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Oscillation and variation inequalities for the commutators of singular integrals with Lipschitz functions

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Abstract

This paper is devoted to investigating the boundedness of the oscillation and variation operators for the commutators generated by Calderón-Zygmund singular integrals with Lipschitz functions in the weighted Lebesgue spaces and the endpoint spaces in dimension 1. Certain criterions of boundedness are given. As applications, the weighted (L^p, L^q)-estimates for the oscillation and variation operators on the iterated commutators of Hilbert transform and Hermitian Riesz transform, the $(L^p, \dot{\Lambda}_{(\beta-1/p)})$-bounds as well as the endpoint estimates for the oscillation and variation operators of the corresponding first order commutators are established.

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1 Introduction

Let $\mathcal{T} = \{T_\varepsilon\}_\varepsilon$ be a family of operators such that the limit $\lim_{\varepsilon\to 0} T_\varepsilon f(x) = Tf(x)$ exists in some sense. A classical method of measuring the speed of convergence of the family $\{T_\varepsilon\}$ is to consider 'square function' of the type $(\sum_{i=1}^\infty |T_{\varepsilon_i} f - T_{\varepsilon_{i+1}} f|^2)^{1/2}$, where $\varepsilon_i \searrow 0$. Or, more generally, the oscillation operator defined as

$$\mathcal{O}(\mathcal{T}f)(x) = \left(\sum_{i=1}^\infty \sup_{t_{i+1}\le\varepsilon_{i+1}<\varepsilon_i\le t_i} \left|T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)\right|^2\right)^{1/2}$$

with $\{t_i\}$ being a fixed sequence decreasing to zero, and the ρ-variation operator defined by

$$\mathcal{V}_\rho(\mathcal{T}f)(x) = \sup_{\varepsilon\searrow 0}\left(\sum_{i=1}^\infty \left|T_{\varepsilon_{i+1}} f(x) - T_{\varepsilon_i} f(x)\right|^\rho\right)^{1/\rho},$$

where the sup is taken over all sequence $\{\varepsilon_i\}$ decreasing to zero. We also consider the operator

$$\mathcal{O}'(\mathcal{T}f)(x) = \left(\sum_{i=1}^\infty \sup_{t_{i+1}<\delta_i<t_i} \left|T_{t_{i+1}} f(x) - T_{\delta_i} f(x)\right|^2\right)^{1/2}.$$

It is easy to check that

$$\mathcal{O}'(\mathcal{T}f)(x) \leq \mathcal{O}(\mathcal{T}f)(x) \leq 2\mathcal{O}'(\mathcal{T}f)(x). \tag{1.1}$$

The oscillation and variation for martingales and some families of operators have been studied in many resent papers on probability, ergodic theory, and harmonic analysis. We refer the readers to [1–4] and the references therein for more background information. Recently, Liu and Wu [5] gave a criterion on the weighted norm estimate of the oscillation and variation operators for the commutators of Calderón-Zygmund singular integrals with BMO functions in dimension 1. We also point out that the L^p-boundedness for the higher order commutators of singular integrals was obtained by Segovia and Torrea [6] in 1993. The purpose of this paper is to establish some new results concerning the oscillation and ρ-variation operators for the families of commutators generated by Calderón-Zygmund singular integrals with Lipschitz functions. Precisely, we will establish a criterion on the weighted (L^p, L^q)-type estimates of the oscillation and ρ-variation operators for the iterated commutators of Calderón-Zygmund singular integrals with Lipschitz functions for $0 < \beta < 1$ and $1 < p < 1/\beta$ with $1/q = 1/p - \beta$. We will also consider the boundedness of $(L^p, \dot{\wedge}_{(\beta-1/p)})$ type for the corresponding operators related to the first order commutator for $1/\beta < p < \infty$, and the endpoint cases, namely $p = 1/\beta$ or $p = \infty$. As applications, the corresponding boundedness of the oscillation and variation operators for the commutators of Hilbert transform and the Hermitian Riesz transforms will be given.

Before stating our main results, we recall some definitions and notations. Let $K(x,y)$ be the standard kernels with constants δ and A, that is, $K(x,y)$ is defined on $\mathbf{R} \times \mathbf{R} \setminus \{(x,x) : x \in \mathbf{R}\}$ and satisfies the size condition for some $A > 0$

$$\left|K(x,y)\right| \leq \frac{A}{|x-y|}; \tag{1.2}$$

and the regularity conditions, for some $\delta > 0$,

$$\left|K(x,y) - K\left(x',y\right)\right| \leq \frac{A|x-x'|^{\delta}}{(|x-y| + |x'-y|)^{1+\delta}}, \tag{1.3}$$

whenever $2|x-x'| \leq \max(|x-y|, |x'-y|)$ and

$$\left|K(x,y) - K\left(x,y'\right)\right| \leq \frac{A|y-y'|^{\delta}}{(|x-y| + |x-y'|)^{1+\delta}}, \tag{1.4}$$

whenever $2|y-y'| \leq \max(|x-y|, |x-y'|)$. The class of all standard kernels with constants δ and A is denoted by $\mathrm{SK}(\delta, A)$. For a locally integrable function b defined in $\mathbf{R}$, we say b belongs to the space Lip_{β}^{p} for $1 \leq p \leq \infty$, $0 < \beta < 1$, if there is a constant $C > 0$ such that

$$\sup_{I \ni x} \frac{1}{|I|^{\beta}} \left(\frac{1}{|I|} \int_I \left|b(x) - b_I\right|^p dx\right)^{1/p} \leq C. \tag{1.5}$$

The smallest bound C satisfying (1.5) is taken to be the norm of b denoted by $\|b\|_{Lip_{\beta}^{p}}$. Here I is an interval in $\mathbf{R}$ and $b_I = |I|^{-1} \int_I b(x)\,dx$.

Obviously, for the case $p = 1$, Lip_β^p is the homogeneous Lipschitz space $\dot{\wedge}_\beta$. García-Cuerva [7] proved that the spaces Lip_β^p coincide, and the norms of $\|\cdot\|_{Lip_\beta^p}$ are equivalent with respect to different values of p provided that $1 \leq p \leq \infty$. For $m \in \mathbf{N}$, $\vec{b} = (b_1, b_2, \dots, b_m) \in \dot{\wedge}_{\vec{\beta}}$, which means that $b_i \in \dot{\wedge}_{\beta_i}$ $(i = 1, \dots, m)$ with $\vec{\beta} = (\beta_1, \dots, \beta_m)$ and $0 < \beta = \beta_1 + \cdots + \beta_m < 1$, we consider the family of operators $\mathcal{T} := \{T_\varepsilon\}_{\varepsilon>0}$ given by

$$T_\varepsilon(f)(x) := \int_{|x-y|>\varepsilon} K(x,y) f(y)\, dy, \tag{1.6}$$

and $\mathcal{T}_{\vec{b}} := \{T_{\varepsilon,\vec{b}}\}_{\varepsilon>0}$, where $T_{\varepsilon,\vec{b}}$ is the iterated commutators T_ε and $\vec{b}$, which is given by

$$[b_1, T_\varepsilon](f)(x) = b_1(x) T_\varepsilon(f)(x) - T_\varepsilon(b_1 f)(x) = \int_{|x-y|>\varepsilon} \bigl[b_1(x) - b_1(y)\bigr] K(x,y) f(y)\, dy \tag{1.7}$$

for $m = 1$, and

$$T_{\varepsilon,\vec{b}}(f)(x) = \bigl[b_m, \dots, \bigl[b_2, [b_1, T_\varepsilon]\bigr]\bigr](f)(x) = \int_{|x-y|>\varepsilon} \prod_{j=1}^{m} \bigl[b_j(x) - b_j(y)\bigr] K(x,y) f(y)\, dy \tag{1.8}$$

for $f \in \bigcup_{1\leq p<\infty} L^p(\mathbf{R})$. When $m = 1$, we also denote $\vec{b}$ by b, $T_{\varepsilon,\vec{b}}$ by $T_{\varepsilon,b}$, and $\mathcal{T}_{\vec{b}}$ by $\mathcal{T}_b$.

In this paper, we will study the behaviors of oscillation and variation operators for the families of commutators defined by (1.7) and (1.8) in Lebesgue spaces. Our main results can be formulated as follows.

Theorem 1.1 *Suppose that $K(x,y)$ satisfies* (1.2)-(1.4), $\vec{b} \in \dot{\wedge}_{\vec{\beta}}$, $0 < \beta = \beta_1 + \cdots + \beta_m < 1$. *Let $\rho > 2$, $\mathcal{T} = \{T_\varepsilon\}_{\varepsilon>0}$ and $\mathcal{T}_{\vec{b}} = \{T_{\varepsilon,\vec{b}}\}_{\varepsilon>0}$ be given by* (1.6) *and* (1.8), *respectively. If $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$ are bounded in $L^{p_0}(\mathbf{R}, dx)$ for some $1 < p_0 < \infty$, then for any $1 < p < 1/\beta$ with $1/q = 1/p - \beta$, $\omega \in A_{(p,q)}$ (the Muckenhoupt classes of fractional type, see the definition below), $\mathcal{O}(\mathcal{T}_{\vec{b}})$ and $\mathcal{V}_\rho(\mathcal{T}_{\vec{b}})$ are bounded from $L^p(\mathbf{R}, \omega(x)^p\, dx)$ to $L^q(\mathbf{R}, \omega(x)^q\, dx)$.*

For $1/\beta \leq p \leq \infty$, we can establish the following un-weighted results only for the oscillation and variation operators related to the first order commutator.

Theorem 1.2 *Suppose that $K(x,y)$ satisfies* (1.2)-(1.4), $b \in \dot{\wedge}_\beta$, $0 < \beta \leq \delta < 1$, *where δ is the same as in* (1.3). *Let $\rho > 2$, $\mathcal{T} = \{T_\varepsilon\}_{\varepsilon>0}$ and $\mathcal{T}_b = \{T_{\varepsilon,b}\}_{\varepsilon>0}$ be given by* (1.6) *and* (1.7), *respectively. If $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$ are bounded in $L^{p_0}(\mathbf{R}, dx)$ for some $1 < p_0 < \infty$, then for any $1/\beta < p < \infty$, there exists a constant $C > 0$ such that for all bounded functions f with compact support,*

$$\bigl\|\mathcal{O}(\mathcal{T}_b)(f)\bigr\|_{\dot{\wedge}_{(\beta-1/p)}} \leq C \|b\|_{\dot{\wedge}_\beta} \|f\|_{L^p}$$

and

$$\bigl\|\mathcal{V}_\rho(\mathcal{T}_b)(f)\bigr\|_{\dot{\wedge}_{(\beta-1/p)}} \leq C \|b\|_{\dot{\wedge}_\beta} \|f\|_{L^p}.$$

Theorem 1.3 *Suppose that $K(x,y)$ satisfies* (1.2)-(1.4), $b \in \dot{\wedge}_\beta$, $0 < \beta \leq \delta < 1$, *where δ is the same as in* (1.3). *Let $\rho > 2$, $\mathcal{T} = \{T_\varepsilon\}_{\varepsilon>0}$ and $\mathcal{T}_b = \{T_{\varepsilon,b}\}_{\varepsilon>0}$ be given by* (1.6) *and* (1.7), *respectively. If $\mathcal{O}(\mathcal{T})$ and $\mathcal{V}_\rho(\mathcal{T})$ are bounded in $L^{p_0}(\mathbf{R},dx)$ for some $1 < p_0 < \infty$, then for $p = 1/\beta$, there exists a constant $C > 0$ such that for all bounded functions f with compact support,*

$$\|\mathcal{O}(\mathcal{T}_b)(f)\|_{BMO} \le C\|b\|_{\dot{\wedge}_\beta}\|f\|_{L^{1/\beta}}$$

and

$$\|\mathcal{V}_\rho(\mathcal{T}_b)(f)\|_{BMO} \le C\|b\|_{\dot{\wedge}_\beta}\|f\|_{L^{1/\beta}}.$$

Remark 1.1 We remark that our arguments in proving Theorems 1.2 and 1.3 do not work for the cases of high order commutators $\mathcal{T}_{\vec{b}}$ ($m > 1$). It is not clear whether the corresponding results for $\mathcal{O}(\mathcal{T}_{\vec{b}})$ and $\mathcal{V}_\rho(\mathcal{T}_{\vec{b}})$ for $m > 1$ also hold, which is very interesting. We also remark that in our theorems, we deal only with $\rho > 2$ for the variation operators, since in the case $\rho \le 2$ the variation is often not bounded (see [1, 8]).

The rest of this paper is organized as follows. In Section 2, we will recall some basic facts concerning weights, maximal functions, sharp maximal functions and characterization of the space $\dot{\wedge}_\beta$. The weighted (L^p, L^q)-type estimates of the oscillation and variation operators for the iterated commutators will be given in Section 3. In Section 4, we will show the $(L^p, \dot{\wedge}_{(\beta-1/p)})$-bounds of the oscillation and variation operators for the first order commutator $\mathcal{T}_b$ in the cases $1/\beta < p < \infty$ and the endpoint. Finally, as applications, the corresponding results of the oscillation and variation operators related to the commutators of Hilbert transform and Hermitian Riesz transforms as well as the λ-jump operators and the number of up-crossing for these operators will be obtained in Section 5. We remark that our works and ideas are greatly motivated by [3, 9].

Throughout the rest of the paper, $C > 0$ always denotes a constant that is independent of main parameters involved but whose value may differ from line to line. For any index $p \in [1,\infty]$, we denote by p' its conjugate index, namely $1/p + 1/p' = 1$.

2 Preliminaries

2.1 Weights

By a weight we mean a non-negative measurable function. We recall that a weight ω belongs to the class A_p, $1 < p < \infty$, if

$$\sup_I \left(\frac{1}{|I|}\int_I \omega(y)\,dy\right)\left(\frac{1}{|I|}\int_I \omega(y)^{1-p'}\,dy\right)^{p-1} < \infty,$$

where I denotes the term in $\mathbf{R}$, $p' = p/(p-1)$. This number is called the A_p constant of ω and is denoted by $[\omega]_{A_p}$. A weight ω belongs to the class A_1 if there is a constant C such that

$$\frac{1}{|I|}\int_I \omega(y)\,dy \le C \inf_{y\in I}\omega(y)$$

and the infimum of this constant C is called the A_1 constant of ω and is denoted by $[\omega]_{A_1}$. Since the A_p classes are increasing with respect to p, the A_∞ class of weights is defined in a natural way by $A_\infty = \bigcup_{p\geq 1} A_p$ and the A_∞ constant of $\omega \in A_\infty$ is the smallest of the infimum of the A_p constant such that $\omega \in A_p$.

A weight $\omega(x)$ is said to belong to the class $A_{(p,q)}$, $1 < p \leq q < \infty$, if

$$\sup_I \left(\frac{1}{|I|} \int_I \omega(y)^q \, dy \right)^{1/q} \left(\frac{1}{|I|} \int_I \omega(y)^{-p'} \, dy \right)^{1/p'} < \infty.$$

It is well known that

$$w \in A_{(p,q)} \iff w^q \in A_{q(1-\alpha)} \iff w^{-p'} \in A_{p'(1-\alpha)} \iff w^q \in A_s, \tag{2.1}$$

$$\begin{aligned} w \in A_{(p,q)} &\implies w^q \in A_q \quad \text{and} \quad w^p \in A_p \\ &\iff w^q \in A_q \quad \text{and} \quad w^{-p'} \in A_{p'}, \end{aligned} \tag{2.2}$$

where $0 < \alpha < 1$, $1 \leq p < 1/\alpha$, $1/q = 1/p - \alpha$ and $s = 1 + q/p'$. The following result, which can be found in Theorem 3.10 of [10], will be used below.

Lemma 2.1 ([10]) *Let* $1 < p \leq q < \infty$. *If* $\omega \in A_{(p,q)}$, *then there exists* $r \in (1, p)$ *such that* $w^r \in A_{(p/r,q/r)}$.

2.2 Maximal functions and sharp maximal functions

We recall the definitions of the Hardy-Littlewood maximal function

$$M(f)(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \, dy$$

and the sharp maximal function

$$M^\sharp(f)(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y) - f_I| \, dy \approx \sup_{I \ni x} \inf_c \frac{1}{|I|} \int_I |f(y) - c| \, dy, \tag{2.3}$$

where $f_I = |I|^{-1} \int_I f(y) \, dy$. A well-known result obtained by Muckenhoupt [11] is that M is bounded on $L^p(\omega)$ if and only if $\omega \in A_p$ for $1 < p < \infty$.

Also, we denote the fractional maximal operator M_β defined by

$$M_\beta(f)(x) := \sup_{I \ni x} \frac{1}{|I|^{1-\beta}} \int_I |f(y)| \, dy,$$

and its variant $M_{\beta,r}$ defined by

$$M_{\beta,r}(f)(x) := \sup_{I \ni x} \left(\frac{1}{|I|^{1-\beta r}} \int_I |f(y)|^r \, dy \right)^{1/r}, \quad r > 0.$$

The following properties will play key roles in the proofs of our main theorems.

Lemma 2.2 ([12]) *Let* $1 < p < \infty$, $\omega \in A_\infty$. *Then*

$$\|M(f)\|_{L^p(\omega)} \leq \|M^\sharp(f)\|_{L^p(\omega)} \tag{2.4}$$

for all f *such that the left-hand side is finite.*

Lemma 2.3 ([13]) *Suppose that* $0 < \beta < 1$, $1 < p < 1/\beta$, $1/q = 1/p - \beta$. *If* $\omega \in A_{(p,q)}$, *then*

$$\left\|M_\beta(f)\right\|_{L^q(\omega^q)} \le \|f\|_{L^p(\omega^p)}.$$

Lemma 2.4 *Suppose that* $0 < \beta < 1$, $1 < r < p < 1/\beta$, $1/q = 1/p - \beta$. *If* $\omega \in A_{(p,q)}$, *then*

$$\left\|M_{\beta,r}(f)\right\|_{L^q(\omega^q)} \le \|f\|_{L^p(\omega^p)}. \tag{2.5}$$

Note that $M_{\beta,r}(f)(x) = (M_{\beta r}(|f|^r)(x))^{1/r}$. Lemma 2.4 immediately follows from Lemmas 2.1 and 2.3. We omit the details.

2.3 Characterization of the space $\dot{\wedge}_\beta$

By the definition of $\dot{\wedge}_\beta$, it is easy to check that for $f \in \dot{\wedge}_\beta$, $0 < \beta \le 1$,

$$\frac{1}{2}\|f\|_{\dot{\wedge}_\beta} \le \sup_{I \ni x} \inf_{C_I} \frac{1}{|I|^{1+\beta}} \int_I \left|f(x) - C_I\right| dx \le \|f\|_{\dot{\wedge}_\beta}. \tag{2.6}$$

3 The weighted (L^p, L^q)-type estimates

This section is devoted to the proof of Theorem 1.1. Let us begin with recalling two previous known results, which will be used below.

Lemma 3.1 ([5]) *Suppose that* $K(x,y)$ *satisfies* (1.2)-(1.4), $\rho > 2$. *Let* $\mathcal{T} = \{T_\varepsilon\}_{\varepsilon>0}$ *be given by* (1.6). *If* $\mathcal{O}(\mathcal{T})$ *and* $\mathcal{V}_\rho(\mathcal{T})$ *are bounded in* $L^{p_0}(\mathbf{R})$ *for some* $1 < p_0 < \infty$, *then for any* $1 < p < \infty$, $\omega \in A_p$,

$$\left\|\mathcal{O}'(\mathcal{T}f)\right\|_{L^p(\omega)} \le \left\|\mathcal{O}(\mathcal{T}f)\right\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)} \tag{3.1}$$

and

$$\left\|\mathcal{V}_\rho(\mathcal{T}f)\right\|_{L^p(\omega)} \le C\|f\|_{L^p(\omega)}. \tag{3.2}$$

The proof of Theorem 1.1 is based on the following sharp maximal function estimate. Before stating the result, we recall some notations. For $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, 2, \dots, m\}$ with j different elements. For any $\sigma \in C_j^m$, the complementary sequence σ' is given by $\sigma' = \{1, 2 \cdots, m\} \setminus \sigma$. For $\vec{\beta} = (\beta_1, \dots, \beta_m)$ with $\beta = \beta_1 + \cdots + \beta_m$, $\vec{b} = (b_1, \dots, b_m)$ with $b_i \in \dot{\wedge}_{\beta_i}$ $(i = 1, \dots, m)$, we denote $\vec{\beta}_\sigma = (\beta_{\sigma(1)}, \dots, \beta_{\sigma(j)})$ with $\beta_\sigma = \beta_{\sigma(1)} + \cdots + \beta_{\sigma(j)}$, $\beta_{\sigma'} = \beta - \beta_\sigma$, and $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$ with

$$\|\vec{b}\|_{\dot{\wedge}_\beta} = \prod_{i=1}^m \|b_i\|_{\dot{\wedge}_{\beta_i}} \quad \text{and} \quad \|\vec{b}_\sigma\|_{\dot{\wedge}_{\beta_\sigma}} = \prod_{i=1}^j \|b_{\sigma(i)}\|_{\dot{\wedge}_{\beta_{\sigma(i)}}}$$

for any $\sigma = \{\sigma(1), \dots, \sigma(j)\}$, $1 \le j \le m$.

Now we state our main lemma as follows.

Lemma 3.2 *Suppose that* $K(x,y)$ *satisfies* (1.2)-(1.4), $\vec{\beta} = (\beta_1, \dots, \beta_m)$ *with* $\beta = \beta_1 + \cdots + \beta_m$ *and* $0 < \beta < 1$, $\vec{b} = (b_1, \dots, b_m)$ *with* $b_i \in \dot{\wedge}_{\beta_i}$ $(i = 1, \dots, m)$. *Then for* $\rho > 2$, $\mathcal{T}$ *and* $\mathcal{T}_{\vec{b}}$ *being as* *in Theorem* 1.1, *we have*

$$M^\sharp\big(\mathcal{O}'(\mathcal{T}_{\vec{b}}f)\big)(x) \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\big\{M_{\beta,r}\big(\mathcal{O}'\big(\mathcal{T}(f)\big)\big)(x)+M_{\beta,r}(f)(x)\big\} + C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}}M_{\beta_\sigma,r}\big(\mathcal{O}'(\mathcal{T}_{\vec{b}_{\sigma'}}f)\big)(x) \tag{3.3}$$

and

$$M^\sharp\big(\mathcal{V}_\rho(\mathcal{T}_{\vec{b}}f)\big)(x) \le C\|\vec{b}\|_{\dot{\Lambda}_\beta}\big\{M_{\beta,r}\big(\mathcal{V}_\rho\big(\mathcal{T}(f)\big)\big)(x)+M_{\beta,r}(f)(x)\big\} + C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec{b}_\sigma\|_{\dot{\Lambda}_{\beta_\sigma}}M_{\beta_\sigma,r}\big(\mathcal{V}_\rho(\mathcal{T}_{\vec{b}_{\sigma'}}f)\big)(x) \tag{3.4}$$

hold for any $r>1$.

Before proving Lemma 3.2, we need to fix some notations. Following [3], we denote by E the mixed norm Banach space of two variable function h defined on $\mathbf{R}\times\mathbf{N}$ such that

$$\|h\|_E \equiv \bigg(\sum_i\Big(\sup_s\big|h(s,i)\big|\Big)^2\bigg)^{1/2}<\infty. \tag{3.5}$$

Given a family of operators $\mathcal{T}:=\{T_t\}_{t>0}$ defined on $L^p(\mathbf{R})$, for a fixed decreasing sequence $\{t_i\}$ with $t_i\searrow 0$, let $J_i=(t_{i+1},t_i]$ and define the operator $U(\mathcal{T}):f\to U(\mathcal{T})f$, where $U(\mathcal{T})f(x)$ is the E-valued function given by

$$U(\mathcal{T})f(x):=\big\{T_{t_{i+1}}f(x)-T_sf(x)\big\}_{s\in J_i,i\in\mathbf{N}}. \tag{3.6}$$

Here the expression $\{T_{t_{i+1}}f(x)-T_sf(x)\}_{s\in J_i,i\in\mathbf{N}}$ is a convenient abbreviation for the element of E given by

$$(s,i)\to\big(T_{t_{i+1}}f(x)-T_sf(x)\big)\chi_{J_i}(s).$$

Then

$$\mathcal{O}'(\mathcal{T}f)(x)=\big\|\big\{T_{t_{i+1}}f(x)-T_sf(x)\big\}_{s\in J_i,i\in\mathbf{N}}\big\|_E=\big\|U(\mathcal{T})f(x)\big\|_E. \tag{3.7}$$

On the other hand, let $\Theta=\{\beta:\beta=\{\varepsilon_i\},\varepsilon_i\in\mathbf{R},\varepsilon_i\searrow 0\}$. We consider the set $\mathbf{N}\times\Theta$ and denote by F_ρ the mixed norm space of two variable functions $g(i,\beta)$ such that

$$\|g\|_{F_\rho}\equiv\sup_\beta\bigg(\sum_i\big|g(i,\beta)\big|^\rho\bigg)^{1/\rho}<\infty. \tag{3.8}$$

We also consider the F_ρ-valued operator $V(\mathcal{T}):f\to V(\mathcal{T})f$ on $L^p(\mathbf{R})$ given by

$$V(\mathcal{T})f(x):=\big\{T_{\varepsilon_{i+1}}f(x)-T_{\varepsilon_i}f(x)\big\}_{\beta=\{\varepsilon_i\}\in\Theta}, \tag{3.9}$$

where $\{T_{\varepsilon_{i+1}}f(x)-T_{\varepsilon_i}f(x)\}_{\beta=\{\varepsilon_i\}\in\Theta}$ is an abbreviation for the element of F_ρ given by

$$(i,\beta)=\big(i,\{\varepsilon_k\}\big)\to T_{\varepsilon_{i+1}}f(x)-T_{\varepsilon_i}f(x).$$

This implies that

$$\mathcal{V}_\rho(\mathcal{T}f)(x) = \|V(\mathcal{T})f(x)\|_{F_\rho}. \tag{3.10}$$

Finally, if B is a Banach space and φ is a B-valued function, we define

$$\varphi^\sharp(x) := \sup_{x\in I} \frac{1}{|I|}\int_I \left\|\varphi(y) - \frac{1}{|I|}\int_I \varphi(z)\,dz\right\|_B dy. \tag{3.11}$$

This together with (2.3), (3.7) and (3.10) leads to

$$M^\sharp\big(\mathcal{O}'(\mathcal{T}f)\big)(x) \le 2\big(U(\mathcal{T})f\big)^\sharp(x) \tag{3.12}$$

and

$$M^\sharp\big(\mathcal{V}_\rho(\mathcal{T}f)\big)(x) \le 2\big(V(\mathcal{T})f\big)^\sharp(x). \tag{3.13}$$

Proof of Lemma 3.2 For simplicity and without loss of generality, we consider only the case $m = 2$. By (3.12)-(3.13), it suffices to show the following results:

$$\begin{aligned} U(\mathcal{T}_{b_1,b_2}f)^\sharp(x) &\le C\|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}\big\{M_{\beta,r}\big(\mathcal{O}'\big(\mathcal{T}(f)\big)\big)(x) + M_{\beta,r}(f)(x)\big\} \\ &\quad + C\|b_1\|_{\dot\wedge_{\beta_1}}M_{\beta_1,r}\big(\mathcal{O}'\big(\mathcal{T}_{b_2}(f)\big)(x)\big) \\ &\quad + C\|b_2\|_{\dot\wedge_{\beta_2}}M_{\beta_2,r}\big(\mathcal{O}'\big(\mathcal{T}_{b_1}(f)\big)(x)\big) \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} V(\mathcal{T}_{b_1,b_2}f)^\sharp(x) &\le C\|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}\big\{M_{\beta,r}\big(\mathcal{V}_\rho\big(\mathcal{T}(f)\big)\big)(x) + M_{\beta,r}(f)(x)\big\} \\ &\quad + C\|b_1\|_{\dot\wedge_{\beta_1}}M_{\beta_1,r}\big(\mathcal{V}_\rho\big(\mathcal{T}_{b_2}(f)\big)(x)\big) \\ &\quad + C\|b_2\|_{\dot\wedge_{\beta_2}}M_{\beta_2,r}\big(\mathcal{V}_\rho\big(\mathcal{T}_{b_1}(f)\big)(x)\big). \end{aligned} \tag{3.15}$$

We will prove only inequality (3.14) since (3.15) can be obtained by a similar argument. Fix f and x_0 with an interval $I = (x_0 - l, x_0 + l)$. Define $f_1(y) = f(y)\chi_{4I}$ and $f_2(y) = f(y) - f_1(y)$. Let

$$C_I = \left\{\int_{t_{i+1}<|x_0-y|<s}\big(b_1(y) - (b_1)_I\big)\big(b_2(y) - (b_2)_I\big)K(x_0,y)f_2(y)\,dy\right\}_{s\in J_i, i\in\mathbf{N}},$$

where $b_I = |I|^{-1}\int_I b(x)\,dx$, $(4I)^c$ denotes the complementary set of the interval $4I = (x_0 - 4l, x_0 + 4l)$. By (3.11), it suffices to prove the following inequality:

$$\begin{aligned} &\frac{1}{|I|}\int_I \big\|U(\mathcal{T}_{b_1,b_2})(f)(x) - C_I\big\|_E\,dx \\ &\quad\le C\|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}\big\{M_{\beta,r}\big(\mathcal{O}'\big(\mathcal{T}(f)\big)\big)(x_0) + M_{\beta,r}(f)(x_0)\big\} \\ &\qquad + C\|b_1\|_{\dot\wedge_{\beta_1}}M_{\beta_1,r}\big(\mathcal{O}'\big(\mathcal{T}_{b_2}(f)\big)(x_0)\big) \\ &\qquad + C\|b_2\|_{\dot\wedge_{\beta_2}}M_{\beta_2,r}\big(\mathcal{O}'\big(\mathcal{T}_{b_1}(f)\big)(x_0)\big) \end{aligned}$$

for every $x_0 \in \mathbf{R}$. Since

$$\begin{aligned}
&\frac{1}{|I|}\int_I \big\|U(\mathcal{T}_{b_1,b_2})(f)(x) - C_I\big\|_E\,dx \\
&\quad\le \frac{1}{|I|}\int_I \big\|U\big(\big(b_1(\cdot)-(b_1)_I\big)\big(b_2(\cdot)-(b_2)_I\big)\mathcal{T}f\big)(x)\big\|_E\,dx \\
&\qquad + \frac{1}{|I|}\int_I \big\|U\big(\big(b_1(\cdot)-(b_1)_I\big)\mathcal{T}_{b_2}f(x)\big)\big\|_E \\
&\qquad + \frac{1}{|I|}\int_I \big\|U\big(\big(b_2(\cdot)-(b_2)_I\big)\mathcal{T}_{b_1}f(x)\big)\big\|_E \\
&\qquad + \frac{1}{|I|}\int_I \big\|U(\mathcal{T})\big(\big(b_1-(b_1)_I\big)\big(b_2-(b_2)_I\big)f\big)(x) - C_I\big\|_E\,dx \\
&\quad =: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Now we estimate the above four terms, respectively. Firstly,

$$\begin{aligned}
I_1 &= \frac{1}{|I|}\int_I \big|b_1(x)-(b_1)_I\big|\big|b_2(x)-(b_2)_I\big|\big\|U(\mathcal{T}f)(x)\big\|_E\,dx \\
&\le \left(\frac{1}{|I|}\int_I \big|b_1(x)-(b_1)_I\big|^{2r'}\,dx\right)^{1/2r'}\left(\frac{1}{|I|}\int_I \big|b_2(x)-(b_2)_I\big|^{2r'}\,dx\right)^{1/2r'} \\
&\quad\times\left(\frac{1}{|I|}\int_I \big\|U(\mathcal{T}f)(x)\big\|_E^r\,dx\right)^{1/r} \\
&\le C\|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}} M_{\beta,r}\big(\mathcal{O}'(\mathcal{T}f)\big)(x_0).
\end{aligned}$$

As for I_2, we have

$$\begin{aligned}
I_2 &= \frac{1}{|I|}\int_I \big|b_1(x)-(b_1)_I\big|\big\|U(\mathcal{T}_{b_2}f)(x)\big\|_E\,dx \\
&\le \left(\frac{1}{|I|}\int_I \big|b_1(x)-(b_1)_I\big|^{r'}\,dx\right)^{1/r'}\left(\frac{1}{|I|}\int_I \big\|U(\mathcal{T}_{b_2}f)(x)\big\|_E^r\,dx\right)^{1/r} \\
&\le \left(\frac{1}{|I|}\int_I \big|b_1(x)-(b_1)_I\big|^{r'}\,dx\right)^{1/r'}\left(\frac{1}{|I|}\int_I \big|\mathcal{O}'(\mathcal{T}_{b_2}f)(x)\big|^r\,dx\right)^{1/r} \\
&\le C\|b_1\|_{\dot{\wedge}_{\beta_1}} M_{\beta_1,r}\big(\mathcal{O}'(\mathcal{T}_{b_2}f)\big)(x_0).
\end{aligned}$$

By symmetry, we have

$$I_3 \le C\|b_2\|_{\dot{\wedge}_{\beta_2}} M_{\beta_2,r}\big(\mathcal{O}'(\mathcal{T}_{b_1}f)\big)(x_0).$$

Finally, we deal with I_4 as follows:

$$\begin{aligned}
&\frac{1}{|I|}\int_I \big\|U(\mathcal{T})\big(\big(b_1-(b_1)_I\big)\big(b_2-(b_2)_I\big)f\big)(x) - C_I\big\|_E\,dx \\
&\quad\le \frac{1}{|I|}\int_I \big\|U(\mathcal{T})\big(\big(b_1-(b_1)_I\big)\big(b_2-(b_2)_I\big)f_1\big)(x)\big\|_E\,dx \\
&\qquad + \frac{1}{|I|}\int_I \big\|U(\mathcal{T})\big(\big(b_1-(b_1)_I\big)\big(b_2-(b_2)_I\big)f_2\big)(x) - C_I\big\|_E\,dx \\
&\quad =: E + F.
\end{aligned}$$

Invoking Lemma 3.1, we know that $\mathcal{O}'(\mathcal{T})$ is bounded on $L^t(\mathbf{R})$ for any $1 < t < \infty$. Therefore, for any $r > 1$, let $t = \sqrt{r}$, we get

$$\begin{aligned}
E &\le \left(\frac{1}{|I|}\int_I \bigl\| U(\mathcal{T})\bigl(\bigl(b_1 - (b_1)_I\bigr)\bigl(b_2 - (b_2)_I\bigr)f_1\bigr)(x)\bigr\|_E^t \, dx\right)^{1/t} \\
&\le C\left(\frac{1}{|I|}\int_{\mathbf{R}} \bigl(\bigl|b_1 - (b_1)_I\bigr|\bigl|b_2 - (b_2)_I\bigr|\bigl|f_1(x)\bigr|\bigr)^t \, dx\right)^{1/t} \\
&= C\left(\frac{1}{|I|}\int_{4I} \bigl|f(x)\bigr|^r \, dx\right)^{1/r}\left(\frac{1}{|I|}\int_{4I} \bigl(\bigl|b_1 - (b_1)_I\bigr|\bigl|b_2 - (b_2)_I\bigr|\bigr)^{tt'} \, dx\right)^{1/tt'} \\
&\le M_{\beta,r}(f)(x_0)|I|^{-\beta}\left(\frac{1}{|I|}\int_{4I} \bigl(\bigl|b_1 - (b_1)_{4I}\bigr| + \bigl|(b_1)_{4I} - (b_1)_I\bigr|\bigr)^{2tt'} \, dx\right)^{1/2tt'} \\
&\quad \times \left(\frac{1}{|I|}\int_{4I} \bigl(\bigl|b_2 - (b_2)_{4I} + (b_2)_{4I} - (b_2)_I\bigr|\bigr)^{2tt'} \, dx\right)^{1/2tt'} \\
&\le C\|b_1\|_{\dot{\wedge}_{\beta_1}}\|b_2\|_{\dot{\wedge}_{\beta_2}} M_{\beta,r}(f)(x_0).
\end{aligned}$$

Now we estimate term F. For $x \in I$, we have

$$\begin{aligned}
&\bigl\| U(\mathcal{T})\bigl(\bigl(b_1 - (b_1)_I\bigr)\bigl(b_2 - (b_2)_I\bigr)f_2\bigr)(x) - C_I\bigr\|_E \\
&\quad = \left\| \left\{ \int_{\{t_{i+1}<|x-y|<s\}} \bigl(b_1(y) - (b_1)_I\bigr)\bigl(b_2(y) - (b_2)_I\bigr)K(x,y)f_2(y)\, dy \right.\right. \\
&\qquad \left.\left. - \int_{t_{i+1}<|x_0-y|<s} \bigl(b_1(y) - (b_1)_I\bigr)\bigl(b_2(y) - (b_2)_I\bigr)K(x_0,y)f_2(y)\, dy \right\}_{s\in J_i, i\in\mathbf{N}} \right\|_E \\
&\quad \le \left\| \left\{ \int_{\{t_{i+1}<|x-y|<s\}} \prod_{j=1}^{2} \bigl(b_j(y) - (b_j)_I\bigr)\bigl(K(x,y) - K(x_0,y)\bigr)f_2(y)\, dy \right\}_{s\in J_i, i\in\mathbf{N}} \right\|_E \\
&\qquad + \left\| \left\{ \int_{\mathbf{R}} \bigl(\chi_{\{t_{i+1}<|x-y|<s\}}(y) - \chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\bigr) \right.\right. \\
&\qquad \left.\left. \times \prod_{j=1}^{2} \bigl(b_j(y) - (b_j)_I\bigr)K(x_0,y)f_2(y)\, dy \right\}_{s\in J_i, i\in\mathbf{N}} \right\|_E \\
&\quad =: F_1 + F_2.
\end{aligned}$$

Note that $\|\{\chi_{t_{i+1}<|x-y|<s}\}_{s\in J_i, i\in\mathbf{N}}\|_E \le 1$ and $|x - x_0| \le l \le |x_0 - y|/2$ for $x \in I$, $y \in (4I)^c$. By Minkowski's inequality and (1.3), we have

$$\begin{aligned}
F_1 &\le \int_{\mathbf{R}} \bigl\| \{\chi_{\{t_{i+1}<|x-y|<s\}}\}_{s\in J_i, i\in\mathbf{R}} \bigr\|_E \prod_{j=1}^{2} \bigl|b_j(y) - (b_j)_I\bigr|\bigl|K(x,y) - K(x_0,y)\bigr|\bigl|f_2(y)\bigr|\, dy \\
&\le C\int_{(4I)^c} \prod_{j=1}^{2} \bigl|b_j(y) - (b_j)_I\bigr| \frac{|x - x_0|^\delta}{|x_0 - y|^{1+\delta}} \bigl|f(y)\bigr|\, dy \\
&\le C\sum_{k=0}^{\infty} \int_{2^k 4l<|x_0-y|<2^{k+1}4l} \frac{\prod_{j=1}^{2} |b_j(y) - (b_j)_I||f(y)|l^\delta}{|2^k 4l|^{1+\delta}}\, dy
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{k=0}^{\infty}\frac{1}{2^{(k+2)\delta}}\left(\frac{1}{|2^{k+3}I|}\int_{2^{k+3}I}\prod_{j=1}^{2}\left|b_j(y)-(b_j)_I\right|^{r'}dy\right)^{1/r'}\left(\frac{1}{|2^{k+3}I|}\int_{2^{k+3}I}\left|f(y)\right|^{r}dy\right)^{1/r}\\
&\le C\sum_{k=0}^{\infty}\frac{1}{2^{(k+2)\delta}}\left(\frac{1}{|2^{k+3}I|}\int_{2^{k+3}I}\left|b_1(y)-(b_1)_{2^{k+3}I}+(b_1)_{2^{k+3}I}-(b_1)_I\right|^{2r'}dy\right)^{1/2r'}\\
&\quad\times\left(\frac{1}{|2^{k+3}I|}\int_{|2^{k+3}I|}\left|b_2(y)-(b_2)_{2^{k+3}I}+(b_2)_{2^{k+3}I}-(b_2)_I\right|^{2r'}dy\right)^{1/2r'}\\
&\quad\times M_{\beta,r}f(x_0)\left|2^{k+3}I\right|^{-\beta}\\
&\le C\sum_{k=0}^{\infty}\frac{1}{2^{(k+2)\delta}}\left(\|b_1\|_{\dot\wedge_{\beta_1}}\left|2^{k+3}I\right|^{\beta_1}+\|b_1\|_{\dot\wedge_{\beta_1}}(k+2)\left|2^{k+3}I\right|^{\beta_1}\right)\\
&\quad\times\left(\|b_2\|_{\dot\wedge_{\beta_2}}\left|2^{k+3}I\right|^{\beta_2}+\|b_2\|_{\dot\wedge_{\beta_2}}(k+2)\left|2^{k+3}I\right|^{\beta_2}\right)M_{\beta,r}f(x_0)\left|2^{k+3}I\right|^{-\beta}\\
&\le C\|b_1\|_{\dot\wedge_{\beta_1}}\|b_2\|_{\dot\wedge_{\beta_2}}M_{\beta,r}f(x_0).
\end{aligned}
$$

For F_2, notice that the integral

$$
\int_{\mathbf{R}}\left|\chi_{\{t_{i+1}<|x-y|<s\}}(y)-\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\right|\prod_{j=1}^{2}\left|b_j-(b_j)_I\right|\left|K(x_0,y)\right|\left|f_2(y)\right|dy
$$

will only be non-zero if either $\chi_{\{t_{i+1}<|x-y|<s\}}(y)=1$ and $\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)=0$ or *vice versa*. That means the integral will only be non-zero in the following cases: (i) $t_{i+1}<|x-y|<s$ and $|x_0-y|\le t_{i+1}$; (ii) $t_{i+1}<|x-y|<s$ and $|x_0-y|\ge s$; (iii) $t_{i+1}<|x_0-y|<s$ and $|x-y|\le t_{i+1}$; (iv) $t_{i+1}<|x_0-y|<s$ and $|x-y|\ge s$. In the first case we observe that $t_{i+1}<|x-y|\le|x-x_0|+|x_0-y|<l+t_{i+1}$ as $|x-x_0|<l$. Analogously, in the third case we have $t_{i+1}<|x_0-y|<l+t_{i+1}$. In the second case we have $s<|x_0-y|\le|x_0-x|+|x-y|<l+s$, and in the fourth case $s<|x-y|<l+s$. Using (1.2), we have

$$
\begin{aligned}
&\int_{\mathbf{R}}\left|\chi_{\{t_{i+1}<|x-y|<s\}}(y)-\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\right|\prod_{j=1}^{2}\left|b_j(y)-(b_j)_I\right|\left|K(x_0,y)\right|\left|f_2(y)\right|dy\\
&\le C\int_{\mathbf{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\chi_{\{t_{i+1}<|x-y|<l+t_{i+1}\}}(y)\prod_{j=1}^{2}\left|b_j(y)-(b_j)_I\right|\frac{|f_2(y)|}{|x_0-y|}dy\\
&\quad+C\int_{\mathbf{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\chi_{\{t_{i+1}<|x_0-y|<l+s\}}(y)\prod_{j=1}^{2}\left|b_j(y)-(b_j)_I\right|\frac{|f_2(y)|}{|x_0-y|}dy\\
&\quad+C\int_{\mathbf{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\chi_{\{t_{i+1}<|x_0-y|<l+t_{i+1}\}}(y)\prod_{j=1}^{2}\left|b_j(y)-(b_j)_I\right|\frac{|f_2(y)|}{|x_0-y|}dy\\
&\quad+C\int_{\mathbf{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\chi_{\{s<|x-y|<l+s\}}(y)\prod_{j=1}^{2}\left|b_j(y)-(b_j)_I\right|\frac{|f_2(y)|}{|x_0-y|}dy\\
&\le C\left(\int_{\mathbf{R}}\chi_{\{t_{i+1}<|x-y|<s\}}(y)\frac{\prod_{j=1}^{2}|b_j(y)-(b_j)_I|^t}{|x_0-y|^t}\left|f_2(y)\right|^t dy\right)^{1/t}l^{1/t'}\\
&\quad+C\left(\int_{\mathbf{R}}\chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\frac{\prod_{j=1}^{2}|b_j(y)-(b_j)_I|^t}{|x_0-y|^t}\left|f_2(y)\right|^t dy\right)^{1/t}l^{1/t'},
\end{aligned}
$$

where in the last inequality we have used Hölder's inequality with r being the range $1 < r < \infty$ and recalling that $t = \sqrt{r}$. Returning to our estimation of F_2, we have

$$\begin{aligned}
F_2 &= \biggl\| \biggl\{ \int_{\mathbf{R}} \bigl(\chi_{\{t_{i+1}<|x-y|<s\}}(y) - \chi_{\{t_{i+1}<|x_0-y|<s\}}(y)\bigr) \\
&\quad \times \prod_{j=1}^{2} \bigl(b_j(y) - (b_j)_I\bigr) K(x_0,y) f_2(y)\, dy \biggr\}_{s\in J_i, i\in\mathbf{N}} \biggr\|_E \\
&\leq C l^{1/t'} \biggl\{ \biggl\| \biggl\{ \biggl(\int_{\mathbf{R}} \chi_{\{t_{i+1}<|x-y|<s\}}(y) \frac{\prod_{j=1}^{2} |b_j(y)-(b_j)_I|^t}{|x_0-y|^t} \bigl|f_2(y)\bigr|^t\, dy \biggr)^{1/t} \biggr\}_{s\in J_i, i\in\mathbf{N}} \biggr\|_E \\
&\quad + \biggl\| \biggl\{ \biggl(\int_{\mathbf{R}} \chi_{\{t_{i+1}<|x_0-y|<s\}}(y) \frac{\prod_{j=1}^{2} |b_j(y)-(b_j)_I|^t}{|x_0-y|^t} \bigl|f_2(y)\bigr|^t\, dy \biggr)^{1/t} \biggr\}_{s\in J_i, i\in\mathbf{N}} \biggr\|_E \biggr\} \\
&=: F_2^1 + F_2^2.
\end{aligned}$$

Choosing $1 < r < 4$ with $t = \sqrt{r}$, we have

$$\begin{aligned}
&\biggl\| \biggl\{ \biggl(\int_{\mathbf{R}} \chi_{\{t_{i+1}<|x-y|<s\}}(y) \frac{\prod_{j=1}^{2} |b_j(y)-(b_j)_I|^t}{|x_0-y|^t} \bigl|f_2(y)\bigr|^t\, dy \biggr)^{1/t} \biggr\}_{s\in J_i, i\in\mathbf{N}} \biggr\|_E \\
&\quad = \biggl[\sum_{i\in\mathbf{N}} \sup_{s\in J_i} \biggl(\int_{\mathbf{R}} \chi_{\{t_{i+1}<|x-y|<s\}}(y) \frac{\prod_{j=1}^{2} |b_j(y)-(b_j)_I|^t}{|x_0-y|^t} \bigl|f_2(y)\bigr|^t\, dy \biggr)^{2/t} \biggr]^{1/2} \\
&\quad \leq \biggl(\sum_{i\in\mathbf{N}} \int_{\mathbf{R}} \chi_{\{t_{i+1}<|x-y|<t_i\}}(y) \frac{\prod_{j=1}^{2} |b_j(y)-(b_j)_I|^t}{|x_0-y|^t} \bigl|f_2(y)\bigr|^t\, dy \biggr)^{1/t} \\
&\quad \leq \biggl(\int_{\mathbf{R}} \frac{\prod_{j=1}^{2} |b_j(y)-(b_j)_I|^t}{|x_0-y|^t} \bigl|f_2(y)\bigr|^t\, dy \biggr)^{1/t} \\
&\quad \leq \biggl(\sum_{k=0}^{\infty} \frac{1}{(2^k 4l)^t} \int_{|x_0-y|<2^{k+1}4l} \prod_{j=1}^{2} \bigl|b_j(y)-(b_j)_I\bigr|^t \bigl|f(y)\bigr|^t\, dy \biggr)^{1/t} \\
&\quad \leq C \sum_{k=0}^{\infty} l^{-1/t'} \frac{1}{(2^{k+2})^{1/t'}} \biggl(\frac{1}{|2^{k+3}I|} \int_{2^{k+3}I} \prod_{j=1}^{2} \bigl|b_j(y)-(b_j)_I\bigr|^{tt'}\, dy \biggr)^{1/tt'} \\
&\qquad \times \biggl(\frac{1}{|2^{k+3}I|} \int_{2^{k+3}I} \bigl|f(y)\bigr|^{t^2}\, dy \biggr)^{1/t^2} \\
&\quad \leq C l^{-1/t'} \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} M_{\beta,r} f(x_0).
\end{aligned}$$

Therefore we get

$$F_2^1 \leq C \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} M_{\beta,r} f(x_0).$$

Similarly,

$$F_2^2 \leq C \|b_1\|_{\dot{\wedge}_{\beta_1}} \|b_2\|_{\dot{\wedge}_{\beta_2}} M_{\beta,r} f(x_0).$$

Consequently,

$$F_2 \le C\|b_1\|_{\dot\Lambda_{\beta_1}}\|b_2\|_{\dot\Lambda_{\beta_2}} M_{\beta,r}f(x_0).$$

This completes the proof of Lemma 3.2. □

Proof of Theorem 1.1 We will only prove the result for the operator $\mathcal{O}(\mathcal{T}_{\vec b})$ since a similar proof can be given for the operator $\mathcal{V}_\rho(\mathcal{T}_{\vec b})$. To apply (2.4), we first take it for granted that $\|M(\mathcal{O}'(\mathcal{T}_{\vec b})f)\|_{L^q(\omega^q)}$ is finite. We will check these to the end of the proof.

We proceed by induction on m. Note that $\omega^q \in A_q$ and $\omega^p \in A_p$. For $m = 1$, by (1.1), Lemmas 2.2, 2.4, 3.1 and 3.2, we have

$$\begin{aligned}
\|\mathcal{O}(\mathcal{T}_b f)\|_{L^q(\omega^q)} &\le C\|\mathcal{O}'(\mathcal{T}_b f)\|_{L^q(\omega^q)} \\
&\le C\|M(\mathcal{O}'(\mathcal{T}_b f))\|_{L^q(\omega^q)} \le C\|M^\sharp(\mathcal{O}'(\mathcal{T}_b f))\|_{L^q(\omega^q)} \\
&\le C\|b\|_{\dot\Lambda_\beta}\bigl(\|M_{\beta,r}(\mathcal{O}'(\mathcal{T}f))\|_{L^q(\omega^q)} + \|M_{\beta,r}(f)\|_{L^q(w^q)}\bigr) \\
&\le C\|b\|_{\dot\Lambda_\beta}\bigl(\|\mathcal{O}'(\mathcal{T}f)\|_{L^p(\omega^p)} + \|f\|_{L^p(\omega^p)}\bigr) \le C\|b\|_{\dot\Lambda_\beta}\|f\|_{L^p(\omega^p)}.
\end{aligned}$$

Now we consider the case $m \ge 2$. Suppose that for $m - 1$ the theorem is true, and let us prove it for m. The same argument as used above and the induction hypothesis yield that

$$\begin{aligned}
\|\mathcal{O}(\mathcal{T}_{\vec b} f)\|_{L^q(\omega^q)} &\le C\|\mathcal{O}'(\mathcal{T}_{\vec b} f)\|_{L^q(\omega^q)} \\
&\le C\|M(\mathcal{O}'(\mathcal{T}_{\vec b} f))\|_{L^q(\omega^q)} \le C\|M^\sharp(\mathcal{O}'(\mathcal{T}_{\vec b} f))\|_{L^q(\omega^q)} \\
&\le C\|\vec b\|_{\dot\Lambda_\beta}\bigl\{\|M_{\beta,r}(\mathcal{O}'(\mathcal{T}f))\|_{L^q(\omega^q)} + \|M_{\beta,r}(f)\|_{L^q(\omega^q)}\bigr\} \\
&\quad + C\sum_{i=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec b_\sigma\|_{\dot\Lambda_{\beta_\sigma}}\|M_{\beta_\sigma,r}(\mathcal{O}'(\mathcal{T}_{\vec b_{\sigma'}}f))\|_{L^q(\omega^q)} \\
&\le C\|\vec b\|_{\dot\Lambda_\beta}\bigl\{\|\mathcal{O}'(\mathcal{T}f)\|_{L^p(\omega^p)} + \|f\|_{L^p(\omega^p)}\bigr\} \\
&\quad + C\sum_{i=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec b_\sigma\|_{\dot\Lambda_{\beta_\sigma}}\|\mathcal{O}'(\mathcal{T}_{\vec b_{\sigma'}}f)\|_{L^{p_\sigma}(\omega^{p_\sigma})} \\
&\le C\|\vec b\|_{\dot\Lambda_\beta}\|f\|_{L^p(\omega^p)} + C\sum_{i=1}^{m-1}\sum_{\sigma\in C_j^m}\|\vec b_\sigma\|_{\dot\Lambda_{\beta_\sigma}}\|\vec b_{\sigma'}\|_{\dot\Lambda_{\beta_{\sigma'}}}\|f\|_{L^p(\omega^p)} \\
&\le C\|\vec b\|_{\dot\Lambda_\beta}\|f\|_{L^p(\omega^p)},
\end{aligned}$$

where $\beta_{\sigma'} = \beta - \beta_\sigma$, $1/q = 1/p_\sigma - \beta_\sigma = 1/p - \beta_\sigma - \beta_{\sigma'} = 1/p - \beta$.

It remains to check that $\|M(\mathcal{O}(\mathcal{T}_{\vec b}f))\|_{L^q(\omega^q)} < \infty$ for any $m \ge 1$. By the weighted L^q-boundedness of M, it is reduced to checking that $\|\mathcal{O}'(\mathcal{T}_{\vec b}f)\|_{L^q(\omega^q)} < \infty$. For simplicity, we will check only that $\|\mathcal{O}'(\mathcal{T}_b f)\|_{L^q(\omega^q)} < \infty$ since the others are similar. Suppose that b and ω are all bounded functions. Notice that

$$\mathcal{O}'(\mathcal{T}_b f)(x) = \|U(\mathcal{T}_b)f(x)\|_E \le |b(x)|\,\|U(\mathcal{T})f(x)\|_E + \|U(\mathcal{T})bf(x)\|_E,$$

we have

$$\begin{aligned}\left\|\mathcal{O}'(\mathcal{T}_b f)\right\|_{L^q(\omega^q)} &\le \|b\|_\infty\|\omega\|_\infty\left\|U(\mathcal{T})f\right\|_{L^q} + \|\omega\|_\infty\left\|U(\mathcal{T})bf\right\|_{L^q} \\ &\le C\bigl(\|b\|_\infty\|\omega\|_\infty\|f\|_{L^q} + \|\omega\|_\infty\|bf\|_{L^q}\bigr) \\ &\le C\|b\|_\infty\|\omega\|_\infty\|f\|_{L^q} < \infty\end{aligned}$$

for all $f \in C_0^\infty(\mathbf{R})$, where in the second inequality we have used the result of (3.1) in Lemma 3.1.

For the general case, we will truncate $\vec{b}$ and ω as follows. For $N \in \mathbf{N}$, we define $\omega^N(x) = \inf\{\omega(x), N\}$, and for $\vec{b} = (b_1, b_2, \ldots, b_m)$, $\vec{b}^N = (b_1^N, b_2^N, \ldots, b_m^N)$,

$$b_j^N(x) = \begin{cases} N, & b_j(x) > N; \\ b_j(x), & |b_j(x)| \le N; \\ -N, & b_j(x) < -N. \end{cases}$$

It is easy to check that

$$\left\|\vec{b}^N\right\|_{\dot{\wedge}_\beta} \le C\|\vec{b}\|_{\dot{\wedge}_\beta} \quad \text{and} \quad \bigl[(\omega^N)^q\bigr]_{A_q} \le C\bigl[\omega^q\bigr]_{A_q}. \tag{3.16}$$

Then the results of Theorem 1.1 hold for the operators family $\mathcal{T}_{\vec{b}^N} = \{T_{\varepsilon,\vec{b}^N}\}_{\varepsilon>0}$ and the weights ω^N. On the other hand, notice that

$$\lim_{N\to\infty} T_{\varepsilon,\vec{b}^N} f(x) = T_{\varepsilon,\vec{b}^N} f(x), \quad \forall\varepsilon > 0.$$

It is not difficult to check that

$$\mathcal{O}'(\mathcal{T}_{\vec{b}} f)(x) \le \lim_{N\to\infty} \mathcal{O}'(T_{\vec{b}^N} f)(x).$$

This together with (3.16) and Fatou's lemma implies that the theorem holds for the general case. Theorem 1.1 is proved. □

4 The $(L^p, \dot{\wedge}_{(\beta-1/p)})$-type estimates

In this section, we will prove Theorems 1.2-1.3, which need the un-weighted results of Theorem 1.1.

Proof of Theorem 1.2 For any interval $I \subset \mathbf{R}$ satisfying $|I| = 2l$, define $f_1(y) = f(y)\chi_{4I}$ and $f_2(y) = f(y) - f_1(y)$. Let

$$C_I = \frac{1}{|I|}\int_I \left\{\int_{\{t_{i+1}<|z-y|<s\}} \bigl(b(z) - b(y)\bigr)K(z,y)f_2(y)\,dy\right\}_{s\in J_i, i\in\mathbf{N}} dz,$$

where $(4I)^c$ denotes the complementary set of the interval $4I$. By (2.6), it suffices to prove that

$$\frac{1}{|I|}\int_I \left\|U(\mathcal{T}_b)(f)(x) - C_I\right\|_E\,dx \le C\|b\|_{\dot{\wedge}_\beta}\|f\|_{L^p}|I|^{\beta-1/p}.$$

We write

$$
\begin{aligned}
&\frac{1}{|I|}\int_I \big\| U(\mathcal{T}_b)(f)(x) - C_I \big\|_E \, dx \\
&\quad \le \frac{1}{|I|}\int_I \big\| U(\mathcal{T}_b)(f_1)(x) + U(\mathcal{T}_b)(f_2)(x) - C_I \big\|_E \, dx \\
&\quad \le \frac{1}{|I|}\int_I \big\| U(\mathcal{T}_b)(f_1)(x) \big\|_E \, dx + \frac{1}{|I|}\int_I \big\| U(\mathcal{T}_b)(f_2)(x) - C_I \big\|_E \, dx \\
&\quad =: A_1 + A_2.
\end{aligned}
$$

Choose $1 < p_1 < 1/\beta < p$ and q_1 with $1/q_1 = 1/p_1 - \beta$. Then, by Theorem 1.1,

$$
\begin{aligned}
A_1 &= \frac{1}{|I|}\int_I \mathcal{O}'(\mathcal{T}_b f_1)(x)\, dx \le \frac{1}{|I|}\left(\int_I \big|\mathcal{O}'(\mathcal{T}_b f_1)(x)\big|^{q_1}\, dx\right)^{1/q_1} |I|^{1-1/q_1} \\
&\le C\|b\|_{\dot{\wedge}_\beta} \frac{1}{|I|}\left(\int_{\mathbf{R}} \big|f_1(x)\big|^{p_1}\, dx\right)^{1/p_1} |I|^{1-1/q_1} \\
&= C\|b\|_{\dot{\wedge}_\beta} \frac{1}{|I|}\left(\int_{4I} \big|f(x)\big|^{p_1}\, dx\right)^{1/p_1} |I|^{1-1/q_1} \\
&\le C\|b\|_{\dot{\wedge}_\beta} \frac{1}{|I|}\left(\int_{\mathbf{R}} \big|f(x)\big|^{p}\, dx\right)^{1/p} |I|^{1-1/q_1} |I|^{1/p_1-1/p} \\
&\le \|b\|_{\dot{\wedge}_\beta} \|f\|_{L^p} |I|^{\beta-1/p}
\end{aligned}
$$

and

$$
\begin{aligned}
A_2 &= \frac{1}{|I|}\int_I \left\| U(\mathcal{T}_b f_2)(x) - \frac{1}{|I|}\int_I U(\mathcal{T}_b)(f_2)(z)\, dz \right\|_E dx \\
&\le \frac{1}{|I|^2}\int_{I\times I} \big\| U(\mathcal{T}_b)(f_2)(x) - U(\mathcal{T}_b)(f_2)(z) \big\|_E \, dz\, dx.
\end{aligned}
$$

We write

$$
\begin{aligned}
&\big\| U(\mathcal{T}_b)(f_2)(x) - U(\mathcal{T}_b)(f_2)(z) \big\|_E \\
&\quad = \bigg\| \bigg\{ \int_{\{t_{i+1}<|x-y|<s\}} \big(b(x)-b(y)\big) K(x,y) f_2(y)\, dy \\
&\qquad - \int_{\{t_{i+1}<|z-y|<s\}} \big(b(z)-b(y)\big) K(z,y) f_2(y)\, dy \bigg\} \bigg\| \\
&\quad \le \bigg\| \bigg\{ \int_{\{t_{i+1}<|x-y|<s\}} \big(K(x,y)-K(z,y)\big)\big(b(x)-b(y)\big) f_2(y)\, dy \bigg\}_{s\in J_i, i\in\mathbf{R}} \bigg\|_E \\
&\qquad + \bigg\| \bigg\{ \int_{\mathbf{R}} (\chi_{\{t_{i+1}<|x-y|<s\}} - \chi_{\{t_{i+1}<|z-y|<s\}}) K(z,y) \\
&\qquad \times \big(b(x)-b(z)\big) f_2(y)\, dy \bigg\}_{s\in J_i, i\in\mathbf{R}} \bigg\|_E \\
&\quad =: A_{21} + A_{22}.
\end{aligned}
$$

Note that $\|\chi_{\{t_{i+1}<|x-y|<s\}}\|_E \le 1$ and $|x-z| \le 2l \le 2|z-y|/3$ for $x \in I$, $y \in (4I)^c$, $z \in I$. We have

$$\begin{aligned}
A_{21} &\le \int_{\mathbf{R}} \big|K(x,y) - K(z,y)\big|\big|b(x) - b(y)\big|\big|f_2(y)\big| \big\|\{\chi_{\{t_{i+1}<|x-y|<s\}}\}_{s\in J_i, i\in\mathbf{N}}\big\| \, dy \\
&\le \int_{(4I)^c} \big|K(x,y) - K(z,y)\big|\big|b(x) - b(y)\big|\big|f(y)\big| \, dy \\
&\le C \int_{(4I)^c} \frac{|x-z|^\delta}{|z-y|^{1+\delta}} \big|f(y)\big|\big|b(x) - b(y)\big| \, dy \\
&\le C \sum_{k=0}^{\infty} \int_{2^{k+1}4I\setminus 2^k 4I} \big|f(y)\big| \|b\|_{\dot{\wedge}_\beta} |x-y|^\beta \frac{|x-z|^\delta}{|z-y|^{1+\delta}} \, dy \\
&\le C\|b\|_{\dot{\wedge}_\beta} \sum_{k=0}^{\infty} \frac{(2l)^\delta (2^{k+1}5l)^\beta}{(2^k 3l)^{1+\delta}} \int_{2^{k+3}I} \big|f(y)\big| \, dy \\
&\le C\|b\|_{\dot{\wedge}_\beta} \sum_{k=0}^{\infty} \frac{(2l)^\delta (2^{k+1}5l)^\beta}{(2^k 3l)^{1+\delta}} \left(\int_{\mathbf{R}} \big|f(y)\big|^p \, dy\right)^{1/p} \big|2^{k+3}I\big|^{1-1/p} \\
&\le C\|b\|_{\dot{\wedge}_\beta} \|f\|_{L^p} |I|^{\beta-1/p}.
\end{aligned}$$

For the second term A_{22}, as the proof of term F_2 in Lemma 3.2, we get

$$\begin{aligned}
A_{22} &= \bigg\| \bigg\{ \int_{\mathbf{R}} (\chi_{\{t_{i+1}<|x-y|<s\}} - \chi_{\{t_{i+1}<|z-y|<s\}}) K(z,y) \\
&\qquad \times \big(b(x) - b(z)\big) f_2(y) \, dy \bigg\}_{s\in J_i, i\in\mathbf{R}} \bigg\|_E \\
&\le C(2l)^{1/p'} \bigg\| \bigg\{ \int_{\mathbf{R}} \chi_{\{t_{i+1}<|x-y|<s\}}(y) \big|b(x) - b(z)\big|^p \frac{|f_2(y)|^p}{|z-y|^p} \, dy \bigg\}_{s\in J_i, i\in\mathbf{R}} \bigg\|_E \\
&\quad + (2l)^{1/p'} \bigg\| \bigg\{ \int_{\mathbf{R}} \chi_{\{t_{i+1}<|z-y|<s\}}(y) \big|b(x) - b(z)\big|^p \frac{|f_2(y)|^p}{|z-y|^p} \, dy \bigg\}_{s\in J_i, i\in\mathbf{R}} \bigg\|_E \\
&=: A_{22}^1 + A_{22}^2.
\end{aligned}$$

Note that

$$\begin{aligned}
&\bigg\| \bigg\{ \int_{\mathbf{R}} \chi_{\{t_{i+1}<|z-y|<s\}}(y) \big|b(x) - b(z)\big|^p \frac{|f_2(y)|^p}{|z-y|^p} \, dy \bigg\}_{s\in J_i, i\in\mathbf{R}} \bigg\|_E \\
&\quad = \bigg[\sum_{i\in\mathbf{N}} \sup_{s\in J_i} \bigg(\int_{\mathbf{R}} \chi_{\{t_{i+1}<|z-y|<s\}}(y) \big|b(x) - b(z)\big|^p \frac{|f_2(y)|^p}{|z-y|^p} \, dy \bigg)^{2/p} \bigg]^{1/2} \\
&\quad \le \bigg(\sum_{i\in\mathbf{N}} \int_{\mathbf{R}} \chi_{\{t_{i+1}<|z-y|<t_i\}}(y) \big|b(x) - b(z)\big|^p \frac{|f_2(y)|^p}{|z-y|^p} \, dy \bigg)^{1/p} \\
&\quad \le \bigg(\sum_{k=0}^{\infty} \int_{2^{k+1}4I\setminus 2^k 4I} \big|b(x) - b(z)\big|^p \big|f(y)\big|^p \frac{1}{|z-y|^p} \, dy \bigg)^{1/p} \\
&\quad \le \|b\|_{\dot{\wedge}_\beta} \sum_{k=0}^{\infty} \frac{(2l)^\beta}{2^k 3l} \bigg(\int_{|z-y|<2^{k+1}5l} \big|f(y)\big|^p \, dy \bigg)^{1/p} \\
&\quad \le \|b\|_{\dot{\wedge}_\beta} \|f\|_{L^p} (2l)^{\beta-1}.
\end{aligned}$$

We get

$$A_{22}^1 \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}|I|^{\beta-1/p}.$$

Similarly,

$$A_{22}^2 \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}|I|^{\beta-1/p}.$$

Consequently,

$$A_2 \le C\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p}|I|^{\beta-1/p},$$

which completes the proof of Theorem 1.2. □

Proof of Theorem 1.3 Theorem 1.3 can be regarded as the case of the endpoint $p = 1/\beta$ in Theorem 1.2. By similar arguments as those in proving Theorem 1.2, we can get Theorem 1.3. Here, we omit the details. □

5 Applications

In this section, we will give certain applications of our main theorems.

5.1 On the oscillation and variation related to the commutators of Hilbert transform and Hermitian Riesz transform

Let $\mathcal{T} = \{T_\varepsilon\}$ be composed by truncations of the Hilbert transform $\mathcal{H} = \{H_\varepsilon\}_\varepsilon$ given by

$$H_\varepsilon f(x) = \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\,dy.$$

In 2000, Campbell *et al.* [14] proved the strong (p,p)-boundedness in the range $1 < p < \infty$ and the weak type $(1,1)$-boundedness of the oscillation operator $\mathcal{O}(\mathcal{H})$ and the ρ-variation operator $\mathcal{V}_\rho(\mathcal{H})$ for $\rho > 2$. Subsequently, in [15], the aforementioned authors extended the above results to the higher dimensional cases. In 2004, Gillespie and Torrea [3] showed that both $\mathcal{O}(\mathcal{H})$ and $\mathcal{V}_\rho(\mathcal{H})$ with $\rho > 2$ are bounded on $L^p(\mathbf{R},\omega(x)\,dx)$ for $\omega(x) \in A_p$, $1 < p < \infty$. Recently, Crecimbeni *et al.* [2] proved that both $\mathcal{O}(\mathcal{H})$ and $\mathcal{V}_\rho(\mathcal{H})$ with $\rho > 2$ map $L^1(\mathbf{R},\omega(x)\,dx)$ into $L^{1,\infty}(\mathbf{R},\omega(x)\,dx)$ for $\omega \in A_1$; moreover, they also showed that both $\mathcal{O}(\mathcal{R}^\pm)$ and $\mathcal{V}_\rho(\mathcal{R}^\pm)$ with $\rho > 2$ map $L^p(\mathbf{R},\omega(x)\,dx)$ for $\omega(x) \in A_p$ in the range $1 < p < \infty$, and map $L^1(\mathbf{R},\omega(x)\,dx)$ into $L^{1,\infty}(\mathbf{R},\omega(x)\,dx)$ for $\omega \in A_1$, where $\mathcal{R}^\pm$ are the Hermitian Riesz transforms, that is, the Riesz transform associated with the harmonic oscillator

$$\mathcal{L} = \bigl(A^*A + AA^*\bigr)/2, \qquad A = \frac{d}{dx} + x \quad \text{and} \quad A^* = -\frac{d}{dx} + x.$$

Precisely, $\mathcal{R}^\pm$ are bounded from $L^p(\mathbf{R},dx)$ into itself for $1 < p < \infty$, and from $L^1(\mathbf{R},dx)$ into $L^{1,\infty}(\mathbf{R},dx)$ (see [16, 17]). Moreover, $\mathcal{R}^\pm$ are principal value operators, that is,

$$\mathcal{R}^\pm(f)(x) = \lim_{\varepsilon\to 0}\mathcal{R}_\varepsilon^\pm(f)(x) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon}\mathcal{R}^\pm(x,y)f(y)\,dy, \quad \text{a.e. } x, f \in L^1(\mathbf{R},dx),$$

where $\mathcal{R}^{\pm}(x,y)$ are the appropriated kernels whose expressions can be found in [16]. In particular, by Proposition 3.1 in [16], the kernels $\mathcal{R}^{\pm}(x,y)$ of the Riesz transform $\mathcal{R}^{\pm}$ are the standard Calderón-Zygmund kernels satisfying (1.2)-(1.4). We consider oscillation and variation operators for commutators of the Hilbert transform and Hermitian Riesz transform. Let $\vec{b} = (b_1,\dots,b_m)$ be a locally integrable function on $\mathbf{R}$, $\mathcal{H}_{\vec{b}} = \{H_{\varepsilon,\vec{b}}\}_\varepsilon$ and $\mathcal{R}^{\pm}_{\vec{b}} = \{\mathcal{R}^{\pm}_{\varepsilon,\vec{b}}\}_\varepsilon$, where

$$H_{\varepsilon,\vec{b}}(f)(x) = [\vec{b},H_\varepsilon](f)(x) = \int_{|x-y|>\varepsilon} \prod_{j=1}^{m} \bigl(b_j(x)-b_j(y)\bigr)\frac{f(y)}{x-y}\,dy$$

and

$$\mathcal{R}^{\pm}_{\varepsilon,\vec{b}}(f)(x) = \bigl[\vec{b},\mathcal{R}^{\pm}_{\varepsilon}\bigr](f)(x) = \int_{|x-y|>\varepsilon} \prod_{j=1}^{m} \bigl(b_j(x)-b_j(y)\bigr)\mathcal{R}^{\pm}(x,y)f(y)\,dy.$$

Then applying Theorems 1.1-1.3 to $\mathcal{H}$ and $\mathcal{R}^{\pm}$, we get the following results.

Theorem 5.1 *Let $\mathcal{T} = \{\mathcal{T}_\varepsilon\}$ be either the truncations of the Hilbert transform $\mathcal{H} = \{H_\varepsilon\}_\varepsilon$ or the truncations of Hermitian Riesz transforms $\mathcal{R}^{\pm} = \{\mathcal{R}^{\pm}_\varepsilon\}_\varepsilon$, and $\mathcal{T}_{\vec{b}} = \{\mathcal{T}_{\varepsilon,\vec{b}}\}_\varepsilon$ the corresponding iterated commutators with $\vec{b} = (b_1,\dots,b_m)$. If $b_i \in \dot{\wedge}_{\beta_i}$ $(i=1,\dots,m)$ with $0<\beta=\beta_1+\cdots+\beta_m<1$, $\rho>2$, then for $1<p<1/\beta$ with $1/q = 1/p-\beta$ and $\omega \in A_{(p,q)}$,*

$$\bigl\|\mathcal{O}(\mathcal{T}_{\vec{b}}f)\bigr\|_{L^q(w^q)} \leq C\|\vec{b}\|_{\dot{\wedge}_\beta}\|f\|f_{L^p(\omega^p)}$$

and

$$\bigl\|\mathcal{V}_\rho(\mathcal{T}_{\vec{b}}f)\bigr\|_{L^q(\omega^q)} \leq C\|\vec{b}\|_{\dot{\wedge}_\beta}\|f\|_{L^p(\omega^p)}.$$

Theorem 5.2 *Let $\mathcal{T} = \{\mathcal{T}_\varepsilon\}$ be either the truncations of the Hilbert transform $\mathcal{H} = \{H_\varepsilon\}_\varepsilon$ or the truncations of Hermitian Riesz transforms $\mathcal{R}^{\pm} = \{\mathcal{R}^{\pm}_\varepsilon\}_\varepsilon$, and $\mathcal{T}_b = \{\mathcal{T}_{\varepsilon,b}\}_\varepsilon$ the corresponding commutators with $b \in \dot{\wedge}_\beta$ and $0<\beta<1$. Then, for $\rho>2$, $1/\beta<p<\infty$,*

$$\bigl\|\mathcal{O}(\mathcal{T}_b f)\bigr\|_{\dot{\wedge}_{(\beta-1/p)}} \leq C\|b\|_{\dot{\wedge}_\beta}\|f\|_{L^p}, \qquad \bigl\|\mathcal{V}_\rho(\mathcal{T}_b f)\bigr\|_{\dot{\wedge}_{(\beta-1/p)}} \leq C\|b\|_{\dot{\wedge}_\beta}\|f\|_{L^p}$$

and

$$\bigl\|\mathcal{O}(\mathcal{T}_b f)\bigr\|_{BMO} \leq C\|b\|_{\dot{\wedge}_\beta}\|f\|_{L^{1/\beta}}, \qquad \bigl\|\mathcal{V}_\rho(\mathcal{T}_b f)\bigr\|_{BMO} \leq C\|b\|_{\dot{\wedge}_\beta}\|f\|_{L^{1/\beta}}.$$

Remark 5.1 Obviously, $T_\varepsilon = H_\varepsilon$ and $T_{\varepsilon,\vec{b}} = H_{\varepsilon,\vec{b}}$ for $K(x,y) = 1/(x-y)$ satisfying (1.2)-(1.4) with $\delta = 1$ and $A = 16$. And by Proposition 3.1 in [16], we know that $T_\varepsilon = \mathcal{R}^{\pm}_\varepsilon$ and $T_{\varepsilon,\vec{b}} = \mathcal{R}^{\pm}_{\varepsilon,\vec{b}}$ for $K(x,y) = \mathcal{R}^{\pm}(x,y)$ satisfying (1.2)-(1.4) with $\delta = 1$ and some $A>0$. Also, by Theorems 1.1 and 1.2 in [14] (resp., by Theorem A in [2]), $\mathcal{O}(\mathcal{H})$ and $\mathcal{V}_\rho(\mathcal{H})$ (resp., $\mathcal{O}(\mathcal{R}^{\pm})$ and $\mathcal{V}_\rho(\mathcal{R}^{\pm})$) with $\rho>2$ are bounded on $L^p(\mathbf{R})$ for $1<p<\infty$. Then Theorems 5.1 and 5.2 directly follow from Theorems 1.1-1.3.

5.2 On the λ-jump operators and the number of up-crossing

To the end, as applications of our main results, we consider the λ-jump operators and the number of up-crossing associated with the operators sequence $\{T_\varepsilon\}$, which give certain quantitative information on the convergence of the family $\{T_\varepsilon\}$.

Definition 5.1 The λ-jump operator associated with a sequence $\mathcal{T} = \{T_\varepsilon\}_\varepsilon$ applied to a function f at a point x is denoted by $\Lambda(\mathcal{T}, f, \lambda)(x)$ and defined by

$$\begin{aligned}\Lambda(\mathcal{T}, f, \lambda)(x) := \sup\bigl\{n \in \mathbf{N} : &\text{there exist } s_1 < t_1 \leq s_2 < t_2 < \cdots \leq s_n < t_n \\ &\text{such that } \bigl|T_{s_i} f(x) - T_{t_i} f(x)\bigr| > \lambda \text{ for } i = 1, 2, \ldots, n\bigr\}.\end{aligned} \tag{5.1}$$

Proposition 5.1 ([18]) *If λ-jump operators is finite a.e. for each choice of $\lambda > 0$, then we must have a.e. convergence of our family of operators.*

Proposition 5.2 ([18]) *The λ-jump operators are controlled by the ρ-variation operator. Precisely, we have*

$$\lambda\bigl(\Lambda(\mathcal{T}, f, \lambda)(x)\bigr)^{1/\rho} \leq \mathcal{V}_\rho(\mathcal{T}f)(x).$$

Applying Theorems 1.1-1.3 together with Proposition 5.2, we can get the following results.

Theorem 5.3 *Suppose that $K(x,y)$ satisfies* (1.2)-(1.4), $\vec{b} = (b_1, \ldots, b_m)$ *with* $b_i \in \dot{\Lambda}_{\beta_i}$ $(i = 1, \ldots, m)$ *and* $0 < \beta = \beta_1 + \cdots + \beta_m \leq \delta < 1$, *where δ is the same as in* (1.3), $\rho > 2$. *Let* $\mathcal{T} = \{\mathcal{T}_\varepsilon\}_{\varepsilon>0}$ *and* $\mathcal{T}_{\vec{b}} = \{T_{\varepsilon,\vec{b}}\}_{\varepsilon>0}$ *be given by* (1.6) *and* (1.8), *respectively. If $\mathcal{V}_\rho(\mathcal{T})$ is bounded in $L^r(\mathbf{R}, dx)$ for some $1 < r < \infty$, then for $1 < p < 1/\beta$ with $1/q = 1/p - \beta$ and $\omega \in A_{(p,q)}$, we have*

$$\bigl\|\Lambda(\mathcal{T}_{\vec{b}}, f, \lambda)^{1/\rho}\bigr\|_{L^q(\omega^q)} \leq \frac{C(p,q,\rho)}{\lambda}\|\vec{b}\|_{\dot{\Lambda}_\beta}\|f\|_{L^p(\omega^p)}.$$

Theorem 5.4 *Suppose that $K(x,y)$ satisfies* (1.2)-(1.4), $b \in \dot{\Lambda}_\beta$, $0 < \beta \leq \delta < 1$, *where δ is the same as in* (1.3), $\rho > 2$. *Let* $\mathcal{T} = \{\mathcal{T}_\varepsilon\}_{\varepsilon>0}$ *and* $\mathcal{T}_b = \{T_{\varepsilon,b}\}_{\varepsilon>0}$ *be given by* (1.6) *and* (1.7), *respectively. If $\mathcal{V}_\rho(\mathcal{T})$ is bounded in $L^r(\mathbf{R}, dx)$ for some $1 < r < \infty$, then we have*

$$\bigl\|\Lambda(\mathcal{T}_b, f, \lambda)^{1/\rho}\bigr\|_{\dot{\Lambda}_{(\beta-1/p)}} \leq \frac{C(p,\rho)}{\lambda}\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^p} \quad \textit{for } 1/\beta < p < \infty$$

and

$$\bigl\|\Lambda(\mathcal{T}_b, f, \lambda)^{1/\rho}\bigr\|_{BMO} \leq \frac{C(\rho)}{\lambda}\|b\|_{\dot{\Lambda}_\beta}\|f\|_{L^{1/\beta}}.$$

Also, for fixed $0 < \alpha < \gamma$, we consider the number of up-crossing associated with a sequence $\mathcal{T} = \{T_\varepsilon\}_\varepsilon$ applied to a function f at a point x, which is defined by

$$\begin{aligned}N(\mathcal{T}, f, \alpha, \gamma, x) = \sup\bigl\{n \in \mathbf{N} : &\text{there exist } s_1 < t_1 < s_2, t_2 < \cdots < s_n < t_n \\ &\text{such that } T_{s_i} f(x) < \alpha, T_{t_i} f(x) > \gamma \text{ for } i = 1, 2, \ldots, n\bigr\}.\end{aligned} \tag{5.2}$$

It is easy to check that

$$N(\mathcal{T},f,\alpha,\gamma,x) \le \Lambda(\mathcal{T},f,\gamma-\alpha)(x). \tag{5.3}$$

This together with Theorems 5.3 and 5.4 directly leads to the following results.

Theorem 5.5 *Under the same assumptions as in Theorem* 5.3 *or Theorem* 5.4, *we have*

$$\big\|N(\mathcal{T}_{\vec{b}},f,\alpha,\gamma,\cdot)^{1/\rho}\big\|_{L^q(\omega^q)} \le \frac{C(p,q,\rho)}{\gamma-\alpha}\|\vec{b}\|_{\dot\wedge_\beta}\|f\|_{L^p(\omega^q)}$$

or

$$\big\|N(\mathcal{T}_b,f,\alpha,\gamma,\cdot)^{1/\rho}\big\|_{\dot\wedge_{(\beta-1/p)}} \le \frac{C(p,\rho)}{\gamma-\alpha}\|b\|_{\dot\wedge_\beta}\|f\|_{L^p} \quad \textit{for } 1/\beta < p < \infty$$

and

$$\big\|N(\mathcal{T}_b,f,\alpha,\gamma,\cdot)^{1/\rho}\big\|_{BMO} \le \frac{C(\rho)}{\gamma-\alpha}\|b\|_{\dot\wedge_\beta}\|f\|_{L^{1/\beta}}.$$

Finally, by Remark 5.1 and Theorems 5.3-5.5, we have the following.

Theorem 5.6 *Let* $\mathcal{T}=\{\mathcal{T}_\varepsilon\}$ *be either the truncations of the Hilbert transform* $\mathcal{H}=\{H_\varepsilon\}_\varepsilon$ *or the truncations of Hermitian Riesz transforms* $\mathcal{R}^{\pm}=\{\mathcal{R}_\varepsilon^{\pm}\}_\varepsilon$, *and* $\mathcal{T}_{\vec{b}}=\{\mathcal{T}_{\varepsilon,\vec{b}}\}_\varepsilon$, *or* $\mathcal{T}_b=\{\mathcal{T}_{\varepsilon,b}\}$, *the corresponding commutators with* $\vec{b}\in\dot\wedge_{\vec{\beta}}$, *or* $b\in\dot\wedge_\beta$, $0<\beta<1$. *Then the corresponding conclusions of Theorems* 5.3-5.5 *hold*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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