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Sharpened versions of the Erdös-Mordell inequality

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Abstract
In this paper, we present two sharpened versions of the Erdös-Mordell inequality and extend them to the cases with one parameter. As applications of our results, the Walker inequality and a new inequality in non-obtuse triangles are obtained. We also propose three interesting conjectures as open problems.

MSC: 51M16

Keywords: Erdös-Mordell inequality; triangle; interior point; Walker's inequality

1 Introduction

Throughout this paper, let ABC be a triangle and P be its interior point. Denote the distances from P to the vertices A, B, C by R_1, R_2, R_3, and the distances from P to the sides BC, CA, AB by r_1, r_2, r_3, respectively. The famous Erdös-Mordell inequality [1], p.313 states that

$$R_1 + R_2 + R_3 \geq 2(r_1 + r_2 + r_3), \tag{1.1}$$

with equality holding if and only if the triangle ABC is equilateral and P is its center.

Many authors have given proofs for this inequality by using different tools; see, for example, [2–7]. On the other hand, this inequality has been extended in various directions, we refer the reader to [1, 8–11]. Some other related results can be found in several papers; see [12–20] and references therein.

In [21], to prove Oppenheim's inequality [12] (see also [22], inequality 12.22),

$$R_2R_3 + R_3R_1 + R_1R_2 \geq (r_3 + r_1)(r_1 + r_2) + (r_1 + r_2)(r_2 + r_3) + (r_2 + r_3)(r_3 + r_1), \tag{1.2}$$

the author presented the following new inequality as a lemma:

$$R_2 + R_3 \geq 2r_1 + \frac{(r_2 + r_3)^2}{R_1}, \tag{1.3}$$

with equality holding if and only if $CA = AB$ and P is the circumcenter of triangle ABC.

It is clear that $R_1 + \frac{(r_2+r_3)^2}{R_1} \geq 2(r_2 + r_3)$ follows from the arithmetic-geometric mean inequality, thus inequality (1.3) implies the Erdös-Mordell inequality (1.1).

Motivated by inequality (1.3), we shall establish in this paper two sharpened versions of the Erdös-Mordell inequality. We shall also extend them to the cases with one parameter.

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2 Two results

We state the first main result in the following.

Theorem 1 *Let P be an interior point of the triangle ABC (P may lie on the boundary except the vertices of ABC), then*

$$\frac{(r_2+r_3)^2}{R_1}+\frac{(r_3+r_1)^2}{R_2}+\frac{(r_1+r_2)^2}{R_3}\le R_1+R_2+R_3, \tag{2.1}$$

with equality holding if and only if $\triangle ABC$ is equilateral and P is its center or $\triangle ABC$ is a right isosceles triangle and P is its circumcenter.

The Erdös-Mordell inequality (1.1) can easily be obtained from (2.1) and the above-mentioned inequality $R_1+\frac{(r_2+r_3)^2}{R_1}\ge 2(r_2+r_3)$. Therefore, although the value of the left hand of (2.1) is not always greater than or equal to $2(r_1+r_2+r_3)$, inequality (2.1) can still be regarded as a sharpened version of the Erdös-Mordell inequality.

The proof of Theorem 1 needs the following well-known lemma, which will be used in other results of this note.

Lemma 1 [2, 5] *Let a, b, c be the sides BC, CA, AB of the triangle ABC, respectively, then for any interior point P*

$$aR_1\ge br_3+cr_2,\qquad bR_2\ge cr_1+ar_3,\qquad cR_3\ge ar_2+br_1. \tag{2.2}$$

Each equality in (2.2) *holds if and only if P lies on the line AO, BO, CO, respectively, where O is the circumcenter of the triangle ABC.*

We now prove Theorem 1.

Proof By Lemma 1, to prove inequality (2.1), we only need to prove that

$$\frac{br_3+cr_2}{a}+\frac{cr_1+ar_3}{b}+\frac{ar_2+br_1}{c}\ge\frac{a(r_2+r_3)^2}{br_3+cr_2}+\frac{b(r_3+r_1)^2}{cr_1+ar_3}+\frac{c(r_1+r_2)^2}{ar_2+br_1}, \tag{2.3}$$

which is equivalent to

$$\begin{aligned}&(br_3+cr_2)(cr_1+ar_3)(ar_2+br_1)\bigl[bc(br_3+cr_2)+ca(cr_1+ar_3)\\&\quad+ab(ar_2+br_1)\bigr]-a^2bc(cr_1+ar_3)(ar_2+br_1)(r_2+r_3)^2\\&\quad-b^2ca(ar_2+br_1)(br_3+cr_2)(r_3+r_1)^2-c^2ab(br_3+cr_2)(cr_1+ar_3)(r_1+r_2)^2\\&\ge 0.\end{aligned} \tag{2.4}$$

Expanding and arranging gives the following inequality (required for the proof):

$$\begin{aligned}&a^4(b-c)^2r_2^2r_3^2+b^4(c-a)^2r_3^2r_1^2+c^4(a-b)^2r_1^2r_2^2\\&\quad+r_1r_2r_3(bcr_1+car_2+abr_3)\bigl[a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2\bigr]\\&\ge 0,\end{aligned} \tag{2.5}$$

which is obviously true and inequality (2.1) is proved.

We now consider the equality condition of (2.1). If P lies inside $\triangle ABC$, then we have strict inequalities $r_1 > 0$, $r_2 > 0$, and $r_3 > 0$. Thus, the equality in (2.5) holds only when $a = b = c$. Furthermore, by Lemma 1 we conclude that the equality in (2.1) holds if and only if $\triangle ABC$ is equilateral and P is its center. If P lies on the boundary (except the vertices) of $\triangle ABC$, then one of r_1, r_2, r_3 is equal to zero. Thus, we deduce that $\triangle ABC$ must be isosceles when the equality in (2.5) holds. By Lemma 1 we further deduce that the equality in (2.1) holds if and only if $\triangle ABC$ is a right isosceles triangle and P is its circumcenter. Combining the arguments of the above two cases, we obtain the equality condition of (2.1) as stated in Theorem 1. This completes the proof of Theorem 1. □

As an interesting application of Theorem 1, we shall next derive an important inequality for non-obtuse triangles, *i.e.*, the Walker inequality. As usually, we shall denote by A, B, C the angles of $\triangle ABC$ and denote by s, R, r the semi-perimeter, the circumradius, and the inradius of triangle ABC, respectively. Suppose that $\triangle ABC$ is non-obtuse and P is its circumcenter, then we have $R_1 = R_2 = R_3 = R$, $r_1 = R\cos A$, $r_2 = R\cos B$, $r_3 = R\cos C$, and it follows from (2.1) that

$$(\cos B + \cos C)^2 + (\cos C + \cos A)^2 + (\cos A + \cos B)^2 \le 3, \tag{2.6}$$

i.e.,

$$\cos^2 A + \cos^2 B + \cos^2 C + \cos B\cos C + \cos C\cos A + \cos A\cos B \le \frac{3}{2}.$$

Using the following known identities (see [1], pp.55-56):

$$\cos^2 A + \cos^2 B + \cos^2 C = \frac{6R^2 + 4Rr + r^2 - s^2}{2R^2}, \tag{2.7}$$

$$\cos B\cos C + \cos C\cos A + \cos A\cos B = \frac{s^2 + r^2 - 4R^2}{4R^2}, \tag{2.8}$$

we further obtain the following Walker inequality (*cf.* [1], pp.247-250).

Corollary 1 *If ABC is a non-obtuse triangle, then*

$$s^2 \ge 2R^2 + 8Rr + 3r^2. \tag{2.9}$$

Equality holds iff $\triangle ABC$ is equilateral or right isosceles.

Next, we give a result similar to Theorem 1.

Theorem 2 *Let P be an interior point of the triangle ABC (P may lie on the boundary except the vertices of ABC), then*

$$\frac{(r_2 + r_3)^2}{R_1 + r_2 + r_3} + \frac{(r_3 + r_1)^2}{R_2 + r_3 + r_1} + \frac{(r_1 + r_2)^2}{R_3 + r_1 + r_2} \le r_1 + r_2 + r_3 \tag{2.10}$$

with equality holding if and only if $\triangle ABC$ is equilateral and P is its center or $\triangle ABC$ is a right isosceles triangle and P is its circumcenter.

Evidently, inequality (2.10) can be regarded as an extension of the Erdös-Mordell inequality. One the other hand, it is also a sharpened version of the Erdös-Mordell inequality. Since we have, by the arithmetic-geometric mean inequality,

$$R_1 + r_2 + r_3 + \frac{4(r_2 + r_3)^2}{R_1 + r_2 + r_3} \geq 4(r_2 + r_3),$$

or

$$\frac{4(r_2 + r_3)^2}{R_1 + r_2 + r_3} \geq 3(r_2 + r_3) - R_1.$$

By this and its two analogs, we immediately obtain the Erdös-Mordell inequality (1.1) from (2.10).

We now prove Theorem 2.

Proof By Lemma 1, to prove inequality (2.10) we need only to prove that

$$r_1 + r_2 + r_3 \geq \frac{(r_2 + r_3)^2}{\frac{br_3+cr_2}{a} + r_2 + r_3} + \frac{(r_3 + r_1)^2}{\frac{cr_1+ar_3}{b} + r_3 + r_1} + \frac{(r_1 + r_2)^2}{\frac{ar_2+br_1}{c} + r_1 + r_2}, \tag{2.11}$$

or

$$r_1 + r_2 + r_3 \geq \frac{a(r_2 + r_3)^2}{(c + a)r_2 + (a + b)r_3} + \frac{b(r_3 + r_1)^2}{(a + b)r_3 + (b + c)r_1} + \frac{c(r_1 + r_2)^2}{(b + c)r_1 + (c + a)r_2},$$

which is equivalent to

$$\begin{aligned}
&(r_1 + r_2 + r_3)\big[(c + a)r_2 + (a + b)r_3\big] \\
&\quad \cdot \big[(a + b)r_3 + (b + c)r_1\big]\big[(b + c)r_1 + (c + a)r_2\big] \\
&\quad - a(r_2 + r_3)^2\big[(a + b)r_3 + (b + c)r_1\big]\big[(b + c)r_1 + (c + a)r_2\big] \\
&\quad - b(r_3 + r_1)^2\big[(b + c)r_1 + (c + a)r_2\big]\big[(c + a)r_2 + (a + b)r_3\big] \\
&\quad - c(r_1 + r_2)^2\big[(c + a)r_2 + (a + b)r_3\big]\big[(a + b)r_3 + (b + c)r_1\big] \geq 0.
\end{aligned} \tag{2.12}$$

This can be simplified as

$$\begin{aligned}
&a(b - c)^2 r_2^2 r_3^2 + b(c - a)^2 r_3^2 r_1^2 + c(a - b)^2 r_1^2 r_2^2 \\
&\quad + \frac{1}{2} r_1 r_2 r_3 \big[(b + c)r_1 + (c + a)r_2 + (a + b)r_3\big] \\
&\quad \cdot \big[(b - c)^2 + (c - a)^2 + (a - b)^2\big] \geq 0,
\end{aligned} \tag{2.13}$$

which is clearly true. Therefore, inequalities (2.11) and (2.10) are proved.

Using similar arguments in the proof of Theorem 1, we easily deduce that the equality in (2.10) holds only when the following two cases occur: the $\triangle ABC$ is equilateral and P is its center or $\triangle ABC$ is a right isosceles triangle and P is its circumcenter. This completes the proof of Theorem 2. □

In Theorem 2, if we let $\triangle ABC$ be a non-obtuse triangle and let P be its circumcenter, then we can obtain the following trigonometric inequality:

$$\frac{(\cos B+\cos C)^2}{1+\cos B+\cos C}+\frac{(\cos C+\cos A)^2}{1+\cos C+\cos A}+\frac{(\cos A+\cos B)^2}{1+\cos A+\cos B} \le \cos A+\cos B+\cos C. \tag{2.14}$$

From (2.14), it is not difficult to obtain the following inequality (we omit the details).

Corollary 2 *If ABC is a non-obtuse triangle, then*

$$s^2 \ge \frac{(2R+r)(2R^3+R^2r+3Rr^2+r^3)}{R^2+Rr-r^2}. \tag{2.15}$$

Equality holds iff $\triangle ABC$ is equilateral or right isosceles.

Remark 1 Inequality (2.15) is incomparable with Walker's inequality (2.9).

3 Generalizations of Theorem 1 and Theorem 2

In this section, we present generalizations of Theorem 1 and Theorem 2.

Theorem 3 *Let P be an interior point of the triangle ABC (P may lie on the boundary except the vertices of ABC) and let $k \ge 0$ be a real number, then*

$$\frac{(kr_1+r_2+r_3)^2}{R_1+kr_1}+\frac{(kr_2+r_3+r_1)^2}{R_2+kr_2}+\frac{(kr_3+r_1+r_2)^2}{R_3+kr_3} \le \frac{k+2}{2}(R_1+R_2+R_3). \tag{3.1}$$

If $k=0$, the equality in (3.1) holds if and only if $\triangle ABC$ is equilateral and P is its center or $\triangle ABC$ is a right isosceles triangle and P is its circumcenter. If $k>0$, the equality in (3.1) holds if and only if $\triangle ABC$ is equilateral and P is its center.

When $k=0$, then the above theorem reduces to Theorem 1. In order to prove this theorem, we first give the following lemma.

Lemma 2 *In any triangle ABC, we let*

$$\begin{aligned}
Q_1 &= b\left(2c^2-2cb+b^2\right)a^2+2c^3(c-2b)a+b^3c^2,\\
Q_2 &= (b+c)a^3+2(b-c)^2a^2+(b+c)\left(b^2-5bc+c^2\right)a+4b^2c^2,\\
Q_3 &= 2\left(b^2-bc+c^2\right)a^3+2bc(b+c)a^2-2bc\left(4b^2-bc+4c^2\right)a\\
&\quad+\left(b^2-bc+c^2\right)(b+c)^3,\\
Q_4 &= 2\left(b^2+c^2\right)a^2-4abc(b+c)+bc(b+c)^2,\\
Q_5 &= \left(b^2+c^2\right)a^3-2abc\left(2b^2-bc+2c^2\right)+2bc(b+c)\left(b^2-bc+c^2\right),\\
Q_6 &= 4\left(b^2+c^2\right)a^4-8bc(b+c)a^3+bc\left(3b^2+4bc+3c^2\right)a^2\\
&\quad+2a(b+c)\left(b^4-3c^3b+3b^2c^2-3cb^3+c^4\right)+2b^3c^3.
\end{aligned}$$

Then

$$Q_i \geq 0, \tag{3.2}$$

where $i = 1, 2, 3, 4, 5, 6$. *All the equalities in* (3.2) *hold if and only if the triangle ABC is equilateral.*

Proof Q_1 can be rewritten as

$$Q_1 = a\left(ab + 2c^2\right)(b-c)^2 + bc^2(a-b)^2, \tag{3.3}$$

so that $Q_1 \geq 0$.

It is easy to check that

$$2Q_2 = a(b+c)\left[(a-b)^2 + (a-c)^2\right] + a(b+c-a)(b-c)^2 + X_1, \tag{3.4}$$

where

$$X_1 = \left(7b^2 - 6bc + 7c^2\right)a^2 - 8bc(b+c)a + 8b^2c^2.$$

Let $\frac{1}{2}(b+c-a) = x$, $\frac{1}{2}(c+a-b) = y$, and $\frac{1}{2}(a+b-c) = z$, then $x > 0$, $y > 0$, $z > 0$, and

$$\begin{cases} a = y + z, \\ b = z + x, \\ c = x + y. \end{cases} \tag{3.5}$$

Also it is easy to obtain

$$X_1 = 8x^4 - 8\left(y^2 + z^2\right)x^2 + 7y^4 - 6y^2z^2 + 7z^4.$$

Note that X_1 is a quadratic function of x^2 with the following discriminant:

$$F_1 = -160(y+z)^2(y-z)^2 \leq 0$$

and $7y^4 - 6y^2z^2 + 7z^4 > 0$. Thus, $X_1 \geq 0$ holds true and then $Q_2 \geq 0$ follows from (3.4).

Using the substitution (3.5), we obtain the following equality:

$$\begin{aligned} Q_3 = {} & 8x^5 + 6(y+z)x^4 + 2\left(y^2 - 10yz + z^2\right)x^3 + (y+z)\left(5y^2 - 22yz + 5z^2\right)x^2 \\ & + \left(3y^2 - 4yz + 3z^2\right)(y+z)^2x + 3(y+z)\left(y^2 + z^2\right)\left(y^2 - yz + z^2\right), \end{aligned}$$

which can be rewritten as follows:

$$Q_3 = x\left[4x^2 + 7(y+z)x + 6y^2 + 4yz + 6z^2\right]\left[(x-y)^2 + (x-z)^2\right] + X_2, \tag{3.6}$$

where

$$\begin{aligned} X_2 = {} & \left(10y^3 + 10z^3 - 4yz^2 - 4y^2z\right)x^2 - \left(3y^4 + 2y^3z + 14y^2z^2 + 2yz^3 + 3z^4\right)x \\ & + 3y^5 + 3z^5 + 3y^3z^2 + 3y^2z^3. \end{aligned}$$

Since $10y^3 + 10z^3 - 4yz^2 - 4y^2z > 0$ and it is easy to obtain the quadratic discriminant F_2 of X_2:

$$F_2 = -\big(111y^6 + 162zy^5 + 197y^4z^2 + 356y^3z^3 + 197z^4y^2 + 162z^5y + 111z^6\big)(y-z)^2 \le 0.$$

Thus we have $X_2 \ge 0$ and then inequality $Q_3 \ge 0$ follows from (3.6).

Inequality $Q_4 \ge 0$ can easily be proved. Indeed, Q_4 can be viewed a quadratic function of a with positive quadratic coefficient and positive constant term, and its discriminant is given by $F_3 = -8bc(b+c)^2(b-c)^2$. Hence, we have $Q_4 \ge 0$.

We now prove inequality $Q_5 \ge 0$. It is easy to check the following identity:

$$4Q_5 = (a+b+c)\big(b^2+c^2\big)(2a-b-c)^2 + (b-c)^2X_3, \tag{3.7}$$

where

$$X_3 = a\big(3b^2 - 4bc + 3c^2\big) - (b+c)\big(b^2 - 4bc + c^2\big).$$

Under the substitution (3.5), X_3 can be written as

$$X_3 = 4x^3 + 8(y+z)x^2 + 2x\big(y^2 + 8yz + z^2\big) + 2(y+z)\big(y^2+z^2\big). \tag{3.8}$$

Thus, inequality $X_3 > 0$ holds strictly and $Q_5 \ge 0$ follows from (3.7).

Finally, we prove $Q_6 \ge 0$. Using the substitution (3.5), we obtain

$$\begin{aligned} Q_6 &= 2x^6 + 2(y+z)x^5 + 6\big(y^2 + 3yz + z^2\big)x^4 + 2(y+z)\big(y^2 - 4yz + z^2\big)x^3 \\ &\quad + \big(y^4 - 18y^3z - 18yz^3 - 32y^2z^2 + z^4\big)x^2 + (y+z)\big(9y^4 - 12y^3z - 12yz^3 \\ &\quad + 8y^2z^2 + 9z^4\big)x + 6\big(y^6 + z^6\big) + 9yz\big(y^4 + z^4\big) + 2y^2z^2\big(y^2 + z^2\big). \end{aligned}$$

Through analysis, we find the equality

$$4Q_6 = (y-z)^2X_4 + \big[(x-y)^2 + (x-z)^2\big]X_5, \tag{3.9}$$

where

$$\begin{aligned} X_4 &= 13\big(y^4 + z^4\big) + 40x\big(y^3 + z^3\big) + 52yz\big(y^2 + z^2\big) + 56xyz(y+z) + 62y^2z^2, \\ X_5 &= 4x^4 + 8(y+z)x^3 + \big(18y^2 + 52yz + 18z^2\big)x^2 + 18x(y+z)^3 \\ &\quad + 11\big(y^4 + z^4\big) + 10yz\big(y^2 + z^2\big) + 26y^2z^2. \end{aligned}$$

Thus, we have inequality $Q_6 \ge 0$.

Form the above proofs of $Q_i \ge 0$, we easily conclude that the equalities in $Q_i \ge 0$ ($i = 1, 2, \ldots, 6$) are all valid if and only if $a = b = c$, *i.e.*, $\triangle ABC$ is equilateral. This completes the proof of Lemma 2. □

In the following, we shall prove Theorem 3. For brevity, we shall, respectively, denote cyclic sums and products over triples (a, b, c), (r_1, r_2, r_3), and (x, y, z) by $\sum$ and $\prod$.

Proof According to Lemma 1, for proving inequality (3.1) it suffices to prove that

$$\frac{k+2}{2}\sum\frac{br_3+cr_2}{a}\geq\sum\frac{(kr_1+r_2+r_3)^2}{\frac{br_3+cr_2}{a}+kr_1}. \tag{3.10}$$

If we set $r_1 = x$, $r_2 = y$, $r_3 = z$, then inequality (3.10) becomes

$$(k+2)\sum bc(zb+yc)\geq 2abc\sum\frac{a(kx+y+z)^2}{kxa+zb+yc}, \tag{3.11}$$

where $x \geq 0$, $y \geq 0$, $z \geq 0$, and at most one of x, y, z is equal to zero.

Putting

$$\begin{aligned}E_0 &= (k+2)\sum bc(zb+yc)\prod(kxa+zb+yc)\\&\quad-2abc\sum a(kyb+xc+za)(kzc+ya+xb)(kx+y+z)^2,\end{aligned}$$

then we see that inequality (3.11) is equivalent to

$$E_0\geq 0. \tag{3.12}$$

With the help of the famous mathematical software Maple (we used Maple 15), we can obtain the following identity:

$$E_0 = e_1k^4+e_2k^3+(e_3+e_4+e_5+e_6)k^2+(e_7+e_8+e_9)k+e_{10}+e_{11}, \tag{3.13}$$

where

$$\begin{aligned}
&k\geq 0,\\
&e_1 = xyzabc\sum xa(b-c)^2,\\
&e_2 = \Big[4xyzabc+\sum yzbc(zb+yc)\Big]\sum xa(b-c)^2,\\
&e_3 = abc\sum a(b-c)^2x^4,\\
&e_4 = \sum a\big\{y\big[b^2c^3+2a(b-2c)b^3+c\big(2b^2-2bc+c^2\big)a^2\big]\\
&\qquad+z\big[c^2b^3+2a(c-2b)c^3+b\big(2c^2-2cb+b^2\big)a^2\big]\big\}x^3,\\
&e_5 = \sum bcy^2z^2\big[(b+c)a^3+2(b-c)^2a^2+(b+c)\big(b^2-5bc+c^2\big)a+4b^2c^2\big],\\
&e_6 = xyz\sum xa\big[2\big(b^2-bc+c^2\big)a^3+2(b+c)bca^2-2\big(4b^2-bc+4c^2\big)bca\\
&\qquad+\big(b^2-bc+c^2\big)(b+c)^3\big],\\
&e_7 = \sum a(zb+yc)\big[2\big(b^2+c^2\big)a^2-4bc(b+c)a+bc(b+c)^2\big]x^3,\\
&e_8 = \sum a\big[\big(b^2+c^2\big)a^3-2bc\big(2b^2-bc+2c^2\big)a+2bc(b+c)\big(b^2-bc+c^2\big)\big]y^2z^2,\\
&e_9 = xyz\sum x\big[4\big(b^2+c^2\big)a^4-8bc(b+c)a^3+bc\big(3b^2+4bc+3c^2\big)a^2\\
&\qquad+2(b+c)\big(b^4-3cb^3+3b^2c^2-3bc^3+c^4\big)a+2b^3c^3\big],
\end{aligned}$$

$$e_{10} = 2\sum a^4(b-c)^2y^2z^2,$$
$$e_{11} = 2xyz\sum xbc\sum a^2(b-c)^2.$$

Clearly, inequalities $e_1 \geq 0$, $e_2 \geq 0$, $e_3 \geq 0$, $e_{10} \geq 0$, and $e_{11} \geq 0$ hold for any triangle ABC and non-negative real numbers x, y, z. Also, by Lemma 2, we have $e_4 \geq 0$, $e_5 \geq 0$, $e_6 \geq 0$, $e_7 \geq 0$, $e_8 \geq 0$, and $e_9 \geq 0$. Thus, from identity (3.13) we see that $E_0 \geq 0$ holds for $x \geq 0$, $y \geq 0$, $z \geq 0$, and $k \geq 0$. Therefore, inequalities (3.12), (3.10), and (3.1) are proved.

When $k = 0$, inequality (3.1) becomes (2.1) and we have obtained the equality conditions (as stated in Theorem 1). When $k > 0$, by Lemma 1 and identity (3.13) we conclude that the equality (3.1) holds if and only if P is the circumcenter of ABC and the equalities in $e_i \geq 0$ ($i = 1, 2, \ldots, 11$) are all valid. Note that at most one of x, y, z is equal to zero. Thus, the equalities of $e_2 \geq 0$, $e_3 \geq 0$, $e_4 \geq 0$, $e_5 \geq 0$, $e_7 \geq 0$, and $e_8 \geq 0$ occur only when $a = b = c$. We further deduce that the equality in (3.1) holds if and only if $\triangle ABC$ is equilateral and P is its center. This completes the proof of Theorem 3. □

We now state and prove the following generalization of Theorem 2.

Theorem 4 *Let P be an interior point of the triangle ABC (P may lie on the boundary except the vertices of ABC) and let $k \geq 1$ be a real number, then*

$$\frac{(r_2+r_3)^2}{R_1+k(r_2+r_3)}+\frac{(r_3+r_1)^2}{R_2+k(r_3+r_1)}+\frac{(r_1+r_2)^2}{R_3+k(r_1+r_2)} \leq \frac{2}{k+1}(r_1+r_2+r_3). \tag{3.14}$$

If $k = 1$, the equality in (3.14) *holds if and only if $\triangle ABC$ is equilateral and P is its center or $\triangle ABC$ is a right isosceles triangle and P is its circumcenter. If $k > 1$, the equality in* (3.14) *holds if and only if $\triangle ABC$ is equilateral and P is its center.*

Proof We still denote cyclic sums and products by $\sum$ and $\prod$, respectively. If we let $k = 1 + t$, then $t \geq 0$ by the assumption $k \geq 1$. According to Lemma 1, for proving inequality (3.14) we have only to prove that

$$\sum \frac{(r_2+r_3)^2}{\frac{br_3+cr_2}{a}+(t+1)(r_2+r_3)} \leq \frac{2}{t+2}\sum r_1. \tag{3.15}$$

Let $r_1 = x$, $r_2 = y$, and $r_3 = z$, then the above inequality becomes

$$\sum \frac{(y+z)^2}{\frac{zb+yc}{a}+(t+1)(y+z)} \leq \frac{2}{t+2}\sum x,$$

or

$$\sum \frac{a(y+z)^2}{zb+yc+(t+1)(y+z)a} \leq \frac{2}{t+2}\sum x, \tag{3.16}$$

where $x \geq 0$, $y \geq 0$, $z \geq 0$, $t \geq 0$, and at most one of x, y, z is equal to zero.

We set

$$M_0 = 2\sum x\prod\big[zb+yc+(t+1)(y+z)a\big]$$
$$-(t+2)\sum a\big[xc+za+(t+1)(z+x)b\big]\big[ya+xb+(t+1)(x+y)c\big](y+z)^2,$$

then (3.16) is equivalent to

$$M_0 \geq 0. \tag{3.17}$$

With the help of the Maple software, we easily obtain the following identity:

$$M_0 = m_1 t^2 + (m_2 + m_3 + m_4)t + m_5 + m_6, \tag{3.18}$$

where

$$\begin{aligned}
&t \geq 0, \\
&m_1 = \prod(y+z)\sum xa(b-c)^2, \\
&m_2 = \bigl[a(y+z) + bz + yc\bigr](b-c)^2 x^3, \\
&m_3 = \sum\bigl[2a^3 - (b+c)a^2 + 2\bigl(b^2+c^2-3bc\bigr)a + bc(b+c)\bigr]y^2z^2, \\
&m_4 = xyz\sum x\bigl[3(b+c)a^2 + \bigl(b^2 - 14bc + c^2\bigr)a + (b+c)\bigl(2b^2 - bc + 2c^2\bigr)\bigr], \\
&m_5 = 2\sum y^2z^2 a(b-c)^2, \\
&m_6 = xyz\sum(b+c)x\sum(b-c)^2.
\end{aligned}$$

It is clear that inequalities $m_1 \geq 0$, $m_2 \geq 0$, $m_5 \geq 0$, and $m_6 \geq 0$ hold for any triangle ABC and non-negative real numbers x, y, z. In addition, by the following identity:

$$\begin{aligned}
&2a^3 - (b+c)a^2 + 2\bigl(b^2+c^2-3bc\bigr)a + bc(b+c) \\
&\quad = a(b-c)^2 + (a+b)(c-a)^2 + (a+c)(a-b)^2,
\end{aligned} \tag{3.19}$$

one sees that $m_3 \geq 0$. Also, by the identity

$$\begin{aligned}
&4\bigl[3(b+c)a^2 + \bigl(b^2 - 14bc + c^2\bigr)a + (b+c)\bigl(2b^2 - bc + 2c^2\bigr)\bigr] \\
&\quad = 3(b+c)(b+c-2a)^2 + (16a + 5b + 5c)(b-c)^2,
\end{aligned} \tag{3.20}$$

we have $m_4 \geq 0$. Therefore, inequality $M_0 \geq 0$ follows from (3.18) and then inequalities (3.15) and (3.14) are proved.

When $k = 1$, inequality (3.14) reduces to (2.10) and we have pointed out the equality conditions in Theorem 2. When $k > 1$, we have $t > 0$ from the assumption. In this case, by Lemma 1 and (3.18) we conclude that the equality in (3.14) holds if and only if P is the circumcenter of ABC and the equalities in $m_i \geq 0$ ($i = 1, 2, \ldots, 6$) are all valid. Note that at most one of x, y, z is equal to zero. We further deduce that the equality in (3.14) holds if and only if $\triangle ABC$ is equilateral and P is its center. The proof of Theorem 4 is completed. □

4 Open problems

The author of this paper has found some sharpened versions of the Erdös-Mordell inequality, which have not been proved at present but have been checked by computer. We introduce here three of them as open problems.

A sharpened version of the Erdös-Mordell inequality similar to the inequalities of Theorem 1 and Theorem 2 is as follows.

Conjecture 1 *For any interior point P of* $\triangle ABC$, *we have*

$$\frac{(2r_1+r_2+r_3)^2}{R_2+R_3}+\frac{(2r_2+r_3+r_1)^2}{R_3+R_1}+\frac{(2r_3+r_1+r_2)^2}{R_1+R_2}\le 4(r_1+r_2+r_3). \tag{4.1}$$

The two conjectured inequalities below are obvious sharpened versions of the Erdös-Mordell inequality.

Conjecture 2 *For any interior point P of* $\triangle ABC$, *we have*

$$R_1+R_2+R_3\ge 2\left(\frac{m_a}{w_a}r_1+\frac{m_b}{w_b}r_2+\frac{m_c}{w_c}r_3\right), \tag{4.2}$$

where m_a, m_b, m_c *are the corresponding medians of triangle ABC and* w_a, w_b, w_c *the bisectors*.

Since we have inequality $m_a \ge w_a$ *etc.*, thus (4.2) is stronger than the Erdös-Mordell inequality.

Conjecture 3 *For any interior point P of* $\triangle ABC$, *we have*

$$R_1+R_2+R_3\ge \frac{w_a+h_a}{h_a}r_1+\frac{w_b+h_b}{h_b}r_2+\frac{w_c+h_c}{h_c}r_3, \tag{4.3}$$

where w_a, w_b, w_c *are the corresponding bisectors of triangle ABC and* h_a, h_b, h_c *the altitudes*.

From the fact that $w_a \ge h_a$ *etc.*, we can see that (4.3) is stronger than the Erdös-Mordell inequality.

Competing interests

The author declares that he has no competing interests.

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