Tuo *Journal of Inequalities and Applications* (2015) 2015:201
DOI 10.1186/s13660-015-0719-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Generalizations of Cauchy-Schwarz inequality in unitary spaces

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Abstract
In this paper, we give a generalization of Cauchy-Schwarz inequality in unitary spaces and obtain its integral analogs. As an application, we establish an inequality for covariances.

MSC: 26D15; 62J10

Keywords: Cauchy-Schwarz inequality; unitary space; positive measure

1 Introduction

Let u and v be two vectors in a unitary space $\mathbb{H}$. The Cauchy-Schwarz inequality is well known,

$$\bigl|\langle u,v\rangle\bigr| \leq \|u\| \cdot \|v\|, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the inner product and norm in $\mathbb{H}$, respectively. Its integral form in the space of real-valued functions $L^2[a,b]$ is

$$\left(\int_a^b f\cdot g\,d\mu\right)^2 \leq \left(\int_a^b f^2\,d\mu\right)\left(\int_a^b g^2\,d\mu\right). \tag{1.2}$$

The Cauchy-Schwarz inequality is one of the most important inequalities in mathematics. To date, a large number of generalizations and refinements of the inequalities (1.1) and (1.2) have been investigated in the literature (see [1] and references therein, also see [2–9]).

In this note, we will present some new generalizations of the Cauchy-Schwarz inequality (1.1).

Suppose that $\mathbb{H}$ is a unitary space (complex inner product space) with standard inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, namely $\langle x,y\rangle = x^T\overline{y}$ and $\|x\| = \sqrt{\langle x,x\rangle}$ (see [10]). Let $X = (x_1,x_2,\dots,x_n)$ denote the n-tuple of vectors $x_i \in \mathbb{H}$, $i = 1,\dots,n$. For two n-tuples $X = (x_1,\dots,x_n)$ and $Y = (y_1,\dots,y_n)$ of $\mathbb{H}$, we define this A-product of vector x_i and y_i for X and Y by

$$x_i \otimes_A y_i = \langle x_i,y_i\rangle - \langle x_i,b\rangle - \langle a,y_i\rangle,$$

where $a = \frac{x_1+\cdots+x_n}{n}$ and $b = \frac{y_1+\cdots+y_n}{n}$.

Our main results are the following theorems.

Theorem 1 *Let $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ be two n-tuples of the unitary space $\mathbb{H}$, then*

$$\sum_{i=1}^{n} |x_i \otimes_A y_i| \le \left(\sum_{i=1}^{n} \|x_i\|^2 \right)^{1/2} \left(\sum_{i=1}^{n} \|y_i\|^2 \right)^{1/2}. \tag{1.3}$$

Equality holds if $y_i = (x_i - 2a)\lambda$ $(i = 1, \dots, n)$ for any $\{x_1, \dots, x_n\}$, where λ is a non-negative constant.

In particular, if $n = 1$, then (1.3) is the Cauchy-Schwarz inequality (1.1).
For complex numbers $\mathbb{C}$, by Theorem 1, we have the following.

Corollary 1 *Suppose that $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are complex numbers. Set*

$$a = \frac{x_1 + \cdots + x_n}{n}, \qquad b = \frac{y_1 + \cdots + y_n}{n},$$

then

$$\left(\sum_{i=1}^{n} |x_i y_i - a y_i - b x_i| \right)^2 \le \left(\sum_{i=1}^{n} |x_i|^2 \right) \left(\sum_{i=}^{n} |y_i|^2 \right).$$

Equality holds if $y_i = (x_i - 2a)\lambda$ $(i = 1, \dots, n)$ for any $\{x_1, \dots, x_n\}$, where λ is a non-negative constant.

Let $H \oplus H \oplus \cdots \oplus H$ denote the direct sum of n unitary space $\mathbb{H}$ with norm $\|X\| = (\sum_{i=1}^{n} \|x_i\|^2)^{\frac{1}{2}}$. Set $f(X, Y) = \sum_{i=1}^{n} x_i \otimes_A y_i$. Since $f(X, X)$ is not always non-negative, $f(X, Y)$ is not an inner product in the above direct sum. Hence, (1.3) is different from the Cauchy-Schwarz inequality in the above direct sum.

If we set $|X \otimes_A Y| = \sum_{i=1}^{n} |x_i \otimes_A y_i|$, then (1.3) can be restated as

$$|X \otimes_A Y| \le \|X\| \cdot \|Y\|.$$

Furthermore, we obtain the following integral form of (1.3) (only consider real-valued functions).

Theorem 2 *Let μ be a positive measure such that $\mu(\Omega) = 1$, f and g be real-valued functions in $L^2(\mu)$, and let*

$$f \otimes_A g(x) = f(x) \cdot g(x) - f(x) \cdot \int_\Omega g \, d\mu - g(x) \cdot \int_\Omega f \, d\mu,$$

then

$$\left(\int_\Omega |f \otimes_A g| \, d\mu \right)^2 \le \left(\int_\Omega f^2 \, d\mu \right) \left(\int_\Omega g^2 \, d\mu \right). \tag{1.4}$$

Equality holds if $g(x) = (f(x) - 2 \int_\Omega f \, d\mu)\lambda$, where λ is a non-negative constant.

2 The proofs of the theorems

Proof of Theorem 1 Using the basic properties of the norm of a unitary space, we get

$$\begin{aligned}\sum_{i=1}^{n}\|y_i-2b\|^2 &= \sum_{i=1}^{n}\langle y_i-2b, y_i-2b\rangle \\ &= \sum_{i=1}^{n}\bigl(4\|b\|^2-2\langle y_i,b\rangle-2\langle b,y_i\rangle+\|y_i\|^2\bigr) \\ &= 4n\|b\|^2-2\left\langle\sum_{i=1}^{n}y_i,b\right\rangle-2\left\langle b,\sum_{i=1}^{n}y_i\right\rangle+\sum_{i=1}^{n}\|y_i\|^2 \\ &= 4n\|b\|^2-2n\|b\|^2-2n\|b\|^2+\sum_{i=1}^{n}\|y_i\|^2 \\ &= \sum_{i=1}^{n}\|y_i\|^2. \end{aligned} \tag{2.1}$$

By (2.1), using the Cauchy-Schwarz inequality (1.1) and the discrete form of the Cauchy-Schwarz inequality, it follows that

$$\begin{aligned}\sum_{i=1}^{n}\bigl|\langle x_i,y_i-2b\rangle\bigr| &\le \sum_{i=1}^{n}\|x_i\|\cdot\|y_i-2b\| \\ &\le \left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\|y_i-2b\|^2\right)^{\frac{1}{2}} \\ &= \left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\|y_i\|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{2.2}$$

Similarly to (2.2), we have

$$\sum_{i=1}^{n}\bigl|\langle x_i-2a,y_i\rangle\bigr| \le \left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\|y_i\|^2\right)^{\frac{1}{2}}. \tag{2.3}$$

Combining (2.2) and (2.3), we infer that

$$\begin{aligned}\sum_{i=1}^{n}|x_i\otimes_A y_i| &= \frac{1}{2}\sum_{i=1}^{n}\bigl|2\langle x_i,y_i\rangle-\langle x_i,2b\rangle-\langle 2a,y_i\rangle\bigr| \\ &= \frac{1}{2}\sum_{i=1}^{n}\bigl|\langle x_i,y_i-2b\rangle+\langle x_i-2a,y_i\rangle\bigr| \\ &\le \frac{1}{2}\sum_{i=1}^{n}\bigl|\langle x_i,y_i-2b\rangle\bigr|+\frac{1}{2}\sum_{i=1}^{n}\bigl|\langle x_i-2a,y_i\rangle\bigr| \\ &\le \left(\sum_{i=1}^{n}\|x_i\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}\|y_i\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

This is the inequality (1.3), as desired. □

Proof of Theorem 2 We first prove the following inequality:

$$\int_\Omega \left| f\left(g - 2\int_\Omega g\,d\mu\right)\right| d\mu \le \left(\int_\Omega f^2\,d\mu\right)^{\frac{1}{2}} \left(\int_\Omega g^2\,d\mu\right)^{\frac{1}{2}}. \tag{2.4}$$

In fact, by the Cauchy-Schwarz inequality (1.2), we obtain

$$\begin{aligned}
&\int_\Omega \left| f\left(g - 2\int_\Omega g\,d\mu\right)\right| d\mu \\
&\quad\le \left(\int_\Omega f^2\,d\mu\right)^{\frac{1}{2}} \left(\int_\Omega \left(g - 2\int_\Omega g\,d\mu\right)^2 d\mu\right)^{\frac{1}{2}} \\
&\quad= \left(\int_\Omega f^2\,d\mu\right)^{\frac{1}{2}} \cdot \left(\int_\Omega \left(4\left(\int_\Omega g\,d\mu\right)^2 - 4\left(\int_\Omega g\,d\mu\right)\cdot g + g^2\right) d\mu\right)^{\frac{1}{2}} \\
&\quad= \left(\int_\Omega f^2\,d\mu\right)^{\frac{1}{2}} \cdot \left(4\left(\int_\Omega g\,d\mu\right)^2 \mu(\Omega) - 4\left(\int_\Omega g\,d\mu\right)^2 + \int_\Omega g^2\,d\mu\right)^{\frac{1}{2}} \\
&\quad= \left(\int_\Omega f^2\,d\mu\right)^{\frac{1}{2}} \left(\int_\Omega g^2\,d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

This is the inequality (2.4).

Similarly, we have

$$\int_\Omega \left| g\left(f - 2\int_\Omega f\,d\mu\right)\right| d\mu \le \left(\int_\Omega f^2\,d\mu\right)^{\frac{1}{2}} \left(\int_\Omega g^2\,d\mu\right)^{\frac{1}{2}}. \tag{2.5}$$

From (2.4) and (2.5), we find that

$$\begin{aligned}
&\int_\Omega |f \otimes_A g|\,d\mu \\
&\quad= \frac{1}{2}\int_\Omega \left| \left(g - 2\int_\Omega g\,d\mu\right) f + \left(f - 2\int_\Omega f\,d\mu\right) g \right| d\mu \\
&\quad\le \frac{1}{2}\left(\int_\Omega \left| f\left(g - 2\int_\Omega g\,d\mu\right)\right| d\mu + \int_\Omega \left| g\left(f - 2\int_\Omega f\,d\mu\right)\right| d\mu\right) \\
&\quad\le \left(\int_\Omega f^2\,d\mu\right)^{\frac{1}{2}} \left(\int_\Omega g^2\,d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

The inequality (1.4) follows. □

3 An application

Let $(a_1, b_1), \ldots, (a_n, b_n)$ be n items of bivariate real data, $x = \{a_1, \ldots, a_n\}$ and $y = \{b_1, \ldots, b_n\}$, then their covariance $\operatorname{Cov}(x, y)$ is defined as [11]

$$\operatorname{Cov}(x, y) = \frac{1}{n}\sum_{i=1}^{n}(a_i - a)(b_i - b),$$

where $a = \frac{a_1 + \cdots + a_n}{n}$ and $b = \frac{b_1 + \cdots + b_n}{n}$.

For the covariance $\mathrm{Cov}(x,y)$, it is well known that Pearson's product moment inequality is

$$\bigl|\mathrm{Cov}(x,y)\bigr| \le \mathrm{SD}(x)\cdot \mathrm{SD}(y),$$

where $\mathrm{SD}(x) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(a_i - a)^2}$ and $\mathrm{SD}(y) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(b_i - b)^2}$.

Similarly, now we define this covariance of two n-tuples $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ of the unitary space $\mathbb{H}$ as

$$\mathrm{Cov}(X,Y) = \frac{1}{n}\sum_{i=1}^{n}\langle x_i - a, y_i - b\rangle,$$

where $a = \frac{x_1+\cdots+x_n}{n}$ and $b = \frac{y_1+\cdots+y_n}{n}$.

Set $\alpha_i = x_i - a$ and $\beta_i = y_i - b$, $i = 1, \ldots, n$. Note that

$$\begin{aligned} x_i \otimes_A y_i &= \langle \alpha_i + a, \beta_i + b\rangle - \langle \alpha_i + a, b\rangle - \langle a, \beta_i + b\rangle \\ &= \langle \alpha_i, \beta_i\rangle - \langle a, b\rangle \\ &= \langle x_i - a, y_i - b\rangle - \langle a, b\rangle. \end{aligned}$$

Hence, (1.3) can be written in the following form:

$$\sum_{i=1}^{n}\bigl|\langle x_i - a, y_i - b\rangle - \langle a, b\rangle\bigr| \le \|X\|\cdot\|Y\|, \tag{3.1}$$

where $\|X\| = (\sum_{i=1}^{n}\|x_i\|^2)^{1/2}$ and $\|Y\| = (\sum_{i=1}^{n}\|y_i\|^2)^{1/2}$.

Using the triangle inequality on the left side of (3.1), we obtain

$$\bigl|n\,\mathrm{Cov}(X,Y) - n\langle a, b\rangle\bigr| \le \|X\|\cdot\|Y\|.$$

Finally, we can simply state the above result, as follows.

Theorem 3 *Let $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two n-tuples of the unitary space $\mathbb{H}$, then*

$$\bigl|\mathrm{Cov}(X,Y) - \langle a, b\rangle\bigr| \le \frac{1}{n}\|X\|\cdot\|Y\|. \tag{3.2}$$

Competing interests
The author declares to have no competing interests.

Acknowledgements
This work was partially supported by the National Natural Science Foundation of China (Grant No. 11171053).

Received: 10 December 2014 Accepted: 28 May 2015 Published online: 17 June 2015

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