Deng et al. *Journal of Inequalities and Applications* (2015) 2015:186
DOI 10.1186/s13660-015-0699-z

Journal of Inequalities and Applications
a SpringerOpen Journal

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Sharp inequalities and asymptotic expansion associated with the Wallis sequence

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Abstract

We present asymptotic expansion of function involving the ratio of gamma functions and provide a recurrence relation for determining the coefficients of the asymptotic expansion. As a consequence, we obtain asymptotic expansion of the Wallis sequence. Also, we establish sharp inequalities for the Wallis sequence.

MSC: Primary 40A05; secondary 33B15; 41A60; 26D15

Keywords: Wallis sequence; gamma function; psi function; polygamma function; inequality; asymptotic expansion

1 Introduction

The Wallis sequence to which the title refers is

$$W_n = \prod_{k=1}^{n} \frac{4k^2}{4k^2-1}, \quad n \in \mathbb{N} := \{1, 2, 3, \ldots\}. \tag{1.1}$$

Wallis (1616-1703) discovered that

$$\prod_{k=1}^{\infty} \frac{4k^2}{4k^2-1} = \frac{2}{1}\frac{2}{3}\frac{4}{3}\frac{4}{5}\frac{6}{5}\frac{6}{7}\frac{8}{7} = \cdots = \frac{\pi}{2} \tag{1.2}$$

(see [1], p.68). Several elementary proofs of (1.2) can be found (see, for example, [2–4]). An interesting geometric construction produces (1.2) [5]. Many formulas exist for the representation of π, and a collection of these formulas is listed [6, 7]. For more history of π see [1, 8–10].

Some inequalities and asymptotic formulas associated with the Wallis sequence W_n can be found (see, for example, [11–24]). Hirschhorn [13] proved that for $n \in \mathbb{N}$,

$$\frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{7}{3}}\right) < W_n < \frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{8}{3}}\right). \tag{1.3}$$

Also in [13], Hirschhorn pointed out that if the c_j are given by

$$\tanh\left(\frac{x}{4}\right) = \sum_{j=0}^{\infty} c_j \frac{x^{2j+1}}{(2j)!}, \tag{1.4}$$

then, as $n \to \infty$,

$$W_n \sim \frac{\pi}{2}\left(1+\frac{1}{2n}\right)^{-1}\prod_{j\geq 0}\exp\left(\frac{c_j}{n^{2j+1}}\right) = \frac{\pi}{2}\left(1+\frac{1}{2n}\right)^{-1}\exp\left(\sum_{j=0}^{\infty}\frac{c_j}{n^{2j+1}}\right). \tag{1.5}$$

Very recently, Lin *et al.* [17] found that

$$c_j = \frac{(2^{2j+2}-1)B_{2j+2}}{2^{2j+1}(2j+1)(j+1)}, \quad j \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}, \tag{1.6}$$

where B_n ($n \in \mathbb{N}_0$) are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi.$$

Also in [17], Lin *et al.* derived

$$W_n = \frac{\pi}{2}\left(1-\frac{1}{4n+\frac{5}{2}}\right)^{1-\frac{3}{64n^2}+\frac{3}{64n^3}-\frac{23}{1{,}024n^4}+O(n^{-5})}, \quad n \to \infty. \tag{1.7}$$

The gamma function is defined for $x > 0$ by

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\,\mathrm{d}t.$$

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called psi (or digamma) function, and $\psi^{(k)}(x)$ ($k \in \mathbb{N}$) are called polygamma functions. These functions play an important role in various branches of mathematics as well as in physics and engineering. For the various properties of these functions, please refer to [25], pp.255-260.

Define the function $W(x)$ by

$$W(x) = \frac{\pi}{2}\left(1+\frac{1}{2x}\right)^{-1}\frac{1}{x}\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2. \tag{1.8}$$

It is easy to see that

$$W_n = W(n).$$

The first aim of present paper is to establish sharp inequalities for W_n. More precisely, we determine the best possible constants α, β, λ, and μ such that the double inequalities

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\alpha}\right) < W_n \leq \frac{\pi}{2}\left(1-\frac{1}{4n+\beta}\right)$$

and

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\frac{5}{2}}\right)^{\lambda} < W_n \leq \frac{\pi}{2}\left(1-\frac{1}{4n+\frac{5}{2}}\right)^{\mu}$$

hold for all $n \in \mathbb{N}$. The second aim of present paper is to develop the formula (1.7) to produce a complete asymptotic expansion. More precisely, we provide a recurrence relation for determining the coefficients r_j ($j \in \mathbb{N}_0$) such that

$$W(x) \sim \frac{\pi}{2}\left(1 - \frac{1}{4x + \frac{5}{2}}\right)^{\sum_{j=0}^{\infty} r_j x^{-j}}, \quad x \to \infty.$$

2 Lemmas

The following lemmas are required in our present investigation.

Lemma 1 ([26], Corollary 1) *Let $m, n \in \mathbb{N}$. Then for $x > 0$,*

$$\begin{aligned} &\sum_{j=1}^{2m}\left(1 - \frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}} \\ &\quad < (-1)^n\left(\psi^{(n-1)}(x+1) - \psi^{(n-1)}\left(x + \frac{1}{2}\right)\right) + \frac{(n-1)!}{2x^n} \\ &\quad < \sum_{j=1}^{2m-1}\left(1 - \frac{1}{2^{2j}}\right)\frac{2B_{2j}}{(2j)!}\frac{(2j+n-2)!}{x^{2j+n-1}}, \end{aligned} \tag{2.1}$$

where B_n are the Bernoulli numbers.

It follows from (2.1) that, for $x > 0$,

$$\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6} < \psi(x+1) - \psi\left(x + \frac{1}{2}\right) < \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} \tag{2.2}$$

and

$$-\frac{1}{2x^2} + \frac{1}{4x^3} - \frac{1}{16x^5} < \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) < -\frac{1}{2x^2} + \frac{1}{4x^3} - \frac{1}{16x^5} + \frac{3}{64x^7}. \tag{2.3}$$

Lemma 2 *For all $x \geq 1$,*

$$\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\right]^2 < \frac{1}{x} - \frac{3}{4x^2} + \frac{17}{32x^3} - \frac{45}{128x^4} + \frac{443}{2{,}048x^5}. \tag{2.4}$$

Proof We consider the function $G(x)$ defined by

$$\begin{aligned} G(x) = {}& 2\ln\Gamma(x+1) - 2\left[\ln\Gamma\left(x + \frac{1}{2}\right) + \ln\left(x + \frac{1}{2}\right)\right] \\ &- \ln\left(\frac{1}{x} - \frac{3}{4x^2} + \frac{17}{32x^3} - \frac{45}{128x^4} + \frac{443}{2{,}048x^5}\right). \end{aligned}$$

From the asymptotic expansion ([25], p.257):

$$\begin{aligned} x^{b-a}\frac{\Gamma(x+a)}{\Gamma(x+b)} = {}& 1 + \frac{(a-b)(a+b-1)}{2x} \\ &+ \frac{1}{12}\binom{a-b}{2}\left(3(a+b-1)^2 - a + b - 1\right)\frac{1}{x^2} + \cdots \quad \text{as } x \to \infty, \end{aligned} \tag{2.5}$$

we conclude that

$$\lim_{x\to\infty} G(x) = 0.$$

Differentiating and applying the first inequality in (2.2) yield, for $x \geq 1$,

$$\begin{aligned}
G'(x) &= 2\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right] \\
&\quad - \frac{4{,}096x^5 - 2{,}048x^4 + 896x^3 - 384x^2 + 222x - 2{,}215}{x(2x+1)(2{,}048x^4 - 1{,}536x^3 + 1{,}088x^2 - 720x + 443)} \\
&> 2\left(\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4} - \frac{1}{128x^6}\right) \\
&\quad - \frac{4{,}096x^5 - 2{,}048x^4 + 896x^3 - 384x^2 + 222x - 2{,}215}{x(2x+1)(2{,}048x^4 - 1{,}536x^3 + 1{,}088x^2 - 720x + 443)} \\
&= \bigl(158{,}193 + 797{,}514(x-1) + 1{,}606{,}106(x-1)^2 + 1{,}619{,}020(x-1)^3 \\
&\quad + 816{,}432(x-1)^4 + 164{,}640(x-1)^5\bigr)/\bigl(64x^6(2x+1)\bigl(1{,}323 + 5{,}040(x-1) \\
&\quad + 8{,}768(x-1)^2 + 6{,}656(x-1)^3 + 2{,}048(x-1)^4\bigr)\bigr) \\
&> 0.
\end{aligned}$$

This leads to

$$\begin{aligned}
G(x) &= \ln\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\right]^2 - \ln\left(\frac{1}{x} - \frac{3}{4x^2} + \frac{17}{32x^3} - \frac{45}{128x^4} + \frac{443}{2{,}048x^5}\right) \\
&< \lim_{x\to\infty} G(x) = 0, \quad x \geq 1.
\end{aligned}$$

The proof of Lemma 2 is complete. □

By (2.2), we obtain

$$\begin{aligned}
(2x+1)\left[\psi(x+1) - \psi\left(x+\frac{1}{2}\right)\right] - 1 &< (2x+1)\left(\frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{64x^4}\right) - 1 \\
&= \frac{1}{4x} - \frac{1}{8x^2} + \frac{1}{32x^3} + \frac{1}{64x^4}.
\end{aligned} \tag{2.6}$$

By (2.4), we get

$$\begin{aligned}
1 - \left(x+\frac{1}{2}\right)\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\right]^2 &> 1 - \left(x+\frac{1}{2}\right)\left(\frac{1}{x} - \frac{3}{4x^2} + \frac{17}{32x^3} - \frac{45}{128x^4} + \frac{443}{2{,}048x^5}\right) \\
&= \frac{1}{4x} - \frac{5}{32x^2} + \frac{11}{128x^3} - \frac{83}{2{,}048x^4} - \frac{443}{4{,}096x^5}.
\end{aligned} \tag{2.7}$$

The proof of Theorem 1 makes use of (2.6) and (2.7).

Lemma 3 ([27]) *Let $-\infty \leq a < b \leq \infty$. Let f and g be differentiable functions on an interval (a,b). Assume that either $g' > 0$ everywhere on (a,b) or $g' < 0$ on (a,b). Suppose that $f(a+) = g(a+) = 0$ or $f(b-) = g(b-) = 0$. Then*

(1) *if* $\frac{f'}{g'}$ *is increasing on* (a,b), *then* $(\frac{f}{g})' > 0$ *on* (a,b);
(2) *if* $\frac{f'}{g'}$ *is decreasing on* (a,b), *then* $(\frac{f}{g})' < 0$ *on* (a,b).

3 Sharp inequalities

Theorem 1 *For all* $n \in \mathbb{N}$,

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\alpha}\right) < W_n \le \frac{\pi}{2}\left(1-\frac{1}{4n+\beta}\right) \tag{3.1}$$

with the best possible constants

$$\alpha = \frac{5}{2} \quad \text{and} \quad \beta = \frac{32-9\pi}{3\pi-8} = 2.614909986\ldots.$$

Equality in (3.1) *occurs for* $n = 1$.

Proof The inequality (3.1) can be written as

$$\alpha \le F(n) < \beta,$$

where

$$F(x) = \frac{1}{1-\frac{1}{x+1/2}[\frac{\Gamma(x+1)}{\Gamma(x+1/2)}]^2} - 4x.$$

Using (2.5), we conclude that

$$\lim_{x\to\infty} F(x) = \frac{5}{2}.$$

Differentiating $F(x)$ and applying (2.4), (2.6), and (2.7) yield, for $x \ge 6$,

$$\begin{aligned}
&\left(1-\left(x+\frac{1}{2}\right)\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\right]^2\right)^2 F'(x) \\
&\quad = \left\{(2x+1)\left[\psi(x+1)-\psi\left(x+\frac{1}{2}\right)\right]-1\right\}\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\right]^2 \\
&\qquad - 4\left(1-\left(x+\frac{1}{2}\right)\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{3}{2})}\right]^2\right)^2 \\
&\quad < \left(\frac{1}{4x}-\frac{1}{8x^2}+\frac{1}{32x^3}+\frac{1}{64x^4}\right)\left(\frac{1}{x}-\frac{3}{4x^2}+\frac{17}{32x^3}-\frac{45}{128x^4}+\frac{443}{2{,}048x^5}\right) \\
&\qquad - 4\left(\frac{1}{4x}-\frac{5}{32x^2}+\frac{11}{128x^3}-\frac{83}{2{,}048x^4}-\frac{443}{4{,}096x^5}\right)^2 \\
&\quad = -\frac{1}{4{,}194{,}304x^{10}}\bigl(248{,}771{,}713+769{,}183{,}956(x-6)+510{,}154{,}660(x-6)^2 \\
&\qquad + 149{,}038{,}464(x-6)^3+22{,}221{,}824(x-6)^4 \\
&\qquad + 1{,}658{,}880(x-6)^5+49{,}152(x-6)^6\bigr) \\
&\quad < 0.
\end{aligned}$$

Straightforward calculation produces

$$\begin{aligned}
F(1) &= \frac{32-9\pi}{3\pi-8} = 2.6149\ldots, \\
F(2) &= \frac{-315\pi+1{,}024}{45\pi-128} = 2.5724\ldots, \\
F(3) &= \frac{-1{,}925\pi+6{,}144}{175\pi-512} = 2.5526\ldots, \\
F(4) &= \frac{-165{,}375\pi+524{,}288}{11{,}025\pi-32{,}768} = 2.5412\ldots, \\
F(5) &= \frac{-829{,}521\pi+2{,}621{,}440}{43{,}659\pi-131{,}072} = 2.5338\ldots, \\
F(6) &= \frac{-15{,}954{,}939\pi+50{,}331{,}648}{693{,}693\pi-2{,}097{,}152} = 2.5286\ldots.
\end{aligned}$$

Thus, the sequence $(F(n))_{n\in\mathbb{N}}$ is strictly decreasing. This leads to

$$\frac{5}{2} < \lim_{x\to\infty} F(x) < F(n) \le F(1) = \frac{32-9\pi}{3\pi-8}, \quad n \in \mathbb{N}.$$

The proof of Theorem 1 is complete. □

Remark 1 In fact, Elezović *et al.* [12] have previously shown that $\frac{5}{2}$ is the best possible constant for a lower bound of W_n of the type $\frac{\pi}{2}(1-\frac{1}{4n+\alpha})$. Moreover, the authors pointed out that

$$W_n = \frac{\pi}{2}\left(1-\frac{1}{4n+\frac{5}{2}}\right) + O\left(\frac{1}{n^3}\right), \quad n \to \infty.$$

Theorem 2 *For all* $n \in \mathbb{N}$,

$$\frac{\pi}{2}\left(1-\frac{1}{4n+\frac{5}{2}}\right)^{\lambda} < W_n \le \frac{\pi}{2}\left(1-\frac{1}{4n+\frac{5}{2}}\right)^{\mu} \tag{3.2}$$

with the best possible constants

$$\lambda = 1 \quad \text{and} \quad \mu = \frac{\ln(3\pi/8)}{\ln(13/11)} = 0.98112316\ldots.$$

Equality in (3.2) *occurs for* $n = 1$.

Proof Inequality (3.2) can be written as

$$\lambda > x_n \ge \mu,$$

where the sequence $(x_n)_{n\in\mathbb{N}}$ is defined by

$$x_n = \frac{\ln(\frac{1}{n+\frac{1}{2}}(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})})^2)}{\ln(1-\frac{1}{4n+\frac{5}{2}})}.$$

We are now in a position to show that the sequence $(x_n)_{n\in\mathbb{N}}$ is strictly increasing. To this end, we consider the function $f(x)$ defined by

$$f(x)=\frac{2\ln\Gamma(x+1)-2\ln\Gamma(x+\frac{1}{2})-\ln(x+\frac{1}{2})}{\ln(1-\frac{1}{4x+\frac{5}{2}})}=\frac{f_1(x)}{f_2(x)},$$

where

$$f_1(x)=2\ln\Gamma(x+1)-2\ln\Gamma\left(x+\frac{1}{2}\right)-\ln\left(x+\frac{1}{2}\right)$$

and

$$f_2(x)=\ln\left(1-\frac{1}{4x+\frac{5}{2}}\right).$$

We conclude from the asymptotic formula of $\ln\Gamma(z)$ ([25], p.257) that

$$f_1(\infty)=\lim_{x\to\infty}f_1(x)=0.$$

Elementary calculations show that

$$\frac{8f_1'(x)}{f_2'(x)}=\left(64x^2+64x+15\right)\left[\psi(x+1)-\psi\left(x+\frac{1}{2}\right)-\frac{1}{2x+1}\right]=:f_3(x).$$

By using inequalities (2.2) and (2.3), we obtain, for $x\geq 2$,

$$\begin{aligned}f_3'(x)&=(128x+64)\left[\psi(x+1)-\psi\left(x+\frac{1}{2}\right)-\frac{1}{2x+1}\right]\\&\quad+\left(64x^2+64x+15\right)\left[\psi'(x+1)-\psi'\left(x+\frac{1}{2}\right)+\frac{2}{(2x+1)^2}\right]\\&>(128x+64)\left[\frac{1}{2x}-\frac{1}{8x^2}+\frac{1}{64x^4}-\frac{1}{128x^6}-\frac{1}{2x+1}\right]\\&\quad+\left(64x^2+64x+15\right)\left[-\frac{1}{2x^2}+\frac{1}{4x^3}-\frac{1}{16x^5}+\frac{2}{(2x+1)^2}\right]\\&=\frac{202+4{,}881(x-2)+7{,}860(x-2)^2+4{,}896(x-2)^3+1{,}368(x-2)^4+144(x-2)^5}{16x^6(2x+1)^2}\\&>0.\end{aligned}$$

Hence, $f_3(x)$ and $\frac{f_1'(x)}{f_2'(x)}$ are both strictly increasing for $x\geq 2$. By Lemma 3, the function

$$f(x)=\frac{f_1(x)}{f_2(x)}=\frac{f_1(x)-f_1(\infty)}{f_2(x)-f_2(\infty)}$$

is strictly increasing for $x\geq 2$. Therefore, the sequence (x_n) is strictly increasing for $n\geq 2$. Direct computation would yield

$$x_1=\frac{\ln(3\pi/8)}{\ln(13/11)}=0.9811\ldots,\qquad x_2=\frac{-7\ln 2+2\ln 3+\ln\pi+\ln 5}{-\ln 19+\ln 3+\ln 7}=0.9927\ldots.$$

Consequently, the sequence $(x_n)_{n\in\mathbb{N}}$ is strictly increasing. This leads to

$$\lim_{n\to\infty} x_n > x_n \geq x_1 = \frac{\ln(3\pi/8)}{\ln(13/11)} \quad \text{for } n \in \mathbb{N}.$$

It remains to prove that

$$\lim_{n\to\infty} x_n = 1. \tag{3.3}$$

We conclude from the asymptotic formula of $\ln\Gamma(z)$ ([25], p.257) that

$$f(x) = \frac{1+O(x^{-1})}{1+O(x^{-1})} \to 1 \quad \text{as } x \to \infty,$$

which implies (3.3). This completes the proof of Theorem 2. □

4 Asymptotic expansion

Theorem 3 *The function* $W(x)$, *defined by* (1.8), *has the following asymptotic expansion*:

$$W(x) \sim \frac{\pi}{2}\left(1 - \frac{1}{4x+\frac{5}{2}}\right)^{\sum_{j=0}^{\infty} r_j x^{-j}}, \quad x \to \infty, \tag{4.1}$$

with the coefficients r_j *given by the recurrence relation*

$$r_0 = 1, \qquad r_j = 4\sum_{k=0}^{j-1} r_k q_{j-k+1} - 4p_{j+1}, \quad j \in \mathbb{N}, \tag{4.2}$$

where

$$p_j = (-1)^{j-1}\left(-\frac{1}{j2^j} + \frac{2((-1)^{j+1} - (2^{-j}-1))B_{j+1}}{j(j+1)}\right), \quad j \in \mathbb{N} \tag{4.3}$$

and

$$q_j = -\sum_{k=0}^{j-1} \frac{1}{(k+1)\cdot 4^{k+1}} \binom{j-1}{j-k-1}\left(-\frac{5}{8}\right)^{j-k-1}, \quad j \in \mathbb{N}. \tag{4.4}$$

Here, B_n *are the Bernoulli numbers*.

Proof Write (4.1) as

$$\frac{\ln(\frac{2}{\pi}W(x))}{\ln(1 - \frac{1}{4x+\frac{5}{2}})} \sim \sum_{j=0}^{\infty} \frac{r_j}{x^j}, \quad x \to \infty. \tag{4.5}$$

The logarithm of gamma function has asymptotic expansion (see [28], p.32):

$$\ln\Gamma(x+t) \sim \left(x+t-\frac{1}{2}\right)\ln x - x + \frac{1}{2}\ln(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}\frac{1}{x^n} \tag{4.6}$$

as $x \to \infty$, where $B_n(t)$ denotes the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(t)\frac{x^n}{n!}.$$

From (4.6), we obtain, as $x \to \infty$,

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)} \sim x\exp\left(\frac{1}{t-s}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}(B_{j+1}(t)-B_{j+1}(s))}{j(j+1)}\frac{1}{x^j}\right). \tag{4.7}$$

Setting $(s,t) = (\frac{1}{2},1)$ in (4.7) and noting that

$$B_n(0) = (-1)^n B_n(1) = B_n \quad \text{and} \quad B_n\left(\frac{1}{2}\right) = \left(2^{1-n}-1\right)B_n \quad \text{for } n \in \mathbb{N}_0$$

(see [25], p.805), we obtain, as $x \to \infty$,

$$\left[\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}\right]^2 \sim x\exp\left(\sum_{j=1}^{\infty}\frac{2(1-(-1)^{j+1}(2^{-j}-1))B_{j+1}}{j(j+1)}\frac{1}{x^j}\right). \tag{4.8}$$

By using the Maclaurin expansion of $\ln(1+t)$,

$$\ln(1+t) = \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}t^j \quad \text{for } -1 < t \le 1,$$

we obtain

$$\left(1+\frac{1}{2x}\right)^{-1} \sim \exp\left(\sum_{j=1}^{\infty}\frac{(-1)^j}{j2^j}\frac{1}{x^j}\right) \quad \text{as } x \to \infty. \tag{4.9}$$

Applying (4.8) and (4.9) yields

$$\ln\left(\frac{2}{\pi}W(x)\right) \sim \sum_{j=1}^{\infty}\frac{p_j}{x^j}, \quad x \to \infty \tag{4.10}$$

with

$$p_j = (-1)^{j-1}\left(-\frac{1}{j2^j}+\frac{2((-1)^{j+1}-(2^{-j}-1))B_{j+1}}{j(j+1)}\right), \quad j \in \mathbb{N}. \tag{4.11}$$

The Maclaurin expansion of $\ln(1+t)$ with $t = -\frac{1}{4x+\frac{5}{2}}$, yields

$$\begin{aligned}\ln\left(1-\frac{1}{4x+\frac{5}{2}}\right) &\sim -\sum_{j=1}^{\infty}\frac{1}{j\cdot 4^j x^j}\left(1+\frac{5}{8x}\right)^{-j}\\ &\sim -\sum_{j=1}^{\infty}\frac{1}{j\cdot 4^j x^j}\sum_{k=0}^{\infty}\binom{-j}{k}\frac{5^k}{8^k x^k}\end{aligned}$$

$$\sim -\sum_{j=1}^{\infty} \frac{1}{j \cdot 4^j x^j} \sum_{k=0}^{\infty} (-1)^k \binom{k+j-1}{k} \frac{5^k}{8^k x^k}$$
$$\sim -\sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \frac{1}{(k+1) \cdot 4^{k+1}} \binom{j-1}{j-k-1} \left(-\frac{5}{8}\right)^{j-k-1} \frac{1}{x^j}.$$

That is,

$$\ln\left(1 - \frac{1}{4x + \frac{5}{2}}\right) \sim \sum_{j=1}^{\infty} \frac{q_j}{x^j}$$

with

$$q_j = -\sum_{k=0}^{j-1} \frac{1}{(k+1) \cdot 4^{k+1}} \binom{j-1}{j-k-1} \left(-\frac{5}{8}\right)^{j-k-1}, \quad j \in \mathbb{N}.$$

It follows from (4.5) that

$$\frac{\sum_{j=1}^{\infty} p_j x^{-j}}{\sum_{j=1}^{\infty} q_j x^{-j}} \sim \sum_{j=0}^{\infty} \frac{r_j}{x^j},$$
$$\sum_{j=1}^{\infty} \frac{p_j}{x^j} \sim \sum_{j=0}^{\infty} \frac{r_j}{x^j} \sum_{k=1}^{\infty} \frac{q_k}{x^k},$$
$$\sum_{j=1}^{\infty} \frac{p_j}{x^j} \sim \sum_{j=1}^{\infty} \left(\sum_{k=0}^{j-1} r_k q_{j-k} \right) \frac{1}{x^j}.$$

We then obtain

$$p_j = \sum_{k=0}^{j-1} r_k q_{j-k}, \quad j \in \mathbb{N},$$
$$p_j = \sum_{k=0}^{j-2} r_k q_{j-k} + r_{j-1} q_1, \quad j \geq 2.$$

Noting that $q_1 = -\frac{1}{4}$, we obtain

$$r_{j-1} = 4 \sum_{k=0}^{j-2} r_k q_{j-k} - 4p_j, \quad j \geq 2,$$

and an empty sum (as usual) is understood to be nil. Noting that $p_1 = -\frac{1}{4}$, we then obtain the recurrence relation

$$r_0 = 1, \qquad r_j = 4 \sum_{k=0}^{j-1} r_k q_{j-k+1} - 4p_{j+1}, \quad j \in \mathbb{N}.$$

The proof of Theorem 3 is complete. □

Here, from (4.1), we give the following explicit asymptotic expansion:

$$W_n = \frac{\pi}{2}\left(1 - \frac{1}{4n+\frac{5}{2}}\right)^{1-\frac{3}{64n^2}+\frac{3}{64n^3}-\frac{23}{1,024n^4}-\frac{1}{512n^5}+\cdots}, \quad n \to \infty, \tag{4.12}$$

which develops the formula (1.7) to produce a complete asymptotic expansion.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Acknowledgements
The authors thank the referees for their careful reading of the manuscript and insightful comments.

Received: 2 January 2015 Accepted: 18 May 2015 Published online: 10 June 2015

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