middle|\, x^{n-m} \right\rangle \\ &\qquad - \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}P\beta_m^{(k)}(\lambda, y+1-\lambda)\left\langle \frac{t}{(1+\lambda t)^{1/\lambda}-1} \,\middle|\, x^{n-m} \right\rangle \\ &\quad = \frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}\bigl(P\beta_m^{(k-1)}(\lambda, y)B_{n-m} - P\beta_m^{(k)}(\lambda, y+1-\lambda)\beta_{n-m}(\lambda)\bigr). \qquad (3.7) \end{aligned}$$

Thus, if we replace (3.4) by (3.6) and (3.5) by (3.7), we obtain

$$\begin{aligned}&P\beta_n^{(k)}(\lambda,x)-xP\beta_{n-1}^{(k)}(\lambda,x-\lambda)\\&\quad=\frac{1}{n}\sum_{m=0}^{n}\binom{n}{m}\bigl(P\beta_m^{(k-1)}(\lambda,x)B_{n-m}-P\beta_m^{(k)}(\lambda,x+1-\lambda)\beta_{n-m}(\lambda)\bigr),\end{aligned}$$

as claimed. □

4 Connections with families of polynomials

In this section, we present a few examples on the connections with families of polynomials. We start with the connection to *Bernoulli polynomials* $B_n^{(s)}(x)$ *of order* s. Recall that the *Bernoulli polynomials* $B_n^{(s)}(x)$ *of order* s are defined by the generating function $(\frac{t}{e^t-1})^s e^{xt}=\sum_{n\geq0}B_n^{(s)}(x)\frac{t^n}{n!}$, equivalently,

$$B_n^{(s)}(x)\sim\left(\left(\frac{e^t-1}{t}\right)^s,t\right) \tag{4.1}$$

(see [11–13]). In the next result, we express our polynomials $P\beta_n^{(k)}(\lambda,x)$ in terms of *Bernoulli polynomials of order* s. To do that, we recall that the Bernoulli numbers $b_n^{(s)}$ of the second kind of order s are defined as

$$\frac{t^s}{\log^s(1+t)}=\sum_{n\geq0}b_n^{(s)}\frac{t^n}{n!}. \tag{4.2}$$

Theorem 4.1 *For all* $n\geq0$,

$$P\beta_n^{(k)}(\lambda,x)=\sum_{m=0}^{n}\left(\sum_{\ell=m}^{n}\sum_{r=0}^{n-\ell}\sum_{j=0}^{n-\ell-r}\sum_{i=0}^{j}\frac{\binom{n}{\ell,r,j,n-\ell-r-j}}{\binom{j+s}{j}}\lambda^{\ell+r+i-m}c_{n,m}(\ell,r,j,i)\right)B_m^{(s)}(x),$$

where $c_{n,m}(\ell,r,j,i)=S_1(\ell,m)S_1(j+s,j-i+s)S_2(j-i+s,s)b_r^{(s)}P\beta_{n-\ell-r-j}^{(k)}(\lambda)$ *and* $\binom{a}{b_1,\ldots,b_m}=\frac{a!}{b_1!\cdots b_m!}$ *is the multinomial coefficient.*

Proof Let $h_s(t)=(\frac{(1+\lambda t)^{1/\lambda}-1}{t})^s$ and $P\beta_n^{(k)}(\lambda,x)=\sum_{m=0}^{n}c_{n,m}B_m^{(s)}(x)$. By (1.8), (1.9), and (4.1), we have

$$\begin{aligned}&m!\lambda^m c_{n,m}\\&\quad=\left\langle\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}\left(\frac{(1+\lambda t)^{1/\lambda}-1}{t}\right)^s\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^s\,\Big|\,\bigl(\log(1+\lambda t)\bigr)^m x^n\right\rangle,\end{aligned}$$

which, by (4.2), implies

$$\begin{aligned}\lambda^m c_{n,m}&=\sum_{\ell=m}^{n}\binom{n}{\ell}\lambda^\ell S_1(\ell,m)\left\langle\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}h_s(t)\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^s\,\Big|\,x^{n-\ell}\right\rangle\\&=\sum_{\ell=m}^{n}\binom{n}{\ell}\lambda^\ell S_1(\ell,m)\left\langle\frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1}h_s(t)\,\Big|\,\left(\frac{\lambda t}{\log(1+\lambda t)}\right)^s x^{n-\ell}\right\rangle\end{aligned}$$

$$= \sum_{\ell=m}^{n} \sum_{r=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{r} \lambda^{\ell+r} S_1(\ell,m) b_r^{(s)} \left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1} h_s(t) \;\middle|\; x^{n-\ell-r} \right\rangle$$

$$= \sum_{\ell=m}^{n} \sum_{r=0}^{n-\ell} \binom{n}{\ell} \binom{n-\ell}{r} \lambda^{\ell+r} S_1(\ell,m) b_r^{(s)} \left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1} \;\middle|\; h_s(t) x^{n-\ell-r} \right\rangle.$$

One can show that

$$h_s(t) = \left(\frac{e^{\frac{1}{\lambda}\log(1+\lambda t)} - 1}{t} \right)^s$$

$$= s! \sum_{j\geq 0} \sum_{i=0}^{j} S_1(j+s, j-i+s) S_2(j-i+s, s) \frac{\lambda^i}{(j+s)!} t^j.$$

Thus, by (1.9), we have

$$c_{n,m} = \sum_{\ell=m}^{n} \sum_{r=0}^{n-\ell} \sum_{j=0}^{n-\ell-r} \sum_{i=0}^{j} \Bigg(s! \binom{n}{\ell} \binom{n-\ell}{r} \lambda^{\ell+r-m} S_1(\ell,m) b_r^{(s)} S_1(j+s, j-i+s)$$

$$\times S_2(j-i+s,s) \frac{\lambda^i}{(j+s)!} (n-\ell-r)_j \left\langle \frac{Li_k(1-e^{-t})}{(1+\lambda t)^{1/\lambda}-1} \;\middle|\; x^{n-\ell-r-j} \right\rangle \Bigg)$$

$$= \sum_{\ell=m}^{n} \sum_{r=0}^{n-\ell} \sum_{j=0}^{n-\ell-r} \sum_{i=0}^{j} \Bigg(\frac{\binom{n}{\ell, r, j, n-\ell-r-j}}{\binom{j+s}{j}} \lambda^{\ell+r+i-m} S_1(\ell,m) S_1(j+s, j-i+s)$$

$$\times S_2(j-i+s,s) b_r^{(s)} P\beta_{n-\ell-r-j}^{(k)}(\lambda) \Bigg),$$

as required. □

Similar techniques as in the proof of the previous theorem, we can express our polynomials $P\beta_n^{(k)}(\lambda,x)$ in terms of other families. Below we present three examples, where we leave the proofs to the interested reader.

The first example is to express our polynomials $P\beta_n^{(k)}(\lambda,x)$ in terms of Frobenius-Euler polynomials. Note that the *Frobenius-Euler polynomials* $H_n^{(s)}(x \mid \mu)$ of order s are defined by the generating function $(\frac{1-\mu}{e^t-\mu})^s e^{xt} = \sum_{n\geq 0} H_n^{(s)}(x \mid \mu) \frac{t^n}{n!}$ $(\mu \neq 1)$, or equivalently, $H_n^{(s)}(x \mid \mu) \sim ((\frac{e^t-\mu}{1-\mu})^s, t)$ (see [10, 14]).

Theorem 4.2 *For all* $n \geq 0$,

$$P\beta_n^{(k)}(\lambda,x) = \sum_{m=0}^{n} \left(\sum_{\ell=m}^{n} \sum_{r=0}^{n-\ell} \sum_{i=0}^{s} \binom{n}{\ell} \binom{n-\ell}{r} \binom{s}{i} \frac{\lambda^{\ell-m}(-\mu)^{s-i}}{(1-\mu)^s} c_{n,m}(\ell,r,i) \right) H_m^{(s)}(x \mid \mu),$$

where $c_{n,m}(\ell,r,i) = S_1(\ell,m)(i \mid \lambda)_{n-\ell-r} P\beta_r^{(k)}(\lambda)$.

If we express our polynomials $P\beta_n^{(k)}(\lambda,x)$ in terms of *falling polynomials* $(x \mid \lambda)_n$, then we get the following result.

Theorem 4.3 *For all* $n \geq 0$, $P\beta_n^{(k)}(\lambda,x) = \sum_{m=0}^{n} \binom{n}{m} P\beta_{n-m}^{(k)}(\lambda)(x \mid \lambda)_m$.

Our last example is to express our polynomials $P\beta_n^{(k)}(\lambda,x)$ in terms of *degenerate Bernoulli polynomials* $\beta_n^{(s)}(\lambda,x)$ of order s. Note that the degenerate Bernoulli polynomials $\beta_n^{(s)}(\lambda,x)$ of order s are given by

$$\left(\frac{t}{(1+\lambda t)^{1/\lambda}-1}\right)^s(1+\lambda t)^{x/\lambda}=\sum_{n\geq 0}\beta_n^{(s)}(\lambda,x)\frac{t^n}{n!}.$$

Theorem 4.4 *For all $n\geq 0$,*

$$P\beta_n^{(k)}(\lambda,x)=\sum_{m=0}^{n}\binom{n}{m}\left(\sum_{j=0}^{n-m}\sum_{i=0}^{j}\frac{\binom{n-m}{j}}{\binom{j+s}{s}}\lambda^i c_{n,m}(j,i)\right)\beta_m^{(s)}(\lambda,x),$$

where $c_{n,m}(j,i)=S_1(j+s,j-i+s)S_2(j-i+s,s)P\beta_{n-m-j}^{(k)}(\lambda)$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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