Sun and Shi *Journal of Inequalities and Applications* (2015) 2015:204
DOI 10.1186/s13660-015-0727-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

The existence of a solution to a class of degenerate parabolic variational inequalities

Yudong Sun[1*] and Yimin Shi[2]

[*]Correspondence: yudongsun@yeah.net
[1]School of Science, Guizhou Minzu Uniwersity, Guiyang, Guizhou 550025, China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the degenerate parabolic variational inequality problem in a bounded domain. First, the weak solutions of the variational inequality are defined. Second, the existence of the solutions in the weak sense are proved by using the penalty method and the reduction method.

MSC: 35B40; 35K35

Keywords: parabolic variational inequality; weak solution; penalty method; existence

1 Introduction

In this article, we consider the initial-boundary problem of the following parabolic variational inequality:

$$\begin{cases} u_t - u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u) - \gamma|\nabla u|^p \le 0, & \text{in } \Omega_T, \\ [u_t - u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u) - \gamma|\nabla u|^p]\cdot(u-u_0(x)) = 0, & \text{in } \Omega_T, \\ u(x,t) \le u_0(x), & \text{in } \Omega_T, \\ u(x,0) = u_0(x), & \text{in } \Omega, \\ u(x,t) = 0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1}$$

where $\Omega_T = \Omega\times(0,T)$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with appropriately smooth boundary $\partial\Omega$, $p\ge 2$, $\gamma>0$, and $u_0(x)$ satisfies

$$0 \le u_0 \in C(\bar{\Omega})\cap W_0^{1,p}(\Omega). \tag{2}$$

Readers can refer to [1] and [2] for the motivation and references about the study of problem (1). The linear parabolic variational inequality problem

$$\begin{cases} \frac{\partial}{\partial\tau}V - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V - (r-\frac{1}{2}\sigma^2)\frac{\partial}{\partial x}V + rV \ge 0, & \text{in } \Omega_T, \\ V \ge g(x), & \text{in } \Omega_T, \\ (\frac{\partial}{\partial\tau}V - \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}V - (r-\frac{1}{2}\sigma^2)\frac{\partial}{\partial x}V + rV)(V-g(x)) = 0, & \text{in } \Omega_T, \\ V(t,x) = 0, & \text{on } \partial\Omega_T, \\ V(x,0) = g(x), & \text{in } \Omega, \end{cases} \tag{3}$$

is similar to (1). The existence of solutions to problem (3) was studied in a series of papers (see [3] and [4] and references therein). Here, r and σ are positive constant. In [5], the authors discussed a general case in which the linear parabolic operator with constant coefficients can be replaced by a quasi-linear one with integro-differential terms. Later, the authors in [6] extended the corresponding conclusions to the $\mathbb{R}^d$-values case in which the existence and uniqueness of solution to parabolic variational inequalities with integro-differential terms were proved by using the penalty method and the reduction method.

However, to the best of our knowledge, the existence of solutions to the variational inequality problem with the degenerate parabolic operators has not been studied. The purpose of this paper is to fill this gap.

In the spirit of [3] and [4], we introduce the following maximal monotone graph:

$$G(\lambda) = \begin{cases} 0, & \lambda > 0, \\ [0, +\infty), & \lambda = 0. \end{cases} \tag{4}$$

In addition, we define a function class for the solution as follows:

$$B = \{u \in L^\infty(\Omega_T) \cap L^p(0, T; W_0^{1,p}(\Omega))\}. \tag{5}$$

Based on the above basic knowledge, we define the weak solution of problem (1) below.

Definition 1 A pair $(u, \xi) \in B \times L^\infty(\Omega_T)$ is called a weak solution of problem (1), if
(a) $u(x,t) \leq u_0(x)$,
(b) $u(x,0) = u_0(x)$,
(c) $\xi \in G(u - u_0)$,
(d) $\forall \varphi \in C_0^\infty(\Omega_T)$,

$$\int_{\Omega_T} \left(-u\varphi_t + u|\nabla u|^{p-2}\nabla u \nabla\varphi + (1-\gamma)|\nabla u|^p \varphi\right) \mathrm{d}x\,\mathrm{d}t + \int_\Omega \xi\phi \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{6}$$

(e) $\lim_{t\to\infty} \int_\Omega |u^\mu(x,t) - u_0^\mu(x)|\,\mathrm{d}x = 0$ holds for some $\mu > 0$.

In Section 2, we prove that for $p \geq 2$, $\gamma \in (0,1)$, problem (1) admits a weak solution in the sense of Definition 1. We end the introduction by showing the following lemma which is used to prove our main results (see [7]).

Lemma 2 *Let* $\theta \geq 0$ *and* $A(\eta) = (\eta^2 + \theta)^{\frac{p-2}{2}}\eta$. *Then*

$$[A(\eta) - A(\eta')] \cdot [\eta - \eta'] \geq C|\eta - \eta'|^p, \quad \forall \eta, \eta' \in \mathbb{R}, \tag{7}$$

where C *is a positive constant only depending on* p.

2 The existence of weak solutions

This section is devoted to the proof of the existence of solutions to problem (1). We prove the following theorem.

Theorem 3 *Let* $p \geq 2$ *and* $\gamma \in (0,1)$. *Under the assumption* (2), *problem* (1) *admits a weak solution* u *with* $\frac{\partial u^\mu}{\partial t} \in L^2(\Omega_T)$ *where*

$$\mu = \frac{\gamma p}{2(p-1)} + \frac{1}{2}. \tag{8}$$

To prove Theorem 3, let us consider the approximation problem

$$\begin{cases} (u_\varepsilon)_t - u_\varepsilon \operatorname{div}(|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) - \gamma|\nabla u_\varepsilon|^p - \beta_\varepsilon(u_{0\varepsilon} - u_\varepsilon) = 0, & \text{in } \Omega_T, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x) = u_0(x) + \varepsilon, & \text{on } \Omega, \\ u_\varepsilon(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{9}$$

where $\beta_\varepsilon(\cdot)$ is the penalty function satisfying

$$\begin{aligned} &0 < \varepsilon \le 1, \qquad \beta_\varepsilon(x) \in C^2(R), \qquad \beta_\varepsilon(x) \le 0, \qquad \beta_\varepsilon(0) = -1, \\ &\beta_\varepsilon'(x) \ge 0, \qquad \beta_\varepsilon''(x) \le 0, \qquad \lim_{\varepsilon\to 0} \beta_\varepsilon(x) = \begin{cases} 0, & x > 0, \\ -\infty, & x < 0. \end{cases} \end{aligned} \tag{10}$$

Definition 4 A nonnegative function u_ε is called a weak solution of problem (9), if

(a) $u_\varepsilon \in L^\infty(\Omega_T) \cap L^p(0,T; W_0^{1,p}(\Omega))$,

(b) $\int_{\Omega_T} (u_\varepsilon\varphi_t + u_\varepsilon|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\varphi + (1-\gamma)|\nabla u_\varepsilon|^p\varphi - \beta_\varepsilon(u_{0\varepsilon} - u_\varepsilon)\varphi)\,\mathrm{d}x\,\mathrm{d}t = 0$ hold, for any $\forall\varphi \in C_0^\infty(\Omega_T)$,

(c) $\lim_{t\to\infty} \int_\Omega |u_\varepsilon^\mu(x,t) - u_{0\varepsilon}^\mu(x)|\,\mathrm{d}x = 0$ holds for some $\mu > 0$.

According to the standard theory for parabolic equations [7], problem (9) admits a weak solution

$$u_\varepsilon \in L^\infty(\Omega_T) \cap L^p\bigl(0,T; W_0^{1,p}(\Omega_T)\bigr) \tag{11}$$

in the sense of Definition 4, which satisfies

$$\bigl(u_\varepsilon^\mu\bigr)_t \in L_2(\Omega_T), \qquad \nabla u_\varepsilon \in L_p(\Omega_T). \tag{12}$$

Further, it follows by the comparison principle and the maximum principle [8, 9] that

$$\varepsilon \le u_\varepsilon \le u_{0\varepsilon} \le |u_0|_\infty + \varepsilon, \qquad u_{\varepsilon_1} \le u_{\varepsilon_2} \quad \text{for } \varepsilon_1 \le \varepsilon_2. \tag{13}$$

Moreover, from (12) and (13), we assert that there exists a subsequence ε (still denoted by ε) such that

$$u_\varepsilon \to u \in L^p\bigl(0,T; W_0^{1,p}(\Omega_T)\bigr) \quad \text{as } \varepsilon \to 0, \tag{14}$$

$$u_\varepsilon \ge u \ge 0 \quad \text{for any } \varepsilon > 0. \tag{15}$$

Lemma 5 *Assume* $Q_c^\varepsilon = \{(x,t) \in \Omega_T; u_\varepsilon \ge c, c > 0\}$, $Q_c = \{(x,t) \in \Omega_T; u \ge c, c > 0\}$ *such that, as* $\varepsilon \to 0$,

$$\int_{Q_c^\varepsilon} |\nabla u_\varepsilon - \nabla u|^p\,\mathrm{d}x\,\mathrm{d}t \to 0, \qquad \int_{Q_c} |\nabla u_\varepsilon - \nabla u|^p\,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{16}$$

Proof Choosing $\varphi = u_\varepsilon^{p-2}(u_\varepsilon^2 - \varepsilon^2 - u^2)$ as the test function in

$$\int_{\Omega_T} \bigl(u_\varepsilon\varphi_t + u_\varepsilon|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\nabla\varphi + (1-\gamma)|\nabla u_\varepsilon|^p\varphi - \beta_\varepsilon(u_{0\varepsilon} - u_\varepsilon)\varphi\bigr)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{17}$$

Then it is easy to see that

$$\begin{aligned}
&\int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{p-2}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -\int_{\Omega_T} |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\big\{u_\varepsilon^{p-1}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\big\}\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\gamma\int_{\Omega_T} u_\varepsilon^{p-2}|\nabla u_\varepsilon|^p\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\int_{\Omega_T} u_\varepsilon^{p-2}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\beta_\varepsilon(u_{0\varepsilon}-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad= -\int_{\Omega_T} u_\varepsilon^{p-1}|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \nabla\big(u_\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\int_{\Omega_T} u_\varepsilon^{p-2}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\beta_\varepsilon(u_{0\varepsilon}-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+(\gamma-p+1)\int_{\Omega_T} u_\varepsilon^{p-2}|\nabla u_\varepsilon|^p\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t. \qquad (18)
\end{aligned}$$

From (15) and the definition of β_ε, we derive

$$\int_{\Omega_T} u_\varepsilon^{p-2}|\nabla u_\varepsilon|^p\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t \ge -\varepsilon^2\int_{\Omega_T} u_\varepsilon^{p-2}|\nabla u_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}t, \qquad (19)$$

$$\begin{aligned}
&\int_{\Omega_T} u_\varepsilon^{p-2}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\beta_\varepsilon(u_{0\varepsilon}-u_\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le \big(|u_0|_\infty+\varepsilon\big)^{p-2}\left|\int_{\Omega_T}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t\right|. \qquad (20)
\end{aligned}$$

Observing that$\gamma-p+1<0$, and combing (18), (19), and (20), we have

$$\begin{aligned}
&\int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{p-2}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le -2^{1-p}\int_{\Omega_T}\big|\nabla u_\varepsilon^2\big|^{p-2}\nabla u_\varepsilon^2\nabla\big(u_\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t + p\varepsilon^2\int_{\Omega_T} u_\varepsilon^{p-2}|\nabla u_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\big(|u_0|_\infty+\varepsilon\big)^{p-2}\left|\int_{\Omega_T}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t\right|. \qquad (21)
\end{aligned}$$

Note that $\varepsilon \le u_\varepsilon \le |u_0|_\infty+\varepsilon$. Thus, it follows by the trigonometrical inequality and the Hölder inequality that

$$\begin{aligned}
&\int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{p-2}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\int_{\Omega_T}\big[\big|\nabla u_\varepsilon^2\big|^{p-2}\nabla u_\varepsilon^2-\big|\nabla u^2\big|^{p-2}\nabla u^2\big]\nabla\big(u_\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad\le -2^{1-p}\int_{\Omega_T}\big|\nabla u^2\big|^{p-2}\nabla u^2\nabla\big(u_\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t + p\varepsilon^2\int_{\Omega_T} u_\varepsilon^{p-2}|\nabla u_\varepsilon|^p\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad+\big(|u_0|_\infty+\varepsilon\big)^{p-2}\left|\int_{\Omega_T}\big(u_\varepsilon^2-\varepsilon^2-u^2\big)\,\mathrm{d}x\,\mathrm{d}t\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^{1-p}\left(\int_{\Omega_T}\left|\nabla u^2\right|^p \mathrm{d}x\,\mathrm{d}t\right)^{\frac{p-1}{p}}\left(\int_{\Omega_T}\left|\nabla\left(u_\varepsilon^2-u^2\right)\right|^p \mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} \\
&\quad+\left(|u_0|+\varepsilon\right)^{p-2}\mu\varepsilon^2\int_{\Omega_T}|\nabla u_\varepsilon|^p \mathrm{d}x\,\mathrm{d}t \\
&\quad+\left(|u_0|_\infty+\varepsilon\right)^{p-2}\left|\int_{\Omega_T}\left(u_\varepsilon^2-\varepsilon^2-u^2\right)\mathrm{d}x\,\mathrm{d}t\right|.
\end{aligned}
\tag{22}
$$

Since $\frac{\partial}{\partial t}u_\varepsilon^\mu \in L^2(\Omega_T)$, using the Hölder inequality, one derives

$$
\begin{aligned}
&\int_{\Omega_T}\left|\frac{\partial u_\varepsilon^{p-1}}{\partial t}\right|\mathrm{d}x\,\mathrm{d}t \\
&\quad=(p-1)\int_{\Omega_T}u_\varepsilon^{p-2}\left|\frac{\partial u_\varepsilon}{\partial t}\right|\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq(p-1)\left(|u_0|_\infty+\varepsilon\right)^{p-2}\int_{\Omega_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq(p-1)\left(|u_0|_\infty+\varepsilon\right)^{p-2}\sqrt{|\Omega_T|}\sqrt{\int_{\Omega_T}\left|\frac{\partial u_\varepsilon}{\partial t}\right|^2\mathrm{d}x\,\mathrm{d}t}<\infty.
\end{aligned}
\tag{23}
$$

From the above equation and (13), we may conclude that, as $\varepsilon \to 0$,

$$
\begin{aligned}
&\left|\int_{\Omega_T}\frac{\partial u_\varepsilon}{\partial t}u_\varepsilon^{p-2}\left(u_\varepsilon^2-\varepsilon^2-u^2\right)\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad\leq\frac{1}{p-1}\int_{\Omega_T}\left|\frac{\partial u_\varepsilon^{p-1}}{\partial t}\left(u_\varepsilon^2-\varepsilon^2-u^2\right)\right|\mathrm{d}x\,\mathrm{d}t \\
&\quad\leq\frac{1}{p-1}\sqrt{\int_{\Omega_T}\left(\frac{\partial u_\varepsilon^{p-1}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t}\cdot\sqrt{\int_{\Omega_T}\left(u_\varepsilon^2-\varepsilon^2-u^2\right)^2\mathrm{d}x\,\mathrm{d}t} \\
&\quad\leq\frac{1}{p-1}\varepsilon^2\sqrt{|\Omega_T|}\sqrt{\int_{\Omega_T}\left(\frac{\partial u_\varepsilon^{p-1}}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t}\to 0,
\end{aligned}
\tag{24}
$$

$$
\left(|u_0|_\infty+\varepsilon\right)^{p-2}\left|\int_{\Omega_T}\left(u_\varepsilon^2-\varepsilon^2-u^2\right)\mathrm{d}x\,\mathrm{d}t\right|\to 0.
\tag{25}
$$

Substituting (24) and (25) into (22) yields

$$
\limsup_{\varepsilon\to 0}\int_{\Omega_T}\left[\left|\nabla u_\varepsilon^2\right|^{p-2}\nabla u_\varepsilon^2-\left|\nabla u^2\right|^{p-2}\nabla u^2\right]\nabla\left(u_\varepsilon^2-u^2\right)\mathrm{d}x\,\mathrm{d}t\leq 0.
\tag{26}
$$

This and Lemma 2 lead to

$$
\lim_{\varepsilon\to 0}\int_{\Omega_T}\left|\nabla u_\varepsilon^2-\nabla u^2\right|^p\mathrm{d}x\,\mathrm{d}t=0.
\tag{27}
$$

Hence, in view of

$$
\nabla u_\varepsilon^2-\nabla u^2=2u_\varepsilon\nabla u_\varepsilon-2u\nabla u=2u_\varepsilon(\nabla u_\varepsilon-\nabla u)+2\nabla u(u_\varepsilon-u),
\tag{28}
$$

$$
\nabla u_\varepsilon^2-\nabla u^2=2u_\varepsilon\nabla u_\varepsilon-2u\nabla u=2u(\nabla u_\varepsilon-\nabla u)+2\nabla u_\varepsilon(u_\varepsilon-u),
\tag{29}
$$

we derive

$$\begin{aligned} &2^p \int_{\Omega_T} u_\varepsilon^p |\nabla u_\varepsilon - \nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le 2^p \int_{\Omega_T} \left|\nabla u_\varepsilon^2 - \nabla u^2\right|^p \,\mathrm{d}x\,\mathrm{d}t + 4^p \int_{\Omega_T} |\nabla u|^p |u_\varepsilon - u|^p \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad (\varepsilon \to 0), \end{aligned} \tag{30}$$

$$\begin{aligned} &2^p \int_{\Omega_T} u^p |\nabla u_\varepsilon - \nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\quad \le 2^p \int_{\Omega_T} \left|\nabla u_\varepsilon^2 - \nabla u^2\right|^p \,\mathrm{d}x\,\mathrm{d}t + 4^p \int_{\Omega_T} |\nabla u_\varepsilon|^p |u_\varepsilon - u|^p \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad (\varepsilon \to 0). \end{aligned} \tag{31}$$

Note that $Q_c^\varepsilon \subset \Omega_T$, $Q_c \subset \Omega_T$. Then it is easy to see that as $\varepsilon \to 0$,

$$c^p \int_{Q_c^\varepsilon} |\nabla u_\varepsilon - \nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} u_\varepsilon^p |\nabla u_\varepsilon - \nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \to 0, \tag{32}$$

$$c^p \int_{Q_c} |\nabla u_\varepsilon - \nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \le \int_{\Omega_T} u^p |\nabla u_\varepsilon - \nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \to 0. \tag{33}$$

Thus, the lemma is proved. □

Lemma 6 *The solution of* (9) *satisfies*

$$\int_{\Omega_T} |\nabla u_\varepsilon|^p u_\varepsilon^{-\alpha} \,\mathrm{d}x\,\mathrm{d}t \le C, \tag{34}$$

where C is independent of ε, $\alpha \in [0, 1-\gamma)$.

Proof Multiply (9) by $u_\varepsilon^{-\alpha}$ and integrate both sides of the equation over Ω_T. After integrating by parts, we obtain

$$\begin{aligned} &\int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{-\alpha} \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{\Omega_T} u_\varepsilon^{1-\alpha} \operatorname{div}\left\{|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon\right\} + \gamma u_\varepsilon^{-\alpha} |\nabla u_\varepsilon|^p + u_\varepsilon^{-\alpha} \beta_\varepsilon(u_{0\varepsilon} - u_\varepsilon) \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T \mathrm{d}t \int_{\partial\Omega} \left\{ u_\varepsilon^{1-\alpha} |\nabla u_\varepsilon|^{p-2} \frac{\partial u_\varepsilon}{\partial \nu} \right\} \mathrm{d}x - (1-\alpha-\gamma) \int_{\Omega_T} u_\varepsilon^{-\alpha} |\nabla u_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\Omega_T} u_\varepsilon^{-\alpha} \beta_\varepsilon(u_{0\varepsilon} - u_\varepsilon) \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{35}$$

where ν denotes the outward normal to $\partial\Omega \times (0, T)$.

Further, putting together (10) and (13) implies

$$\int_{\Omega_T} u_\varepsilon^{-\alpha} \beta_\varepsilon(u_{0\varepsilon} - u_\varepsilon) \,\mathrm{d}x\,\mathrm{d}t \le 0. \tag{36}$$

Since $u_\varepsilon \ge \varepsilon$, we have

$$\frac{\partial u_\varepsilon}{\partial \nu} \le 0, \quad \text{on } \partial\Omega \times (0, T). \tag{37}$$

This leads to

$$\int_0^T \int_{\partial\Omega} \left\{ u_\varepsilon^{1-\alpha} |\nabla u_\varepsilon|^{p-2} \frac{\partial u_\varepsilon}{\partial \nu} \right\} \mathrm{d}x\,\mathrm{d}t \le 0. \tag{38}$$

Now, we drop the non-positive terms (36) and (38) in (35) to get

$$\int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{-\alpha} \,\mathrm{d}x\,\mathrm{d}t \le -(1-\alpha-\gamma) \int_{\Omega_T} u_\varepsilon^{-\alpha} |\nabla u_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t. \tag{39}$$

Clearly, using integration by parts, we derive

$$\int_{\Omega_T} \frac{\partial u_\varepsilon}{\partial t} u_\varepsilon^{-\alpha} \,\mathrm{d}x\,\mathrm{d}t = \frac{1}{1-\alpha} \int_{\Omega} u_\varepsilon^{1-\alpha}(x,T) - u_\varepsilon^{1-\alpha}(x,0) \,\mathrm{d}x. \tag{40}$$

This and (39) lead to

$$\int_{\Omega_T} |\nabla u_\varepsilon|^p u_\varepsilon^{-\alpha} \,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{(1-\alpha-\gamma)(1-\alpha)} \int_{\Omega} u_\varepsilon^{-\alpha}(x,0) \,\mathrm{d}x \le C_1, \tag{41}$$

where $C_1 > 0$ depending on α, γ, Ω, and $|u_0|$. Hence, the proof is completed. □

Lemma 7 *As $\varepsilon \to 0$, we have*

$$\int_{\Omega_T} \big| |\nabla u_\varepsilon|^p - |\nabla u|^p \big| \,\mathrm{d}x\,\mathrm{d}t \to 0, \tag{42}$$

$$\int_{\Omega_T} \big| u_\varepsilon |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - u |\nabla u|^p \nabla u \big| \,\mathrm{d}x\,\mathrm{d}t \to 0, \tag{43}$$

$$\beta_\varepsilon(u_{0\varepsilon} - u_\varepsilon) \to \xi \in G(u_0 - u). \tag{44}$$

Proof Let χ_η and $\chi_\eta^{(\varepsilon)}$ be the characteristic functions of $\{(x,t) \in \Omega_T; u(x,t) < \eta\}$ and $\{(x,t) \in \Omega_T; u_\varepsilon(x,t) < \eta\}$, respectively. Since $u_\varepsilon \to u$, as $\varepsilon \to 0$, $\chi_\eta \le \chi_\eta^{(\varepsilon)}$, we have

$$\begin{aligned}
&\int_{\Omega_T} \big| |\nabla u_\varepsilon|^p - |\nabla u|^p \big| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_{\Omega_T} \big| |\nabla u|^p \chi_\eta - |\nabla u_\varepsilon|^p \chi_\eta^{(\varepsilon)} \big| \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \big| |\nabla u|^p (1-\chi_\eta) - |\nabla u_\varepsilon|^p \big(1-\chi_\eta^{(\varepsilon)}\big) \big| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_{\Omega_T} |\nabla u_\varepsilon|^p \chi_\eta^{(\varepsilon)} \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} |\nabla u|^p \chi_\eta \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_{\Omega_T} |\nabla u|^p \big(\chi_\eta^{(\varepsilon)} - \chi_\eta\big) \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \big| |\nabla u_\varepsilon|^p - |\nabla u|^p \big| \big(1-\chi_\eta^{(\varepsilon)}\big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad = H_1 + H_2 + H_3 + H_4.
\end{aligned} \tag{45}$$

Taking $\alpha = (1-\gamma)/2$ in Lemma 6 one obtains

$$\begin{aligned}
H_1 &= \int_{\Omega_T} |\nabla u_\varepsilon|^p \frac{u_\varepsilon^\alpha}{u_\varepsilon^\alpha} \chi_\eta^{(\varepsilon)} \,\mathrm{d}x\,\mathrm{d}t \\
&\le \eta^\alpha \int_{\Omega_T} |\nabla u_\varepsilon|^p u_\varepsilon^{-\alpha} \chi_\eta^{(\varepsilon)} \,\mathrm{d}x\,\mathrm{d}t \\
&\le \eta^\alpha \int_{\Omega_T} |\nabla u_\varepsilon|^p u_\varepsilon^{-\alpha} \,\mathrm{d}x\,\mathrm{d}t \le C\eta^\alpha \to 0 \quad (\eta \to 0).
\end{aligned} \tag{46}$$

Applying Lemma 5, (46), and the fact that $\chi_\eta \le \chi_\eta^{(\varepsilon)}$ implies

$$\begin{aligned} H_2 &\le \int_{\Omega_T} \chi_\eta^{(\varepsilon)} |\nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_{\Omega_T} \chi_\eta^{(\varepsilon)} |\nabla u_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} \chi_\eta^{(\varepsilon)} |\nabla u_\varepsilon - \nabla u|^p \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad (\eta \to 0), \end{aligned} \tag{47}$$

$$H_4 \to 0 \quad (\eta \to 0). \tag{48}$$

For fixed $\eta > 0$, $\chi_\eta^{(\varepsilon)} \to \chi_\eta$ $(\varepsilon \to 0)$ a.e. in Ω_T, so

$$H_3 \to 0 \quad (\eta \to 0). \tag{49}$$

Putting together (46), (47), (48), and (49), we have

$$\int_{\Omega_T} \bigl| |\nabla u_\varepsilon|^p - |\nabla u|^p \bigr| \,\mathrm{d}x\,\mathrm{d}t \to 0 \quad (\eta \to 0). \tag{50}$$

Thus (42) holds.

Next we prove (43). It follows by the trigonometrical inequality that

$$\begin{aligned} &\int_{\Omega_T} \bigl| u_\varepsilon |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon - u |\nabla u|^{p-2} \nabla u \bigr| \,\mathrm{d}x\,\mathrm{d}t \\ &\quad\le \int_{\Omega_T} |u_\varepsilon - u| |\nabla u_\varepsilon|^{p-1} \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_T} u |\nabla u_\varepsilon|^{p-2} |\nabla u_\varepsilon - \nabla u| \,\mathrm{d}x\,\mathrm{d}t \\ &\qquad + \int_{\Omega_T} u |\nabla u| \cdot \bigl| |\nabla u_\varepsilon|^{p-2} - |\nabla u|^{p-2} \bigr| \,\mathrm{d}x\,\mathrm{d}t \\ &\quad= H_5 + H_6 + H_7. \end{aligned} \tag{51}$$

Using the Hölder inequality and Lemma 6, we obtain

$$H_5 \le C \biggl(\int_{\Omega_T} |u_\varepsilon - u|^p \,\mathrm{d}x\,\mathrm{d}t \biggr)^{\frac{1}{p}} \to 0 \quad \text{as } \varepsilon \to 0. \tag{52}$$

With the inequality $|a^r - b^r| \le |a - b|^r$ $(r \in [0,1], a, b \ge 0)$, the Hölder inequality, and (42), we have

$$\begin{aligned} H_7 &= \int_{\Omega_T} u |\nabla u| \cdot \bigl| |\nabla u_\varepsilon|^{p-2} - |\nabla u|^{p-2} \bigr| \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\Omega_T} u |\nabla u| \cdot \Bigl(\bigl(|\nabla u_\varepsilon|^p \bigr)^{\frac{p-2}{p}} - \bigl(|\nabla u|^p \bigr)^{\frac{p-2}{p}} \Bigr) \,\mathrm{d}x\,\mathrm{d}t \\ &\le C \int_{\Omega_T} |\nabla u| \cdot \bigl| |\nabla u_\varepsilon|^p - |\nabla u|^p \bigr|^{\frac{p-2}{p}} \,\mathrm{d}x\,\mathrm{d}t \\ &\le C \biggl(\int_{\Omega_T} |\nabla u| \cdot \bigl| |\nabla u_\varepsilon|^p - |\nabla u|^p \bigr| \,\mathrm{d}x\,\mathrm{d}t \biggr)^{\frac{p-2}{p}} \\ &\qquad \times \biggl(\int_{\Omega_T} |\nabla u|^{\frac{p}{2}} \,\mathrm{d}x\,\mathrm{d}t \biggr)^{\frac{2}{p}} \to 0 \quad (\varepsilon \to 0). \end{aligned} \tag{53}$$

Finally, we estimate H_6. Again by using trigonometrical inequality, we arrive at

$$\begin{aligned} H_6 &= \int_{\Omega_T} u|\nabla u_\varepsilon|^{p-2} \cdot |\nabla u_\varepsilon - \nabla u|\chi_\eta \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{\Omega_T} u|\nabla u_\varepsilon|^{p-2} \cdot |\nabla u_\varepsilon - \nabla u|(1-\chi_\eta) \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \eta \int_{\Omega_T} |\nabla u_\varepsilon|^{p-2} \cdot |\nabla u_\varepsilon - \nabla u|\chi_\rho \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + C\left(\int_{\Omega_T} |\nabla u_\varepsilon|^{(p-2)p/(p-1)} \,\mathrm{d}x\,\mathrm{d}t\right)^{p/(p-1)} \left(\int_{\Omega_T} |\nabla u_\varepsilon - \nabla u|^p (1-\chi_\rho) \,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \\ &= \eta \int_{\Omega_T} |\nabla u_\varepsilon|^{p-2} \cdot |\nabla u_\varepsilon - \nabla u|\chi_\rho \,\mathrm{d}x\,\mathrm{d}t \\ &\quad + C\left(\int_{\Omega_T} |\nabla u_\varepsilon - \nabla u|^p (1-\chi_\rho) \,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} \\ &= H_8 + H_9. \end{aligned} \tag{54}$$

For all $\delta > 0$ and $\varepsilon \in (0,1)$, let η be small enough and use Lemma 5 such that, as $\varepsilon \to 0$,

$$\begin{aligned} H_8 &\leq \left(\int_{\Omega_T} |\nabla u_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p-2}{p}} \left(\int_{\Omega_T} |\nabla u_\varepsilon - \nabla u|^{\frac{p}{2}} \chi_\rho^{\frac{2}{p}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{p}} \\ &\leq |\Omega_T|^{\frac{1}{p}} \left(\int_{\Omega_T} |\nabla u_\varepsilon|^p \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{p-2}{p}} \left(\int_{\Omega_T} |\nabla u_\varepsilon - \nabla u|^p \chi_\rho \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}} \to 0. \end{aligned} \tag{55}$$

Clearly, for fixed $\eta > 0$, using Lemma 5, we have

$$H_9 \to 0 \quad \text{as } \varepsilon \to 0. \tag{56}$$

Substituting (55) and (56) into (54), we obtain

$$H_6 \to 0 \quad \text{as } \varepsilon \to 0. \tag{57}$$

Hence, (43) is proved by putting together (52), (53), and (57).

Finally we prove (44). Using (13) and the definition of β_ε, we have

$$\beta_\varepsilon(u_\varepsilon - u_{0\varepsilon}) \to \xi \quad \text{as } \varepsilon \to 0. \tag{58}$$

Now we prove $\xi \in G(u_0 - u)$. According to the definition of $G(\cdot)$, we only need to prove that if $u(x_0, t_0) < u_0(x_0)$, $\xi(x_0, t_0) = 0$. In fact, if $u(x_0, t_0) < u_0(x)$, there exist a constant $\lambda > 0$ and a δ neighborhood $B_\delta(x_0, t_0)$ such that, if ε is small enough, we have

$$u_\varepsilon(x,t) \leq u_{0\varepsilon}(x) - \lambda, \quad \forall (x,t) \in B_\delta(x_0, t_0). \tag{59}$$

Thus, if ε is small enough, we have

$$0 \geq \beta_\varepsilon(u_{0\varepsilon} - u_\varepsilon) \geq \beta_\varepsilon(\lambda) = 0, \quad \forall (x,t) \in B_\delta(x_0, t_0). \tag{60}$$

Furthermore, it follows by $\varepsilon \to 0$ that

$$\xi(x,t) = 0, \quad \forall (x,t) \in B_\delta(x_0,t_0). \tag{61}$$

Hence, (44) holds, and the proof of Lemma 7 is completed. □

With Lemma 7 and (14), it is easy to check that u satisfies (c) and (d) in Definition 1. Moreover, applying (13), it is clear that

$$u(x,t) \le u_0(x), \quad \text{in } \Omega_T, \qquad u(x,0) = u_0(x), \quad \text{in } \Omega. \tag{62}$$

Thus (a) and (b) hold.

To show the existence of problem (1), we only need to prove that (e) holds. Define

$$I = \int_\Omega \left| u_\varepsilon^\mu - u_{0\varepsilon}^\mu \right| \,\mathrm{d}x. \tag{63}$$

Applying the Hölder inequality twice, we obtain

$$\begin{aligned} I &= \int_\Omega \left| u_\varepsilon^\mu(x,t) - u_{0\varepsilon}^\mu(x) \right| \,\mathrm{d}x = \int_\Omega \left| \int_0^t \frac{\partial}{\partial s} u_\varepsilon^\mu \,\mathrm{d}x \right| \,\mathrm{d}x \\ &\le \sqrt{t} \int_\Omega \left| \sqrt{\int_0^t \left(\frac{\partial u_\varepsilon^\mu}{\partial s} \right)^2 \mathrm{d}t} \right| \,\mathrm{d}x \le |\Omega|^{\frac{1}{2}} \sqrt{t} \int_\Omega \int_0^t \left(\frac{\partial u_\varepsilon^\mu}{\partial s} \right)^2 \mathrm{d}t \,\mathrm{d}x \\ &\le \sqrt{t} |\Omega|^{\frac{1}{2}} \,\mathrm{d}x \,\mathrm{d}t. \end{aligned} \tag{64}$$

It follows by (12) that

$$\int_\Omega \left| u_\varepsilon^\mu(x,t) - u_{0\varepsilon}^\mu(x) \right| \,\mathrm{d}x \le C\sqrt{t}, \tag{65}$$

where C is independent of ε. Using (14) and letting $\varepsilon \to 0$ yields

$$\int_\Omega \left| u^\mu(x,t) - u_0^\mu(x) \right| \,\mathrm{d}x \le C\sqrt{t}. \tag{66}$$

So

$$\int_\Omega \left| u^\mu(x,t) - u_0^\mu(x) \right| \,\mathrm{d}x \to 0 \quad \text{as } t \to 0. \tag{67}$$

Thus the proof of existence is completed.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this paper and they read and approved the final manuscript.

Author details
[1]School of Science, Guizhou Minzu Uniwersity, Guiyang, Guizhou 550025, China. [2]School of Science, Northwestern Polytechnical University, Xi'an, 710072, China.

Acknowledgements
This work was supported by National Nature Science Foundation of China (Grant Nos. 71171164, 11426176) and the Doctorate Foundation of Northwestern Polytechnical University (Grant No. CX201235). The authors are sincerely grateful to the referee and the Associate Editor handling the paper for their valuable comments.

Received: 21 November 2014 Accepted: 2 June 2015 Published online: 18 June 2015

References

1. Chen, X, Yi, F, Wang, L: American lookback option with fixed strike price 2-D parabolic variational inequality. J. Differ. Equ. **251**, 3063-3089 (2011)
2. Zhou, Y, Yi, F: A free boundary problem arising from pricing convertible bond. Appl. Anal. **89**, 307-323 (2010)
3. Chen, X, Yi, F: Parabolic variational inequality with parameter and gradient constraints. J. Math. Anal. Appl. **385**, 928-946 (2012)
4. Zhou, Y, Yi, F: A variational inequality arising from American installment call options pricing. J. Math. Anal. Appl. **357**, 54-68 (2009)
5. Sun, Y, Shi, Y, Gu, X: An integro-differential parabolic variational inequality arising from the valuation of double barrier American option. J. Syst. Sci. Complex. **27**, 276-288 (2014)
6. Sun, Y, Shi, Y, Wu, M: Second-order integro-differential parabolic variational inequalities arising from the valuation of American option. J. Inequal. Appl. **2014**, 8 (2014)
7. Zhou, W, Wu, Z: Some results on a class of degenerate parabolic equations not in divergence form. Nonlinear Anal. TMA **60**, 863-886 (2005)
8. Ladyzenskaja, OA, Solonnikov, VA, Ural'ceva, NN: Linear and Quasilinear Equations of Parabolic Type. Translations of Mathematical Monographs, vol. 23. Am. Math. Soc., Providence (1968)
9. Dibenedetto, E: Degenerate Parabolic Equations. Springer, New York (2010)