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Generalizations of Steffensen's inequality via Taylor's formula

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Abstract

We generalize Steffensen's inequality to the class of n-convex functions using Taylor's formula. Further, we use inequalities for the Čebyšev functional to obtain bounds for identities related to generalizations of Steffensen's inequality, and we give Ostrowski-type inequalities related to obtained generalizations. Finally, we apply our results to obtain new Stolarsky-type means.

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1 Introduction

In 1918 Steffensen proved the following inequality (see [1]).

Theorem 1.1 *Suppose that f is nonincreasing and g is integrable on $[a,b]$ with $0 \leq g \leq 1$ and $\lambda = \int_a^b g(t)\,dt$. Then we have*

$$\int_{b-\lambda}^{b} f(t)\,dt \leq \int_a^b f(t)g(t)\,dt \leq \int_a^{a+\lambda} f(t)\,dt. \tag{1.1}$$

The inequalities are reversed for f nondecreasing.

Since its appearance many papers have been devoted to generalizations and refinements of Steffensen's inequality and its connection to other important inequalities.

The following identities are the starting point for our generalizations of Steffensen's inequality:

$$\begin{aligned}\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt \\ = \int_a^{a+\lambda} \big[f(t)-f(a+\lambda)\big]\big[1-g(t)\big]\,dt + \int_{a+\lambda}^{b} \big[f(a+\lambda)-f(t)\big]g(t)\,dt\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^{b} f(t)\,dt \\ = \int_a^{b-\lambda} \big[f(t)-f(b-\lambda)\big]g(t)\,dt + \int_{b-\lambda}^{b} \big[f(b-\lambda)-f(t)\big]\big[1-g(t)\big]\,dt.\end{aligned} \tag{1.3}$$

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In [2] Mitrinović stated that the inequalities in (1.1) follow from identities (1.2) and (1.3).

Let us recall the well-known Taylor's formula needed for our generalizations:

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!}(x-a)^i + \frac{1}{(n-1)!}\int_a^x f^{(n)}(t)(x-t)^{n-1}\,dt. \tag{1.4}$$

Similarly, we have Taylor's formula in point b, *i.e.*

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(b)}{i!}(x-b)^i - \frac{1}{(n-1)!}\int_x^b f^{(n)}(t)(x-t)^{n-1}\,dt. \tag{1.5}$$

In this paper we generalize Steffensen's inequality to n-convex functions using Taylor's formula. Further, we use inequalities for the Čebyšev functional to obtain bounds for identities related to generalizations of Steffensen's inequality. We continue with Ostrowski-type inequalities related to obtained generalizations, and we conclude the paper with an application to Stolarsky-type means.

2 Generalizations via Taylor's formula

We begin this section with the proof of some identities related to generalizations of Steffensen's inequality.

Theorem 2.1 *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, and let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $\lambda = \int_a^b g(t)\,dt$ and let the function G_1 be defined by*

$$G_1(x) = \begin{cases} \int_a^x (1-g(t))\,dt, & x \in [a, a+\lambda], \\ \int_x^b g(t)\,dt, & x \in [a+\lambda, b]. \end{cases} \tag{2.1}$$

Then

$$\begin{aligned} &\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i\,dx \\ &\quad = -\frac{1}{(n-2)!}\int_a^b \left(\int_t^b G_1(x)(x-t)^{n-2}\,dx\right) f^{(n)}(t)\,dt \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!}\int_a^b G_1(x)(x-b)^i\,dx \\ &\quad = \frac{1}{(n-2)!}\int_a^b \left(\int_a^t G_1(x)(x-t)^{n-2}\,dx\right) f^{(n)}(t)\,dt. \end{aligned} \tag{2.3}$$

Proof Applying Taylor's formula (1.4) to the function f' and replacing n with $n-1$ ($n \geq 2$), we have

$$f'(x) = \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!}(x-a)^i + \int_a^x f^{(n)}(t)\frac{(x-t)^{n-2}}{(n-2)!}\,dt. \tag{2.4}$$

Applying integration by parts and then using the definition of the function G_1, identity (1.2) becomes

$$\begin{aligned}
&\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt \\
&\quad = -\int_a^{a+\lambda}\left(\int_a^x \bigl(1-g(t)\bigr)\,dt\right)df(x) - \int_{a+\lambda}^b\left(\int_x^b g(t)\,dt\right)df(x) \\
&\quad = -\int_a^b G_1(x)f'(x)\,dx.
\end{aligned}$$

Hence, using (2.4) we obtain

$$\begin{aligned}
\int_a^b G_1(x)f'(x)\,dx &= \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i\,dx \\
&\quad + \frac{1}{(n-2)!}\int_a^b G_1(x)\left(\int_a^x (x-t)^{n-2}f^{(n)}(t)\,dt\right)dx. \qquad (2.5)
\end{aligned}$$

After applying Fubini's theorem to the last term in (2.5) we obtain (2.2).

Similarly, applying Taylor's formula (1.5) to the function f' we obtain (2.3). □

Theorem 2.2 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geq 2$, and let $g:[a,b]\to\mathbb{R}$ be an integrable function such that $0\leq g\leq 1$. Let $\lambda=\int_a^b g(t)\,dt$ and let the function G_2 be defined by*

$$G_2(x)=\begin{cases}\int_a^x g(t)\,dt, & x\in[a,b-\lambda],\\ \int_x^b (1-g(t))\,dt, & x\in[b-\lambda,b].\end{cases} \qquad (2.6)$$

Then

$$\begin{aligned}
&\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_2(x)(x-a)^i\,dx \\
&\quad = -\frac{1}{(n-2)!}\int_a^b\left(\int_t^b G_2(x)(x-t)^{n-2}\,dx\right)f^{(n)}(t)\,dt \qquad (2.7)
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_2(x)(x-b)^i\,dx \\
&\quad = \frac{1}{(n-2)!}\int_a^b\left(\int_a^t G_2(x)(x-t)^{n-2}\,dx\right)f^{(n)}(t)\,dt. \qquad (2.8)
\end{aligned}$$

Proof Similar to the proof of Theorem 2.1 applying integration by parts to identity (1.3) and then using identity (2.4). □

In the following theorem we obtain generalizations of Steffensen's inequality for n-convex functions.

Theorem 2.3 *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, and let $g : [a, b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $\lambda = \int_a^b g(t)\,dt$ and let the function G_1 be defined by* (2.1).

(i) *If f is n-convex, then*

$$\int_a^b f(t)g(t)\,dt \geq \int_a^{a+\lambda} f(t)\,dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i\,dx. \tag{2.9}$$

(ii) *If f is n-convex and*

$$\int_a^t G_1(x)(x-t)^{n-2}\,dx \leq 0, \quad t \in [a, b], \tag{2.10}$$

then

$$\int_a^b f(t)g(t)\,dt \geq \int_a^{a+\lambda} f(t)\,dt + \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i\,dx. \tag{2.11}$$

Proof If the function f is n-convex, without loss of generality we can assume that f is n-times differentiable and $f^{(n)} \geq 0$ (see [3], p.16 and p.293). Since $0 \leq g \leq 1$, the function G_1 is nonnegative, and for every $n \geq 2$ we have

$$\int_t^b G_1(x)(x-t)^{n-2}\,dx \geq 0, \quad t \in [a, b].$$

Hence, we can apply Theorem 2.1 to obtain (2.9) and (2.11) respectively. □

Theorem 2.4 *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, and let $g : [a, b] \to \mathbb{R}$ be an integrable function such that $0 \leq g \leq 1$. Let $\lambda = \int_a^b g(t)\,dt$ and let the function G_2 be defined by* (2.6).

(i) *If f is n-convex, then*

$$\int_a^b f(t)g(t)\,dt \leq \int_{b-\lambda}^{b} f(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i\,dx. \tag{2.12}$$

(ii) *If f is n-convex and*

$$\int_a^t G_2(x)(x-t)^{n-2}\,dx \leq 0, \quad t \in [a, b], \tag{2.13}$$

then

$$\int_a^b f(t)g(t)\,dt \leq \int_{b-\lambda}^{b} f(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i\,dx. \tag{2.14}$$

Proof Similar as in the proof of Theorem 2.4, we can apply Theorem 2.2 to obtain (2.12) and (2.14). Again, since $0 \leq g \leq 1$, the function G_2 is nonnegative and for every $n \geq 2$ we have

$$\int_t^b G_2(x)(x-t)^{n-2}\,dx \geq 0, \quad t \in [a, b].$$

□

Taking $n = 2$ in Theorems 2.3 and 2.4, we obtain the following special cases for convex functions.

Corollary 2.1 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous, let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$ and let $\lambda = \int_a^b g(t)\,dt$.*

(i) *If f is convex, then*

$$\int_a^b f(t)g(t)\,dt \ge \int_a^{a+\lambda} f(t)\,dt + f'(a)\left(\int_a^b tg(t)\,dt - \lambda a - \frac{\lambda^2}{2}\right).$$

(ii) *If f is convex and*

$$\int_a^t (t-x)g(x)\,dx \ge \frac{(t-a)^2}{2}, \quad t \in [a, a+\lambda],$$

$$\int_a^t (t-x)g(x)\,dx \ge \frac{\lambda^2}{2} + \lambda(t-a-\lambda), \quad t \in [a+\lambda, b],$$

then

$$\int_a^b f(t)g(t)\,dt \ge \int_a^{a+\lambda} f(t)\,dt + f'(b)\left(\int_a^b tg(t)\,dt - \lambda a - \frac{\lambda^2}{2}\right).$$

Corollary 2.2 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous, let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$ and let $\lambda = \int_a^b g(t)\,dt$.*

(i) *If f is convex, then*

$$\int_a^b f(t)g(t)\,dt \le \int_{b-\lambda}^{b} f(t)\,dt - f'(a)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)\,dt\right).$$

(ii) *If f is convex and*

$$\int_a^t (t-x)g(x)\,dx \le 0, \quad t \in [a, b-\lambda],$$
$$\int_a^b (b-x)g(x)\,dx \le \frac{(b-t)^2-\lambda^2}{2} + \lambda(t-b+\lambda), \quad t \in [b-\lambda, b],$$

then

$$\int_a^b f(t)g(t)\,dt \le \int_{b-\lambda}^{b} f(t)\,dt - f'(b)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)\,dt\right).$$

3 Bounds for identities related to generalizations of Steffensen's inequality

In the paper the symbol $L_p[a,b]$ ($1 \le p < \infty$) denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_p = \left(\int_a^b \left|f(t)\right|^p\,dt\right)^{\frac{1}{p}},$$

and $L_\infty[a,b]$ denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{t\in[a,b]} \bigl|f(t)\bigr|.$$

For two Lebesgue integrable functions $f,h:[a,b]\to\mathbb{R}$, we consider the Čebyšev functional

$$T(f,h) = \frac{1}{b-a}\int_a^b f(t)h(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b h(t)\,dt.$$

In [4] the authors proved the following theorems.

Theorem 3.1 *Let $f:[a,b]\to\mathbb{R}$ be a Lebesgue integrable function and $h:[a,b]\to\mathbb{R}$ be an absolutely continuous function with $(\cdot-a)(b-\cdot)[h']^2\in L[a,b]$. Then we have the inequality*

$$\bigl|T(f,h)\bigr| \le \frac{1}{\sqrt{2}}\bigl[T(f,f)\bigr]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (x-a)(b-x)\bigl[h'(x)\bigr]^2\,dx\right)^{\frac{1}{2}}. \tag{3.1}$$

The constant $\frac{1}{\sqrt{2}}$ in (3.1) *is best possible.*

Theorem 3.2 *Assume that $h:[a,b]\to\mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is absolutely continuous with $f'\in L_\infty[a,b]$. Then we have the inequality*

$$\bigl|T(f,h)\bigr| \le \frac{1}{2(b-a)}\|f'\|_\infty\int_a^b (x-a)(b-x)\,dh(x). \tag{3.2}$$

The constant $\frac{1}{2}$ in (3.2) *is best possible.*

In the sequel we use the above theorems to obtain generalizations of the results proved in the previous section.

Firstly, let us denote

$$\Phi_i(t) = \int_t^b G_i(x)(x-t)^{n-2}\,dx, \quad i=1,2, \tag{3.3}$$

and

$$\Omega_i(t) = \int_a^t G_i(x)(x-t)^{n-2}\,dx, \quad i=1,2. \tag{3.4}$$

We have that Čebyšev functionals $T(\Phi_i,\Phi_i)$ and $T(\Omega_i,\Omega_i)$, $i=1,2$, are given by:

$$\begin{aligned}
T(\Phi_1,\Phi_1) &= \frac{1}{(n-1)^2(b-a)}\left[\int_a^b \Psi^2(t)\,dt - \frac{2}{n}\int_a^{a+\lambda}(a+\lambda-t)^n\Psi(t)\,dt + \frac{\lambda^{2n+1}}{(2n+1)n^2}\right]\\
&\quad - \frac{1}{(b-a)^2(n-1)^2n^2}\left(\int_a^b g(x)(x-a)^n\,dx - \frac{\lambda^{n+1}}{n+1}\right)^2,\\
T(\Phi_2,\Phi_2) &= \frac{1}{(n-1)^2(b-a)}\left[\frac{(b-a)^{2n+1}-(b-\lambda-a)^{2n+1}}{(2n+1)n^2} + \int_a^b \Psi^2(t)\,dt\right.\\
&\quad \left. - \frac{2}{n}\left(\frac{1}{n}\int_a^{b-\lambda}(b-t)^n(b-\lambda-t)^n\,dt + \int_a^b (b-t)^n\Psi(t)\,dt\right.\right.
\end{aligned}$$

$$
\begin{aligned}
&- \int_a^{b-\lambda} (b-\lambda-t)^n \Psi(t)\,dt\Bigg)\Bigg] \\
&- \frac{1}{(b-a)^2(n-1)^2n^2}\left(\frac{(b-\lambda-a)^{n+1}-(b-a)^{n+1}}{n+1} + \int_a^b g(x)(x-a)^n\,dx\right)^2,
\end{aligned}
$$

where $\Psi(t) = \int_t^b g(x)(x-t)^{n-1}\,dx$,

$$
\begin{aligned}
T(\Omega_1,\Omega_1) = {}& \frac{1}{(n-1)^2(b-a)}\Bigg[\frac{(a+\lambda-b)^{2n+1}-(a-b)^{2n+1}}{(2n+1)n^2} + \int_a^b \Upsilon^2(t)\,dt \\
&- \frac{2}{n}\Bigg(\frac{1}{n}\int_{a+\lambda}^b (a-t)^n(a+\lambda-t)^n\,dt + \int_a^b (a-t)^n\Upsilon(t)\,dt \\
&- \int_{a+\lambda}^b (a+\lambda-t)^n\Upsilon(t)\,dt\Bigg)\Bigg] \\
&- \frac{1}{(b-a)^2(n-1)^2n^2}\left(\frac{(a+\lambda-b)^{n+1}-(a-b)^{n+1}}{n+1} + \int_a^b g(x)(x-b)^n\,dx\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
T(\Omega_2,\Omega_2) = {}& \frac{1}{(n-1)^2(b-a)}\left[\int_a^b \Upsilon^2(t)\,dt - \frac{2}{n}\int_{b-\lambda}^b (b-\lambda-t)^n\Upsilon(t)\,dt + \frac{\lambda^{2n+1}}{(2n+1)n^2}\right] \\
&- \frac{1}{(b-a)^2(n-1)^2n^2}\left(\int_a^b g(x)(x-b)^n\,dx + \frac{(-\lambda)^{n+1}}{n+1}\right)^2,
\end{aligned}
$$

where $\Upsilon(t) = \int_a^t g(x)(x-t)^{n-1}\,dx$.

Theorem 3.3 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function for some $n\geq 2$ with $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2\in L[a,b]$, and let g be an integrable function on $[a,b]$ such that $0\leq g\leq 1$. Let $\lambda=\int_a^b g(t)\,dt$ and let the functions G_1, Φ_1 and Ω_1 be defined by* (2.1), (3.3) *and* (3.4).

(i) *Then*

$$
\begin{aligned}
&\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i\,dx \\
&+ \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n-2)!}\int_a^b \Phi_1(t)\,dt = H_n^1(f;a,b), \qquad (3.5)
\end{aligned}
$$

where the remainder $H_n^1(f;a,b)$ satisfies the estimation

$$
\left|H_n^1(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!}\left[T(\Phi_1,\Phi_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2\,dt\right|^{\frac{1}{2}}. \qquad (3.6)
$$

(ii) *Then*

$$
\begin{aligned}
&\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_1(x)(x-b)^i\,dx \\
&- \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n-2)!}\int_a^b \Omega_1(t)\,dt = H_n^2(f;a,b), \qquad (3.7)
\end{aligned}
$$

where the remainder $H_n^2(f;a,b)$ satisfies the estimation

$$\left|H_n^2(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!}\left[T(\Omega_1,\Omega_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2\,dt\right|^{\frac{1}{2}}.$$

Proof (i) If we apply Theorem 3.1 for $f \to \Phi_1$ and $h \to f^{(n)}$, we obtain

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \Phi_1(t)f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b \Phi_1(t)\,dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right| \\ &\quad\leq \frac{1}{\sqrt{2}}\left[T(\Phi_1,\Phi_1)\right]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left|\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2\,dt\right|^{\frac{1}{2}}.\end{aligned}$$

Therefore we have

$$\frac{1}{(b-a)(n-2)!}\int_a^b \Phi_1(t)\,dt \cdot \frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt = \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n-2)!}\int_a^b \Phi_1(t)\,dt.$$

Now if we add that to the both sides of identity (2.2), we obtain (3.5).

(ii) Similar to the first part. □

Theorem 3.4 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function for some $n\geq 2$ with $(\cdot-a)(b-\cdot)[f^{(n+1)}]^2 \in L[a,b]$, and let g be an integrable function on $[a,b]$ such that $0\leq g\leq 1$. Let $\lambda = \int_a^b g(t)\,dt$ and let the functions G_2, Φ_2 and Ω_2 be defined by* (2.6), (3.3) *and* (3.4).

(i) *Then*

$$\begin{aligned}&\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_2(x)(x-a)^i\,dx \\ &\quad+ \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n-2)!}\int_a^b \Phi_2(t)\,dt = H_n^3(f;a,b),\end{aligned} \tag{3.8}$$

where the remainder $H_n^3(f;a,b)$ satisfies the estimation

$$\left|H_n^3(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!}\left[T(\Phi_2,\Phi_2)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2\,dt\right|^{\frac{1}{2}}.$$

(ii) *Then*

$$\begin{aligned}&\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_2(x)(x-b)^i\,dx \\ &\quad- \frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{(b-a)(n-2)!}\int_a^b \Omega_2(t)\,dt = H_n^4(f;a,b),\end{aligned} \tag{3.9}$$

where the remainder $H_n^4(f;a,b)$ satisfies the estimation

$$\left|H_n^4(f;a,b)\right| \leq \frac{\sqrt{b-a}}{\sqrt{2}(n-2)!}\left[T(\Omega_2,\Omega_2)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[f^{(n+1)}(t)\right]^2\,dt\right|^{\frac{1}{2}}.$$

Proof Similar to the proof of Theorem 3.3. □

Taking $n = 2$ in Theorems 3.3 and 3.4, we obtain the following corollaries.

Corollary 3.1 *Let $f : [a,b] \to \mathbb{R}$ be such that f'' is an absolutely continuous function with $(\cdot - a)(b - \cdot)[f''']^2 \in L[a,b]$, let g be an integrable function on $[a,b]$ such that $0 \le g \le 1$ and let $\lambda = \int_a^b g(t)\,dt$.*

(i) *Then*

$$\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + f'(a)\left(\int_a^b tg(t)\,dt - \lambda a - \frac{\lambda^2}{2}\right) + \frac{f'(b) - f'(a)}{2(b-a)}\left(\int_a^b g(t)(t-a)^2\,dt - \frac{\lambda^3}{3}\right) = H_2^1(f;a,b),$$

where the remainder $H_2^1(f;a,b)$ satisfies the estimation

$$\left|H_2^1(f;a,b)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\phi_1,\phi_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[f'''(t)\right]^2\,dt\right|^{\frac{1}{2}}$$

and

$$\begin{aligned} T(\phi_1,\phi_1) = {}& \frac{1}{b-a}\left[\int_a^b \left(\int_t^b g(x)(x-t)\,dx\right)^2 dt + \frac{\lambda^5}{20} \right. \\ & \left. - \int_a^{a+\lambda} (a+\lambda-t)^2\left(\int_t^b g(x)(x-t)\,dx\right) dt\right] \\ & - \frac{1}{4(b-a)^2}\left(\int_a^b g(x)(x-a)^2\,dx - \frac{\lambda^3}{3}\right)^2. \end{aligned}$$

(ii) *Then*

$$\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + f'(b)\left(\int_a^b tg(t)\,dt - \lambda a - \frac{\lambda^2}{2}\right) - \frac{f'(b) - f'(a)}{2(b-a)}\left(\frac{(b-a)^3 - (b-a-\lambda)^3}{3} - \int_a^b g(t)(b-t)^2\,dt\right) = H_2^2(f;a,b),$$

where the remainder $H_2^2(f;a,b)$ satisfies the estimation

$$\left|H_2^2(f;a,b)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\omega_1,\omega_1)\right]^{\frac{1}{2}}\left|\int_a^b (t-a)(b-t)\left[f'''(t)\right]^2\,dt\right|^{\frac{1}{2}}$$

and

$$\begin{aligned} T(\omega_1,\omega_1) = {}& \frac{1}{b-a}\left[\frac{(a+\lambda-b)^5 - (a-b)^5}{20} - \left(\frac{1}{2}\int_{a+\lambda}^b (a-t)^2(a+\lambda-t)^2\,dt \right.\right. \\ & + \int_a^b (a-t)^2\left(\int_a^t g(x)(x-t)\,dx\right) dt \\ & \left. - \int_{a+\lambda}^b (a+\lambda-t)^2\left(\int_a^t g(x)(x-t)\,dx\right) dt\right) \end{aligned}$$

$$
\begin{aligned}
&+\int_a^b\left(\int_a^t g(x)(x-t)\,dx\right)^2 dt\Bigg] \\
&-\frac{1}{4(b-a)^2}\left(\frac{(a+\lambda-b)^3-(a-b)^3}{3}+\int_a^b g(x)(x-b)^2\,dx\right)^2.
\end{aligned}
$$

Corollary 3.2 *Let $f:[a,b]\to\mathbb{R}$ be such that f'' is an absolutely continuous function with $(\cdot-a)(b-\cdot)[f''']^2\in L[a,b]$, let g be an integrable function on $[a,b]$ such that $0\le g\le 1$ and let $\lambda=\int_a^b g(t)\,dt$.*

(i) *Then*

$$
\begin{aligned}
&\int_a^b f(t)g(t)\,dt-\int_{b-\lambda}^b f(t)\,dt+f'(a)\left(b\lambda-\frac{\lambda^2}{2}-\int_a^b tg(t)\,dt\right) \\
&\quad+\frac{f'(b)-f'(a)}{2(b-a)}\left(\frac{(b-a)^3}{3}-\frac{(b-a-\lambda)^3}{3}-\frac{1}{2}\int_a^b g(x)(x-a)^2\,dx\right)=H_2^3(f;a,b),
\end{aligned}
$$

where the remainder $H_2^3(f;a,b)$ satisfies the estimation

$$
\left|H_2^3(f;a,b)\right|\le\frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\phi_2,\phi_2)\right]^{\frac{1}{2}}\left|\int_a^b(t-a)(b-t)\left[f'''(t)\right]^2 dt\right|^{\frac{1}{2}}
$$

and

$$
\begin{aligned}
T(\phi_2,\phi_2)=\frac{1}{b-a}\Bigg[&\frac{(b-a)^5-(b-\lambda-a)^5}{20} \\
&-\left(\frac{1}{2}\int_a^{b-\lambda}(b-t)^2(b-\lambda-t)^2\,dt\right. \\
&+\int_a^b(b-t)^2\left(\int_t^b g(x)(x-t)\,dx\right)dt \\
&\left.-\int_a^{b-\lambda}(b-\lambda-t)^2\left(\int_t^b g(x)(x-t)\,dx\right)dt\right) \\
&+\int_a^b\left(\int_t^b g(x)(x-t)\,dx\right)^2 dt\Bigg] \\
&-\frac{1}{4(b-a)^2}\left(\frac{(b-\lambda-a)^3-(b-a)^3}{3}+\int_a^b g(x)(x-a)^2\,dx\right)^2.
\end{aligned}
$$

(ii) *Then*

$$
\begin{aligned}
&\int_a^b f(t)g(t)\,dt-\int_{b-\lambda}^b f(t)\,dt+f'(b)\left(b\lambda-\frac{\lambda^2}{2}-\int_a^b tg(t)\,dt\right) \\
&\quad-\frac{f'(b)-f'(a)}{2(b-a)}\left(\int_a^b g(t)(b-t)^2\,dt-\frac{\lambda^3}{3}\right)=H_2^4(f;a,b),
\end{aligned}
$$

where the remainder $H_2^4(f;a,b)$ satisfies the estimation

$$
\left|H_2^4(f;a,b)\right|\le\frac{\sqrt{b-a}}{\sqrt{2}}\left[T(\omega_2,\omega_2)\right]^{\frac{1}{2}}\left|\int_a^b(t-a)(b-t)\left[f'''(t)\right]^2 dt\right|^{\frac{1}{2}},
$$

and

$$T(\omega_2,\omega_2) = \frac{1}{b-a}\left[\int_a^b \left(\int_a^t g(x)(x-t)\,dx\right)^2 dt + \frac{\lambda^5}{20} - \int_{b-\lambda}^b (b-\lambda-t)^2 \left(\int_a^t g(x)(x-t)\,dx\right) dt\right] - \frac{1}{4(b-a)^2}\left(\int_a^b g(x)(x-b)^2\,dx - \frac{\lambda^3}{3}\right)^2.$$

Using Theorem 3.2 we obtain the following Grüss-type inequalities.

Theorem 3.5 *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ $(n\geq 2)$ is an absolutely continuous function and $f^{(n+1)}\geq 0$ on $[a,b]$. Let the functions Φ_i and Ω_i, $i=1,2$, be defined by* (3.3) *and* (3.4).

(i) *Then we have representation* (3.5), *and the remainder $H_n^1(f;a,b)$ satisfies the bound*

$$\left|H_n^1(f;a,b)\right| \leq \frac{1}{(n-2)!}\left\|\Phi_1'\right\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}. \tag{3.10}$$

(ii) *Then we have representation* (3.7), *and the remainder $H_n^2(f;a,b)$ satisfies the bound*

$$\left|H_n^2(f;a,b)\right| \leq \frac{1}{(n-2)!}\left\|\Omega_1'\right\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

(iii) *Then we have representation* (3.8), *and the remainder $H_n^3(f;a,b)$ satisfies the bound*

$$\left|H_n^3(f;a,b)\right| \leq \frac{1}{(n-2)!}\left\|\Phi_2'\right\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

(iv) *Then we have representation* (3.9), *and the remainder $H_n^4(f;a,b)$ satisfies the bound*

$$\left|H_n^4(f;a,b)\right| \leq \frac{1}{(n-2)!}\left\|\Omega_2'\right\|_\infty \left\{\frac{f^{(n-1)}(b)+f^{(n-1)}(a)}{2} - \frac{f^{(n-2)}(b)-f^{(n-2)}(a)}{b-a}\right\}.$$

Proof (i) Applying Theorem 3.2 for $f\to\Phi_1$ and $h\to f^{(n)}$, we obtain

$$\left|\frac{1}{b-a}\int_a^b \Phi_1(t)f^{(n)}(t)\,dt - \frac{1}{b-a}\int_a^b \Phi_1(t)\,dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right| \leq \frac{1}{2(b-a)}\left\|\Phi_1'\right\|_\infty \int_a^b (t-a)(b-t)f^{(n+1)}(t)\,dt. \tag{3.11}$$

Since

$$\int_a^b (t-a)(b-t)f^{(n+1)}(t)\,dt = \int_a^b \left[2t-(a+b)\right]f^{(n)}(t)\,dt = (b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right] - 2\left(f^{(n-2)}(b)-f^{(n-2)}(a)\right),$$

using representation (2.2) and inequality (3.11), we deduce (3.10).

Similarly, we can prove the other parts. □

Taking $n = 2$ in the previous theorem, we obtain the following corollary.

Corollary 3.3 *Let $f : [a,b] \to \mathbb{R}$ be such that f'' is an absolutely continuous function and $f''' \geq 0$ on $[a,b]$. Let g be an integrable function such that $0 \leq g \leq 1$, $\lambda = \int_a^b g(t)\,dt$ and let the functions G_i, $i = 1,2$, be defined by* (2.1) *and* (2.6).

(i) *Then we have*

$$\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + f'(a)\left(\int_a^b tg(t)\,dt - \lambda a - \frac{\lambda^2}{2}\right) + \frac{f'(b)-f'(a)}{b-a}\int_a^b (x-a)G_1(x)\,dx = H_2^1(f;a,b),$$

and the remainder $H_2^1(f;a,b)$ satisfies the bound

$$\left|H_2^1(f;a,b)\right| \leq \left\|\Phi_1'\right\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\},$$

where

$$\Phi_1'(t) = -\int_a^t \bigl(1-g(x)\bigr)\,dx.$$

(ii) *Then we have*

$$\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + f'(b)\left(\int_a^b tg(t)\,dt - \lambda a - \frac{\lambda^2}{2}\right) - \frac{f'(b)-f'(a)}{b-a}\int_a^b (b-x)G_1(x)\,dx = H_2^2(f;a,b),$$

and the remainder $H_2^2(f;a,b)$ satisfies the bound

$$\left|H_2^2(f;a,b)\right| \leq \left\|\Omega_1'\right\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\},$$

where

$$\Omega_1'(t) = \int_t^b g(x)\,dx.$$

(iii) *Then we have*

$$\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + f'(a)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)\,dt\right) + \frac{f'(b)-f'(a)}{b-a}\int_a^b (x-a)G_2(x)\,dx = H_2^3(f;a,b),$$

and the remainder $H_2^3(f;a,b)$ satisfies the bound

$$\left|H_2^3(f;a,b)\right| \leq \left\|\Phi_2'\right\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\},$$

where

$$\Phi_2'(t) = -\int_a^t g(x)\,dx.$$

(iv) *Then we have*

$$\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + f'(b)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)\,dt\right) - \frac{f'(b)-f'(a)}{b-a}\int_a^b (b-x)G_2(x)\,dx = H_2^4(f;a,b),$$

and the remainder $H_2^4(f;a,b)$ *satisfies the bound*

$$\left|H_2^4(f;a,b)\right| \le \left\|\Omega_2'\right\|_\infty \left\{\frac{f'(b)+f'(a)}{2} - \frac{f(b)-f(a)}{b-a}\right\},$$

where

$$\Omega_2'(t) = \int_t^b \bigl(1-g(x)\bigr)\,dx.$$

4 Ostrowski-type inequalities

In this section we give the Ostrowski-type inequalities related to generalizations of Steffensen's inequality obtained in Section 2.

Theorem 4.1 *Suppose that all the assumptions of Theorem* 2.1 *hold. Assume that* (p,q) *is a pair of conjugate exponents, that is,* $1 \le p,q \le \infty$, $1/p + 1/q = 1$. *Let* $f^{(n)} \in L_p[a,b]$ *for some* $n \ge 2$. *Then we have*:

(i)

$$\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i\,dx\right| \le \frac{1}{(n-2)!}\left\|f^{(n)}\right\|_p \left\|\int_t^b G_1(x)(x-t)^{n-2}\,dx\right\|_q. \tag{4.1}$$

The constant on the right-hand side of (4.1) *is sharp for* $1 < p \le \infty$ *and best possible for* $p = 1$.

(ii)

$$\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_1(x)(x-b)^i\,dx\right| \le \frac{1}{(n-2)!}\left\|f^{(n)}\right\|_p \left\|\int_a^t G_1(x)(x-t)^{n-2}\,dx\right\|_q. \tag{4.2}$$

The constant on the right-hand side of (4.2) *is sharp for* $1 < p \le \infty$ *and best possible for* $p = 1$.

Proof (i) Let us denote

$$C(t)=\frac{1}{(n-2)!}\int_t^b G_1(x)(x-t)^{n-2}\,dx.$$

Since $0\leq g\leq 1$, the function G_1 is nonnegative and for every $n\geq 2$ we have $C(t)\geq 0$, $\forall t\in[a,b]$. Using identity (2.2) and applying Hölder's inequality, we obtain

$$\begin{aligned}&\left|\int_a^{a+\lambda} f(t)\,dt-\int_a^b f(t)g(t)\,dt+\sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_1(x)(x-a)^i\,dx\right|\\&\quad=\left|-\int_a^b C(t)f^{(n)}(t)\,dt\right|\leq\left\|f^{(n)}\right\|_p\left\|C(t)\right\|_q.\end{aligned}$$

For the proof of the sharpness, we will find a function f for which the equality in (4.1) is obtained.

For $1<p<\infty$ take f to be such that

$$f^{(n)}(t)=\operatorname{sgn} C(t)\left|C(t)\right|^{\frac{1}{p-1}}.$$

For $p=\infty$, take $f^{(n)}(t)=\operatorname{sgn} C(t)$.

For $p=1$, we prove that

$$\left|\int_a^b C(t)f^{(n)}(t)\,dt\right|\leq\max_{t\in[a,b]}\left|C(t)\right|\left(\int_a^b\left|f^{(n)}(t)\right|dt\right)\tag{4.3}$$

is the best possible inequality. Suppose that $|C(t)|$ attains its maximum at $t_0\in[a,b]$, and we have $C(t_0)>0$. For ε small enough, we define $f_\varepsilon(t)$ by

$$f_\varepsilon(t)=\begin{cases}0, & a\leq t\leq t_0,\\ \frac{1}{\varepsilon n!}(t-t_0)^n, & t_0\leq t\leq t_0+\varepsilon,\\ \frac{1}{n!}(t-t_0)^{n-1}, & t_0+\varepsilon\leq t\leq b.\end{cases}$$

Then, for ε small enough,

$$\left|\int_a^b C(t)f^{(n)}(t)\,dt\right|=\left|\int_{t_0}^{t_0+\varepsilon} C(t)\frac{1}{\varepsilon}\,dt\right|=\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)\,dt.$$

Now from inequality (4.3) we have

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)\,dt\leq C(t_0)\int_{t_0}^{t_0+\varepsilon}\frac{1}{\varepsilon}\,dt=C(t_0).$$

Since

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} C(t)\,dt=C(t_0),$$

the statement follows.

(ii) Here, we denote $C(t) = \frac{1}{(n-2)!}\int_a^t G_1(x)(x-t)^{n-2}\,dx$. Thus we have one more case when $|C(t)|$ attains its maximum at $t_0 \in [a,b]$ and $C(t_0) < 0$. In the case $C(t_0) < 0$, we define $f_\varepsilon(t)$ by

$$f_\varepsilon(t) = \begin{cases} \frac{1}{n!}(t-t_0-\varepsilon)^{n-1}, & a \le t \le t_0, \\ -\frac{1}{\varepsilon n!}(t-t_0-\varepsilon)^{n}, & t_0 \le t \le t_0+\varepsilon, \\ 0, & t_0+\varepsilon \le t \le b. \end{cases}$$

The rest of the proof is the same as above. □

For $n = 2$, we obtain the following result.

Corollary 4.1 *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous, let $g : [a,b] \to \mathbb{R}$ be an integrable function such that $0 \le g \le 1$, and let $\lambda = \int_a^b g(t)\,dt$. Assume that (p,q) is a pair of conjugate exponents, that is, $1 \le p,q \le \infty$, $1/p + 1/q = 1$. Let $f'' \in L_p[a,b]$. Then we have*:

(i)

$$\begin{aligned} &\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + f'(a)\left(\int_a^b tg(t)\,dt - \lambda a - \frac{\lambda^2}{2}\right)\right| \\ &\quad\le \|f''\|_p \left(\int_a^{a+\lambda}\left|t\int_a^t g(x)\,dx + \int_t^b xg(x)\,dx - \lambda a - \frac{\lambda^2}{2} - \frac{(t-a)^2}{2}\right|^q dt \right. \\ &\qquad \left. + \int_{a+\lambda}^b \left|\int_t^b xg(x)\,dx - t\int_t^b g(x)\,dx\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{4.4}$$

The constant on the right-hand side of (4.4) *is sharp for $1 < p \le \infty$ and best possible for $p = 1$.*

(ii)

$$\begin{aligned} &\left|\int_a^{a+\lambda} f(t)\,dt - \int_a^b f(t)g(t)\,dt + f'(b)\left(\int_a^b tg(t)\,dt - \lambda a - \frac{\lambda^2}{2}\right)\right| \\ &\quad\le \|f''\|_p \left(\int_a^{a+\lambda}\left|\frac{(t-a)^2}{2} - \int_a^t (t-x)g(x)\,dx\right|^q dt \right. \\ &\qquad \left. + \int_{a+\lambda}^b \left|\frac{\lambda^2}{2} + \lambda(t-a-\lambda) - \int_a^t (t-x)g(x)\,dx\right|^q dt\right)^{\frac{1}{q}}. \end{aligned} \tag{4.5}$$

The constant on the right-hand side of (4.5) *is sharp for $1 < p \le \infty$ and best possible for $p = 1$.*

Using identities (2.7) and (2.8) we obtain the following result.

Theorem 4.2 *Suppose that all the assumptions of Theorem* 2.2 *hold. Assume (p,q) is a pair of conjugate exponents, that is, $1 \le p,q \le \infty$, $1/p + 1/q = 1$. Let $f^{(n)} \in L_p[a,b]$ for some $n \ge 2$. Then we have*:

(i)

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(a)}{i!}\int_a^b G_2(x)(x-a)^i\,dx\right|$$
$$\leq \frac{1}{(n-2)!}\|f^{(n)}\|_p\left\|\int_t^b G_2(x)(x-t)^{n-2}\,dx\right\|_q. \tag{4.6}$$

The constant on the right-hand side of (4.6) *is sharp for* $1 < p \leq \infty$ *and best possible for* $p = 1$.

(ii)

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + \sum_{i=0}^{n-2}\frac{f^{(i+1)}(b)}{i!}\int_a^b G_2(x)(x-b)^i\,dx\right|$$
$$\leq \frac{1}{(n-2)!}\|f^{(n)}\|_p\left\|\int_a^t G_2(x)(x-t)^{n-2}\,dx\right\|_q. \tag{4.7}$$

The constant on the right-hand side of (4.7) *is sharp for* $1 < p \leq \infty$ *and best possible for* $p = 1$.

Proof Similar to the proof of Theorem 4.1. □

Taking $n = 2$ in the previous theorem, we obtain the following corollary.

Corollary 4.2 *Let* $f : [a,b] \to \mathbb{R}$ *be such that* f' *is absolutely continuous, let* $g : [a,b] \to \mathbb{R}$ *be an integrable function such that* $0 \leq g \leq 1$, *and let* $\lambda = \int_a^b g(t)\,dt$. *Assume that* (p,q) *is a pair of conjugate exponents, that is,* $1 \leq p,q \leq \infty$, $1/p + 1/q = 1$. *Let* $f'' \in L_p[a,b]$. *Then we have*:

(i)

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + f'(a)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)\,dt\right)\right|$$
$$\leq \|f''\|_p\left(\int_a^{b-\lambda}\left|b\lambda - \frac{\lambda^2}{2} - t\int_a^t g(x)\,dx - \int_t^b xg(x)\,dx\right|^q dt\right.$$
$$\left.+\int_{b-\lambda}^b\left|\frac{(b-t)^2}{2} - \int_t^b (x-t)g(x)\,dx\right|^q dt\right)^{\frac{1}{q}}. \tag{4.8}$$

The constant on the right-hand side of (4.8) *is sharp for* $1 < p \leq \infty$ *and best possible for* $p = 1$.

(ii)

$$\left|\int_a^b f(t)g(t)\,dt - \int_{b-\lambda}^b f(t)\,dt + f'(b)\left(b\lambda - \frac{\lambda^2}{2} - \int_a^b tg(t)\,dt\right)\right|$$
$$\leq \|f''\|_p\left(\int_a^{b-\lambda}\left|\int_a^t (t-x)g(x)\,dx\right|^q dt\right.$$
$$\left.+\int_{b-\lambda}^b\left|\int_a^b (b-x)g(x)\,dx - \frac{(b-t)^2-\lambda^2}{2} - \lambda(t-b+\lambda)\right|^q dt\right)^{\frac{1}{q}}. \tag{4.9}$$

The constant on the right-hand side of (4.9) *is sharp for* $1 < p \leq \infty$ *and best possible for* $p = 1$.

5 *k*-Exponential convexity and exponential convexity

In [5] Bernstein defined exponentially convex functions on an open interval I in the following way.

Definition 5.1 A function $\psi : I \to \mathbb{R}$ is *exponentially convex* on I if it is continuous and

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi(x_i + x_j) \geq 0$$

for all $n \in \mathbb{N}$ and all choices $\xi_i \in \mathbb{R}$, $i = 1, \dots, n$, such that $x_i + x_j \in I$, $1 \leq i, j \leq n$.

Proposition 5.1 *Let* $\psi : I \to \mathbb{R}$. *The following statements are equivalent*:

(i) ψ *is exponentially convex on* I;

(ii) ψ *is continuous and*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geq 0 \tag{5.1}$$

for every $\xi_i \in \mathbb{R}$ *and every* $x_i \in I$, $1 \leq i \leq n$.

Remark 5.1 From (5.1) we have the following properties of exponentially convex functions:

(i) if ψ is exponentially convex on I, then $\psi(x) \geq 0$ for all $x \in I$; for any $c \geq 0$, $c\psi$ is again exponentially convex;

(ii) if ψ_1 and ψ_2 are exponentially convex on I, then $\psi_1 + \psi_2$ is also exponentially convex on I;

(iii) if ψ is an exponentially convex function, then for any $d, t \in \mathbb{R}$, $x \to \psi(dx)$ and $x \to \psi(x - t)$ are exponentially convex functions.

Using basic calculus we have the following corollary.

Corollary 5.1 *If* ψ *is exponentially convex on* I, *then the matrix*

$$\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n}$$

is positive semidefinite. *Particularly*,

$$\det\left[\psi\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{n} \geq 0$$

for every $n \in \mathbb{N}$, $x_i \in I$, $i = 1, \dots, n$.

One of the most important properties of exponentially convex functions is their integral representation.

Theorem 5.1 *The function* $\psi : I \to \mathbb{R}$ *is exponentially convex on* I *if and only if*

$$\psi(x) = \int_{-\infty}^{\infty} e^{tx}\, d\sigma(t), \quad x \in I \tag{5.2}$$

for some nondecreasing function $\sigma : \mathbb{R} \to \mathbb{R}$.

Proof See [6], p.211. □

The most obvious example of an exponentially convex function is $x \mapsto ce^{\alpha x}$, where $c \geq 0$ and $\alpha \in \mathbb{R}$ are constants. Other, less obvious, examples can be deduced using integral representation (5.2) and some result from Laplace transformation (for details, see [7]).

For readers' convenience we recall some other notions and definitions of some classes of functions widely used in the literature which are related to the class of exponentially convex functions.

Firstly, let us mention an important subclass of the class of exponentially convex functions called completely monotonic functions. Let $J \subseteq (0, \infty)$ be an open interval.

Definition 5.2 The function $f : J \to \mathbb{R}$ is *completely monotonic on* J if

$$(-1)^k f^{(k)}(x) \geq 0, \quad x \in J, k = 0, 1, \ldots.$$

Theorem 5.2 *The function* $f : J \to \mathbb{R}$ *is completely monotonic on* J *if and only if*

$$f(x) = \int_0^{\infty} e^{-tx}\, d\sigma(t), \quad x \in J,$$

for some nondecreasing bounded function $\sigma : (0, \infty) \to \mathbb{R}$.

Proof See [8], p.160. □

In [9] Bhatia uses the notion *functions of positive type* defined in the following way.

Definition 5.3 A complex-valued function φ on $[0, \infty)$ is said to be *of positive type* if for every positive integer n, we have

$$\sum_{i,j=1}^{n} \varphi(x_i + x_j)\xi_i \overline{\xi_j} \geq 0$$

for every choice of $x_i \in [0, \infty)$ and $\xi_i \in \mathbb{C}$, $i = 1, \ldots, n$.

Remark 5.2 As noted in [9], functions of positive type are characterized as being completely monotonic.

Another widely used term is *positive definite functions*. Survey of positive definite functions is given in [10] and [11].

Definition 5.4 A complex-valued function φ on $\mathbb{R}$ is said to be *positive definite* if for every positive integer n, we have

$$\sum_{i,j=1}^{n} \varphi(x_i - x_j)\xi_i\overline{\xi_j} \geq 0,$$

for every choice of $x_i \in \mathbb{R}$ and $\xi_i \in \mathbb{C}$, $i = 1, \dots, n$; that is, if every square matrix $[\varphi(x_i - x_j)]_{i,j=1}^{n}$ is positive semidefinite.

Positive definite functions satisfy the following characterization (see [12]).

Theorem 5.3 (Bochner) *A continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ is positive definite if and only if it is the Fourier transform of a finite positive measure μ on $\mathbb{R}$, that is,*

$$\varphi(x) = \int_{-\infty}^{\infty} e^{itx}\, d\mu(t).$$

The class of positive definite functions and the class of exponentially convex functions are not necessarily equivalent. However, in the following theorem we give a bijection between exponentially convex and entire positive definite functions.

Theorem 5.4 *If a function $\psi : \mathbb{R} \to \mathbb{R}$ is an exponentially convex function, then it is entire, and $\varphi(t) = \psi(it)$, $t \in \mathbb{R}$ is a positive definite function. Conversely, if $\varphi : \mathbb{R} \to \mathbb{C}$ is an entire positive definite function, then $\psi(t) = \varphi(-it)$, $t \in \mathbb{R}$ is an exponentially convex function.*

Proof See [13]. □

Remark 5.3

(i) Recall that $\psi : \mathbb{R} \to \mathbb{C}$ is entire if it can be extended to a necessarily unique analytic function $\psi : \mathbb{C} \to \mathbb{C}$.

(ii) Conclusions of Theorem 5.4 can be extended to exponentially convex functions defined on any open interval (see [13]).

(iii) The key application of Theorem 5.4 lies in the explicit construction of exponentially convex functions from positive definite functions.

The notion of exponential convexity is, for the sake of applications, refined in [14] in the following way.

Definition 5.5 A function $\psi : I \to \mathbb{R}$ is *k-exponentially convex in the Jensen sense* on I if

$$\sum_{i,j=1}^{k} \xi_i\xi_j\psi\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for all choices $\xi_1, \dots, \xi_k \in \mathbb{R}$ and all choices $x_1, \dots, x_k \in I$. A function $\psi : I \to \mathbb{R}$ is *k-exponentially convex* if it is k-exponentially convex in the Jensen sense and continuous on I.

Remark 5.4 A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is k-exponentially convex in the Jensen sense for all $k \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 5.5 It is known that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Motivated by inequalities (2.9), (2.11), (2.12) and (2.14), under the assumptions of Theorems 2.3 and 2.4, we define the following linear functionals:

$$L_1(f) = \int_a^b f(t)g(t)\,dt - \int_a^{a+\lambda} f(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_1(x)(x-a)^i\,dx, \tag{5.3}$$

$$L_2(f) = \int_a^b f(t)g(t)\,dt - \int_a^{a+\lambda} f(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_1(x)(x-b)^i\,dx, \tag{5.4}$$

$$L_3(f) = \int_{b-\lambda}^b f(t)\,dt - \int_a^b f(t)g(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(a)}{i!} \int_a^b G_2(x)(x-a)^i\,dx, \tag{5.5}$$

$$L_4(f) = \int_{b-\lambda}^b f(t)\,dt - \int_a^b f(t)g(t)\,dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(b)}{i!} \int_a^b G_2(x)(x-b)^i\,dx. \tag{5.6}$$

Remark 5.6 Under the assumptions of Theorems 2.3 and 2.4, it holds $L_i(f) \geq 0$, $i = 1, \ldots, 4$, for all n-convex functions f.

Lagrange- and Cauchy-type mean value theorems related to defined functionals are given in the following theorems. Similar results were proved in [15], so we omit the proof.

Theorem 5.5 *Let $f : [a,b] \to \mathbb{R}$ be such that $f \in C^n[a,b]$. If the inequalities in* (2.10) ($i = 2$) *and* (2.13) ($i = 4$) *hold, then there exist $\xi_i \in [a,b]$ such that*

$$L_i(f) = f^{(n)}(\xi_i)L_i(\varphi), \quad i = 1, \ldots, 4,$$

where $\varphi(x) = \frac{x^n}{n!}$ and L_i, $i = 1, \ldots, 4$, are defined by (5.3)-(5.6).

Theorem 5.6 *Let $f, \hat{f} : [a,b] \to \mathbb{R}$ be such that $f, \hat{f} \in C^n[a,b]$ and $\hat{f}^{(n)} \neq 0$. If the inequalities in* (2.10) ($i = 2$) *and* (2.13) ($i = 4$) *hold, then there exist $\xi_i \in [a,b]$ such that*

$$\frac{L_i(f)}{L_i(\hat{f})} = \frac{f^{(n)}(\xi_i)}{\hat{f}^{(n)}(\xi_i)}, \quad i = 1, \ldots, 4,$$

where L_i, $i = 1, \ldots, 4$, are defined by (5.3)-(5.6).

Now we produce k-exponentially and exponentially convex functions applying defined functionals. We use an idea from [14]. In the sequel I and J will be intervals in $\mathbb{R}$.

Theorem 5.7 *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $p \mapsto [x_0, \dots, x_n; f_p]$ is k-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in I$. Let L_i, $i = 1, \dots, 4$, be linear functionals defined by* (5.3)-(5.6). *Then $p \mapsto L_i(f_p)$ is a k-exponentially convex function in the Jensen sense on J.*

If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is k-exponentially convex on J.

Proof For $\xi_j \in \mathbb{R}$ and $p_j \in J$, $j = 1, \dots, k$, we define the function

$$h(x) = \sum_{j,l=1}^{k} \xi_j \xi_l f_{\frac{p_j+p_l}{2}}(x).$$

Using the assumption that the function $p \mapsto [x_0, \dots, x_n; f_p]$ is k-exponentially convex in the Jensen sense, we have

$$[x_0, \dots, x_n; h] = \sum_{j,l=1}^{k} \xi_j \xi_l [x_0, \dots, x_n; f_{\frac{p_j+p_l}{2}}] \geq 0,$$

which in turn implies that h is an n-convex function on J, so $L_i(h) \geq 0$, $i = 1, \dots, 4$. Hence

$$\sum_{j,l=1}^{k} \xi_j \xi_l L_i(f_{\frac{p_j+p_l}{2}}) \geq 0.$$

We conclude that the function $p \mapsto L_i(f_p)$ is k-exponentially convex on J in the Jensen sense.

If the function $p \mapsto L_i(f_p)$ is also continuous on J, then $p \mapsto L_i(f_p)$ is k-exponentially convex by definition. □

The following corollary is an immediate consequence of the above theorem.

Corollary 5.2 *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in R such that the function $p \mapsto [x_0, \dots, x_n; f_p]$ is exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in I$. Let L_i, $i = 1, \dots, 4$, be linear functionals defined by* (5.3)-(5.6). *Then $p \mapsto L_i(f_p)$ is an exponentially convex function in the Jensen sense on J. If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is exponentially convex on J.*

Now, we prove corollary of Theorem 5.7 which will be used in the next section to obtain new Stolarsky-type means.

Corollary 5.3 *Let $\Lambda = \{f_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$ such that the function $p \mapsto [x_0, \dots, x_n; f_p]$ is 2-exponentially convex in the Jensen sense on J for every $(n+1)$ mutually different points $x_0, \dots, x_n \in I$. Let L_i, $i = 1, \dots, 4$, be linear functionals defined by* (5.3)-(5.6). *Then the following statements hold*:

(i) *If the function $p \mapsto L_i(f_p)$ is continuous on J, then it is a 2-exponentially convex function on J. If $p \mapsto L_i(f_p)$ is additionally strictly positive, then it is also log-convex on J. Furthermore, the following inequality holds true*:

$$\left[L_i(f_s)\right]^{t-r} \le \left[L_i(f_r)\right]^{t-s}\left[L_i(f_t)\right]^{s-r}, \quad i = 1,\dots,4,$$

for every choice $r,s,t \in J$ such that $r < s < t$.

(ii) *If the function $p \mapsto L_i(f_p)$ is strictly positive and differentiable on J, then for every $p,q,u,v \in J$, such that $p \le u$ and $q \le v$, we have*

$$\mu_{p,q}(L_i,\Lambda) \le \mu_{u,v}(L_i,\Lambda), \tag{5.7}$$

where

$$\mu_{p,q}(L_i,\Lambda) = \begin{cases} \left(\frac{L_i(f_p)}{L_i(f_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\frac{d}{dp}L_i(f_p)}{L_i(f_p)}\right), & p = q, \end{cases} \tag{5.8}$$

for $f_p, f_q \in \Lambda$.

Proof (i) This is an immediate consequence of Theorem 5.7 and Remark 5.5.

(ii) Since $p \mapsto L_i(f_p)$ is positive and continuous, by (i) we have that $p \mapsto L_i(f_p)$ is log-convex on J, that is, the function $p \mapsto \log L_i(f_p)$ is convex on J. Hence we get

$$\frac{\log L_i(f_p) - \log L_i(f_q)}{p-q} \le \frac{\log L_i(f_u) - \log L_i(f_v)}{u-v} \tag{5.9}$$

for $p \le u$, $q \le v$, $p \neq q$, $u \neq v$. So, we conclude that

$$\mu_{p,q}(L_i,\Lambda) \le \mu_{u,v}(L_i,\Lambda).$$

Cases $p = q$ and $u = v$ follow from (5.9) as limit cases. □

6 Stolarsky-type means

In this section, we present several families of functions which fulfill the conditions of Theorem 5.7, Corollary 5.2 and Corollary 5.3. This enables us to construct a large family of functions which are exponentially convex. Explicit form of these functions is obtained after calculating an explicit action of functionals on a given family.

Firstly, let us recall Stolarsky means (see [16] and [17]).

Let $p,q \in \mathbb{R}$ and let $0 < x < y < \infty$. The Stolarsky mean $E_{p,q}(x,y)$ is defined by

$$E_{p,q}(x,y) = \begin{cases} \left(\frac{q(y^p - x^p)}{p(y^q - x^q)}\right)^{\frac{1}{p-q}}, & pq(p-q) \neq 0, \\ \left(\frac{y^q - x^q}{q(\log y - \log x)}\right)^{\frac{1}{q}}, & p = 0, q \neq 0, \\ e^{-\frac{1}{q}}\left(\frac{x^{x^q}}{y^{y^q}}\right)^{\frac{1}{x^q - y^q}}, & p = q \neq 0, \\ \sqrt{xy}, & p = q = 0. \end{cases}$$

Stolarsky in [16] and then Leach and Sholander in [18] showed that the function $E_{p,q}(x,y)$ is increasing in both parameters p and q, *i.e.* for $p \leq r$ and $q \leq s$, we have

$$E_{p,q}(x,y) \leq E_{r,s}(x,y).$$

Example 6.1 Let us consider a family of functions

$$\Lambda_1 = \{f_p : \mathbb{R} \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$f_p(x) = \begin{cases} \frac{e^{px}}{p^n}, & p \neq 0, \\ \frac{x^n}{n!}, & p = 0. \end{cases}$$

Since $\frac{d^n f_p}{dx^n}(x) = e^{px} > 0$, the function f_p is n-convex on $\mathbb{R}$ for every $p \in \mathbb{R}$ and $p \mapsto \frac{d^n f_p}{dx^n}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 5.7, we also have that $p \mapsto [x_0, \dots, x_n; f_p]$ is exponentially convex (and so exponentially convex in the Jensen sense). Now, using Corollary 5.2 we conclude that $p \mapsto L_i(f_p)$, $i = 1, \dots, 4$, are exponentially convex in the Jensen sense. It is easy to verify that these mappings are continuous (although the mapping $p \mapsto f_p$ is not continuous for $p = 0$), so they are exponentially convex. For this family of functions, $\mu_{p,q}(L_i, \Lambda_1)$, $i = 1, \dots, 4$, from (5.8), becomes

$$\mu_{p,q}(L_i, \Lambda_1) = \begin{cases} (\frac{L_i(f_p)}{L_i(f_q)})^{\frac{1}{p-q}}, & p \neq q, \\ \exp(\frac{L_i(id \cdot f_p)}{L_i(f_p)} - \frac{n}{p}), & p = q \neq 0, \\ \exp(\frac{1}{n+1} \frac{L_i(id \cdot f_0)}{L_i(f_0)}), & p = q = 0, \end{cases}$$

where id is the identity function. By Corollary 5.3, $\mu_{p,q}(L_i, \Lambda_1)$, $i = 1, \dots, 4$, are monotonic functions in parameters p and q.

Since

$$\left(\frac{\frac{d^n f_p}{dx^n}}{\frac{d^n f_q}{dx^n}} \right)^{\frac{1}{p-q}} (\log x) = x,$$

using Theorem 5.6 it follows that

$$M_{p,q}(L_i, \Lambda_1) = \log \mu_{p,q}(L_i, \Lambda_1), \quad i = 1, \dots, 4,$$

satisfy

$$a \leq M_{p,q}(L_i, \Lambda_1) \leq b, \quad i = 1, \dots, 4.$$

So, $M_{p,q}(L_i, \Lambda_1)$, $i = 1, \dots, 4$, are monotonic means.

Example 6.2 Let us consider a family of functions

$$\Lambda_2 = \{g_p : (0,\infty) \to \mathbb{R} : p \in \mathbb{R}\}$$

defined by

$$g_p(x) = \begin{cases} \frac{x^p}{p(p-1)\cdots(p-n+1)}, & p \notin \{0,1,\ldots,n-1\}, \\ \frac{x^j \log x}{(-1)^{n-1-j} j!(n-1-j)!}, & p = j \in \{0,1,\ldots,n-1\}. \end{cases}$$

Since $\frac{d^n g_p}{dx^n}(x) = x^{p-n} > 0$, the function g_p is n-convex for $x > 0$, and $p \mapsto \frac{d^n g_p}{dx^n}(x)$ is exponentially convex by definition. Arguing as in Example 6.1 we get that the mappings $p \mapsto L_i(g_p)$, $i = 1,\ldots,4$, are exponentially convex. Hence, for this family of functions $\mu_{p,q}(L_i,\Lambda_2)$, $i = 1,\ldots,4$, from (5.8), is equal to

$$\mu_{p,q}(L_i,\Lambda_2) = \begin{cases} \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}}, & p \neq q, \\ \exp((-1)^{n-1}(n-1)!\frac{L_i(g_0 g_p)}{L_i(g_p)} + \sum_{k=0}^{n-1}\frac{1}{k-p}), & p = q \notin \{0,1,\ldots,n-1\}, \\ \exp((-1)^{n-1}(n-1)!\frac{L_i(g_0 g_p)}{2L_i(g_p)} + \sum_{\substack{k=0 \\ k\neq p}}^{n-1}\frac{1}{k-p}), & p = q \in \{0,1,\ldots,n-1\}. \end{cases}$$

Again, using Theorem 5.6 we conclude that

$$a \leq \left(\frac{L_i(g_p)}{L_i(g_q)}\right)^{\frac{1}{p-q}} \leq b, \quad i = 1,\ldots,4.$$

So, $\mu_{p,q}(L_i,\Lambda_2)$, $i = 1,\ldots,4$, are means and by (5.7) they are monotonic.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors jointly worked on the results and they read and approved the final manuscript.

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