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New upper bounds for $\|A^{-1}\|_\infty$ of strictly diagonally dominant M-matrices

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Abstract

A new upper bound for the infinity norm of inverse matrix of a strictly diagonally dominant M-matrix is given, and the lower bound for the minimum eigenvalue of the matrix is obtained. Furthermore, an upper bound for the infinity norm of inverse matrix of a strictly α-diagonally dominant M-matrix is presented. Finally, we give numerical examples to illustrate our results.

MSC: 15A42; 15A45

Keywords: diagonal dominance; M-matrix; infinity norm; upper bound; minimum eigenvalue

1 Introduction

Let $R^{n\times n}$ denote the set of all $n \times n$ real matrices, $N = \{1, 2, \ldots, n\}$ and $A = (a_{ij}) \in R^{n\times n}$ ($n \ge 2$). A matrix A is called a nonsingular M-matrix if there exist a nonnegative matrix B and some real number s such that

$$A = sI - B, \quad s > \rho(B),$$

where I is the identity matrix, $\rho(B)$ is the spectral radius of B. $\tau(A)$ denotes the minimum of all real eigenvalues of the nonsingular M-matrix A.

Very often in numerical analysis, one needs a bound for the condition number of a square $n \times n$ matrix A, $\mathrm{Cond}(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$. Bounding $\|A\|_\infty$ is not usually difficult, but a bound of $\|A^{-1}\|_\infty$ is not usually available unless A^{-1} is known explicitly.

However, if $A = (a_{ij}) \in R^{n\times n}$ is a strictly diagonally dominant matrix, Varah [1] bound $\|A^{-1}\|_\infty$ quite easily by the following result:

$$\|A^{-1}\|_\infty \le \frac{1}{\min_{i\in N}\{|a_{ii}| - \sum_{j\ne i} |a_{ij}|\}}. \tag{1}$$

Remark 1 [2] If the diagonal dominance of A is weak, *i.e.*, $\min_{i\in N}\{|a_{ii}| - \sum_{j\ne i} |a_{ij}|\}$ is small, then using (1) in estimating $\|A^{-1}\|_\infty$, the bound may yield a large value.

In 2007, Cheng and Huang [2] presented the following results.

If $A = (a_{ij})$ is a strictly diagonally dominant M-matrix, then

$$\|A^{-1}\|_\infty \le \frac{1}{a_{11}(1 - u_1 l_1)} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii}(1 - u_i l_i)} \prod_{j=1}^{i-1} \left(1 + \frac{u_j}{1 - u_j l_j}\right)\right]. \tag{2}$$

If $A = (a_{ij})$ is a strictly diagonally dominant M-matrix, then the bound in (2) is sharper than that in Theorem 3.3 in [3], *i.e.*,

$$\frac{1}{a_{11}(1-u_1l_1)} + \sum_{i=2}^{n}\left[\frac{1}{a_{ii}(1-u_il_i)}\prod_{j=1}^{i-1}\left(1+\frac{u_j}{1-u_jl_j}\right)\right] < \sum_{i=1}^{n}\left[a_{ii}\prod_{j=1}^{i}(1-u_j)\right]^{-1}.$$

In 2009, Wang [4] obtained the better result: Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix. Then

$$\|A^{-1}\|_\infty < \frac{1}{a_{11}(1-u_1l_1)} + \sum_{i=2}^{n}\left[\frac{1}{a_{ii}(1-u_il_i)}\prod_{j=1}^{i-1}\frac{1}{1-u_jl_j}\right]. \tag{3}$$

In this paper, we present new upper bounds for $\|A^{-1}\|_\infty$ of a strictly (α-)diagonally dominant M-matrix A, which improved the above results. As an application, a lower bound of $\tau(A)$ is obtained.

For convenience, for $i,j,k \in N$, $j \neq i$, denote

$$R_i(A) = \sum_{j\neq i}|a_{ij}|, \qquad C_i(A) = \sum_{j\neq i}|a_{ji}|, \qquad d_i = \frac{R_i(A)}{|a_{ii}|},$$

$$J(A) = \{i \in N | d_i < 1\}, \qquad u_i = \frac{\sum_{j=i+1}^{n}|a_{ij}|}{|a_{ii}|}, \qquad l_k = \max_{k\le i\le n}\left\{\frac{\sum_{k\le j\le n}|a_{ij}|}{|a_{ii}|}\right\},$$

$$l_n = u_n = 0, \qquad r_{ji} = \frac{|a_{ji}|}{|a_{jj}| - \sum_{k\neq j,i}|a_{jk}|}, \qquad r_i = \max_{j\neq i}\{r_{ji}\},$$

$$\sigma_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_i}{|a_{jj}|}, \qquad h_i = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|\sigma_{ji} - \sum_{k\neq j,i}|a_{jk}|\sigma_{ki}}\right\},$$

$$u_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|\sigma_{ki}h_i}{|a_{jj}|}, \qquad \omega_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|u_{ki}}{|a_{jj}|}.$$

We will denote by $A^{(n_1,n_2)}$ the principal submatrix of A formed from all rows and all columns with indices between n_1 and n_2 inclusively; *e.g.*, $A^{(2,n)}$ is the submatrix of A obtained by deleting the first row and the first column of A.

Definition 1 [3] $A = (a_{ij}) \in R^{n\times n}$ is a weakly chained diagonally dominant if for all $i \in N$, $d_i \le 1$ and $J(A) \neq \phi$, and for all $i \in N$, $i \notin J(A)$, there exist indices $i_1, i_2, \ldots, i_k$ in N with $a_{i_r,i_{r+1}} \neq 0$, $0 \le r \le k-1$, where $i_0 = i$ and $i_k \in J(A)$.

Definition 2 [5] $A = (a_{ij}) \in R^{n\times n}$ is called a strictly α-diagonally dominant matrix if there exists $\alpha \in [0,1]$ such that

$$|a_{ii}| > \alpha R_i(A) + (1-\alpha)C_i(A), \quad \forall i \in N.$$

2 Upper bounds for $\|A^{-1}\|_\infty$ of a strictly diagonally dominant *M*-matrix

In this section, we give several bounds of $\|A^{-1}\|_\infty$ and $\tau(A)$ for a strictly diagonally dominant M-matrix A.

Lemma 1 [2] *Let $A = (a_{ij})$ be a weakly chained diagonally dominant M-matrix, $B = A^{(2,n)}$, $A^{-1} = (\alpha_{ij})$, and $B^{-1} = (\beta_{ij})$. Then, for $i,j = 2,\dots,n$,*

$$\alpha_{11} = \frac{1}{\triangle}, \qquad \alpha_{i1} = \frac{1}{\triangle}\sum_{k=2}^{n}\beta_{ik}(-a_{k1}), \qquad \alpha_{1j} = \frac{1}{\triangle}\sum_{k=2}^{n}\beta_{kj}(-a_{1k}),$$

$$\alpha_{ij} = \beta_{ij} + \alpha_{1j}\sum_{k=2}^{n}\beta_{ik}(-a_{k1}), \qquad \triangle = a_{11} - \sum_{k=2}^{n}a_{1k}\left(\sum_{i=2}^{n}\beta_{ki}a_{i1}\right) > 0.$$

Furthermore, if $J(A) = N$, then

$$\triangle \ge a_{11}(1 - d_1 l_1) \ge a_{11}(1 - d_1).$$

Lemma 2 [2] *If $A = (a_{ij})$ is a strictly diagonally dominant M-matrix, then*

$$\triangle \ge a_{11}(1 - d_1 l_1) > a_{11}(1 - d_1) > 0.$$

Lemma 3 *Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix. Then, for $A^{-1} = (\alpha_{ij})$,*

$$\alpha_{ji} \le \omega_{ji}\alpha_{ii}, \quad i,j \in N, j \ne i.$$

Proof This proof is similar to the one of Lemma 2 in [6]. □

Lemma 4 *Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix. Then, for $A^{-1} = (\alpha_{ij})$,*

$$\frac{1}{a_{ii}} \le \alpha_{ii} \le \frac{1}{a_{ii} - \sum_{j\ne i}|a_{ij}|\omega_{ji}}, \quad i \in N.$$

Proof This proof is similar to the one of Lemma 2.3 in [7]. □

Lemma 5 [3] *Let $A = (a_{ij})$ be a weakly chained diagonally dominant M-matrix, $A^{-1} = (\alpha_{ij})$, and $\tau = \tau(A)$. Then*

$$\tau \le \min_{i\in N}\{a_{ii}\}, \qquad \tau \le \max_{i\in N}\left\{\sum_{j\in N}a_{ij}\right\}, \qquad \tau \ge \min_{i\in N}\left\{\sum_{j\in N}a_{ij}\right\}, \quad \frac{1}{M} \le \tau \le \frac{1}{m},$$

where

$$M = \max_{i\in N}\left\{\sum_{j\in N}\alpha_{ij}\right\} = \|A^{-1}\|_\infty, \qquad m = \min_{i\in N}\left\{\sum_{j\in N}\alpha_{ij}\right\}.$$

Theorem 1 *Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix, $B = A^{(2,n)}$, $A^{-1} = (\alpha_{ij})$, and $B^{-1} = (\beta_{ij})$. Then*

$$\|A^{-1}\|_\infty \le \frac{1}{a_{11} - \sum_{j=2}^{n}|a_{1j}|\omega_{j1}} + \frac{1}{1 - d_1 l_1}\|B^{-1}\|_\infty.$$

Proof Let

$$\eta_i = \sum_{j=1}^{n} \alpha_{ij}, \qquad M_A = \|A^{-1}\|_\infty, \qquad M_B = \|B^{-1}\|_\infty.$$

Then

$$M_A = \max_{i\in N}\{\eta_i\}, \qquad M_B = \max_{2\le i\le n}\left\{\sum_{j=2}^{n}\beta_{ij}\right\}.$$

By Lemma 1, Lemma 2, and Lemma 4,

$$\begin{aligned}\eta_1 &= \alpha_{11} + \sum_{j=2}^{n}\alpha_{1j} = \frac{1}{\triangle} + \frac{1}{\triangle}\sum_{k=2}^{n}(-a_{1k})\sum_{j=2}^{n}\beta_{kj} \le \frac{1}{\triangle} + \frac{1}{\triangle}a_{11}d_1M_B \\ &\le \frac{1}{\triangle} + \frac{d_1M_B}{1-d_1l_1} \le \frac{1}{a_{11}-\sum_{j=2}^{n}|a_{1j}|\omega_{j1}} + \frac{M_B}{1-d_1l_1}.\end{aligned} \tag{4}$$

Let $2 \le i \le n$. Then, by Lemma 1 and Lemma 3,

$$\sum_{k=2}^{n}\beta_{ik}(-a_{k1}) = \triangle\cdot\alpha_{i1} \le \triangle\omega_{i1}\alpha_{11} = \omega_{i1} < 1,$$

$$\alpha_{ij} = \beta_{ij} + \alpha_{1j}\sum_{k=2}^{n}\beta_{ik}(-a_{k1}) \le \beta_{ij} + \alpha_{1j}\omega_{i1} < \beta_{ij} + \alpha_{1j}.$$

Therefore, for $2 \le i \le n$, we have

$$\begin{aligned}\eta_i &= \alpha_{i1} + \sum_{j=2}^{n}\alpha_{ij} \le \alpha_{11}\omega_{i1} + \sum_{j=2}^{n}(\beta_{ij} + \alpha_{1j}\omega_{i1}) = \eta_1\omega_{i1} + M_B \le \eta_1 l_1 + M_B \\ &\le \left(\frac{1}{\triangle} + \frac{d_1M_B}{1-d_1l_1}\right)l_1 + M_B \le \frac{1}{\triangle} + \frac{M_B}{1-d_1l_1} \le \frac{1}{a_{11}-\sum_{j=2}^{n}|a_{1j}|\omega_{j1}} + \frac{M_B}{1-d_1l_1}.\end{aligned} \tag{5}$$

Furthermore, from (4) and (5), we obtain

$$M_A \le \frac{1}{a_{11}-\sum_{j=2}^{n}|a_{1j}|\omega_{j1}} + \frac{1}{1-d_1l_1}\|B^{-1}\|_\infty. \tag{6}$$

The result follows. □

Theorem 2 *Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix. Then*

$$\|A^{-1}\|_\infty \le \frac{1}{a_{11}-\sum_{k=2}^{n}|a_{1k}|\omega_{k1}} + \sum_{i=2}^{n}\left[\frac{1}{a_{ii}-\sum_{k=i+1}^{n}|a_{ik}|\omega_{ki}}\prod_{j=1}^{i-1}\frac{1}{1-u_jl_j}\right]. \tag{7}$$

Proof The result follows by applying the principle of mathematical induction with respect to k on $A^{(k,n)}$ in (6). □

By Lemma 5 and Theorem 1, we can obtain a new bound of $\tau(A)$.

Corollary 1 *If $A = (a_{ij})$ is a strictly diagonally dominant M-matrix, then*

$$\tau(A) \geq \left\{ \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}|\omega_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii} - \sum_{k=i+1}^{n} |a_{ik}|\omega_{ki}} \prod_{j=1}^{i-1} \frac{1}{1-u_j l_j} \right] \right\}^{-1}.$$

Theorem 3 *Let $A = (a_{ij})$ be a strictly diagonally dominant M-matrix. Then the bound in* (7) *is better than that in* (3), *i.e.*,

$$\begin{aligned} &\frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}|\omega_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii} - \sum_{k=i+1}^{n} |a_{ik}|\omega_{ki}} \prod_{j=1}^{i-1} \frac{1}{1-u_j l_j} \right] \\ &\quad \leq \frac{1}{a_{11}(1-u_1 l_1)} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii}(1-u_i l_i)} \prod_{j=1}^{i-1} \frac{1}{1-u_j l_j} \right]. \end{aligned}$$

Proof Since A is a strictly diagonally dominant matrix, so $0 \leq u_j$, $l_j < 1$ for all j. By the definition of u_i, l_i, ω_{ki}, we have $\omega_{ki} \leq l_i$ and $a_{ii}u_i = \sum_{k=i+1}^{n} |a_{ik}|$ for all i. Obviously, the result follows. □

3 Upper bounds for $\|A^{-1}\|_\infty$ of a strictly α-diagonally dominant *M*-matrix

In this section, we present an upper bound of $\|A^{-1}\|_\infty$ for a strictly α-diagonally dominant M-matrix A.

Lemma 6 [8] *Let $A, B \in R^{n\times n}$. If A and $A - B$ are nonsingular, then*

$$(A-B)^{-1} = A^{-1} + A^{-1}B\bigl(I - A^{-1}B\bigr)^{-1}A^{-1}.$$

Lemma 7 *Let $A = (a_{ij}) \in R^{n\times n}$ be a strictly diagonally dominant M-matrix, and $B = (b_{ij}) \in R^{n\times n}$. If $\varphi_0 \cdot \|B\|_\infty < 1$, then $\|A^{-1}B\|_\infty < 1$, where*

$$\varphi_0 = \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}|\omega_{k1}} + \sum_{i=2}^{n} \left[\frac{1}{a_{ii} - \sum_{k=i+1}^{n} |a_{ik}|\omega_{ki}} \prod_{j=1}^{i-1} \frac{1}{1-u_j l_j} \right].$$

Proof By Theorem 2, we get

$$\bigl\|A^{-1}B\bigr\|_\infty \leq \bigl\|A^{-1}\bigr\|_\infty \|B\|_\infty \leq \varphi_0 \|B\|_\infty < 1.$$

The result follows. □

Lemma 8 [8] *If $\|A^{-1}\|_\infty < 1$, then $I - A$ is nonsingular and*

$$\bigl\|(I-A)^{-1}\bigr\|_\infty \leq \frac{1}{1 - \|A\|_\infty}.$$

Theorem 4 *Let $A = (a_{ij}) \in R^{n\times n}$ be a strictly α-diagonally dominant matrix, $\alpha \in (0,1]$ and A be an M-matrix. If $\{i \in N | R_i(A) > C_i(A)\} \neq \emptyset$, and*

$$\varphi_1 < \frac{1}{\max_{1\leq i\leq n} \alpha(R_i(A) - C_i(A))},$$

then

$$\|A^{-1}\|_\infty < \frac{\varphi_1}{1-\varphi_1 \max_{1\le i\le n}\alpha(R_i(A)-C_i(A))}, \tag{8}$$

where

$$\varphi_1 = \frac{1}{\nu_1-\sum_{k=2}^n |a_{1k}|\omega_{k1}} + \sum_{i=2}^n\left[\frac{1}{\nu_i-\sum_{k=i+1}^n |a_{ik}|\omega_{ki}}\prod_{j=1}^{i-1}\frac{1}{1-u_j l_j}\right],$$

$$\nu_i = \max_{1\le i\le n}\bigl\{a_{ii}, a_{ii}+\alpha\bigl(R_i(A)-C_i(A)\bigr)\bigr\}.$$

Proof Let $A = B - C$, where $B = (b_{ij})$, $C = (c_{ij})$, and

$$b_{ij} = \begin{cases} a_{ii}+\alpha(R_i(A)-C_i(A)), & i=j, R_i(A) > C_i(A),\\ a_{ij}, & \text{otherwise},\end{cases}$$

$$c_{ij} = \begin{cases} \alpha(R_i(A)-C_i(A)), & i=j, R_i(A) > C_i(A),\\ 0, & \text{otherwise}.\end{cases}$$

For any $i \in \{i\in N|R_i(A) > C_i(A)\}$, we get

$$b_{ii} = a_{ii}+\alpha\bigl(R_i(A)-C_i(A)\bigr) > R_i(A) = R_i(B).$$

For any $i \in \{i\in N|R_i(A) \le C_i(A)\}$, we have

$$b_{ii} = a_{ii} > \alpha R_i(A) + (1-\alpha)C_i(A) \ge R_i(A) = R_i(B).$$

Thus, B is a strictly diagonal dominant M-matrix. By Lemma 7, we get $\|B^{-1}C\|_\infty < 1$. By Lemma 6, Lemma 8, and Theorem 2, we have

$$\begin{aligned}\|B^{-1}\|_\infty &\le \frac{1}{b_{11}-\sum_{k=2}^n |a_{1k}|\omega_{k1}} + \sum_{i=2}^n\left[\frac{1}{b_{ii}-\sum_{k=i+1}^n |a_{ik}|\omega_{ki}}\prod_{j=1}^{i-1}\frac{1}{1-u_j l_j}\right]\\ &= \frac{1}{\nu_1-\sum_{k=2}^n |a_{1k}|\omega_{k1}} + \sum_{i=2}^n\left[\frac{1}{\nu_i-\sum_{k=i+1}^n |a_{ik}|\omega_{ki}}\prod_{j=1}^{i-1}\frac{1}{1-u_j l_j}\right].\end{aligned}$$

Therefore

$$\|B^{-1}C\|_\infty \le \varphi_1 \max_{1\le i\le n}\alpha\bigl(R_i(A)-C_i(A)\bigr).$$

Furthermore, we have

$$\begin{aligned}\|A^{-1}\|_\infty &= \|(B-C)^{-1}\|_\infty = \|B^{-1}+B^{-1}C\bigl(I-B^{-1}C\bigr)^{-1}B^{-1}\|_\infty\\ &\le \|B^{-1}\|_\infty + \|B^{-1}C\|_\infty \cdot \|\bigl(I-B^{-1}C\bigr)^{-1}\|_\infty \cdot \|B^{-1}\|_\infty\\ &\le \|B^{-1}\|_\infty + \frac{\|B^{-1}C\|_\infty}{1-\|B^{-1}C\|_\infty}\|B^{-1}\|_\infty\end{aligned}$$

$$
\begin{aligned}
&= \frac{\|B^{-1}\|_\infty}{1 - \|B^{-1}C\|_\infty} \\
&\le \frac{\varphi_1}{1 - \varphi_1 \max_{1\le i\le n} \alpha(R_i(A) - C_i(A))}.
\end{aligned}
$$

The result follows. □

4 Numerical examples

In this section, we present numerical examples to illustrate the advantages of our derived results.

Example 1 Let

$$
A = \begin{pmatrix}
37 & -1 & -3 & -1 & -2 & -4 & -2 & -3 & -1 & -5 \\
-4 & 30 & -1 & -2 & -3 & -4 & 0 & -1 & -1 & -3 \\
-1 & -3 & 30 & -4 & 0 & -2 & -3 & -2 & -4 & -5 \\
-3 & -5 & -3 & 40 & -1 & -2 & -3 & -4 & -2 & -4 \\
-5 & -2 & 0 & -5 & 25.01 & -5 & 0 & -1 & -5 & -2 \\
-2 & 0 & -2 & -1 & -4 & 30 & -5 & -2 & -5 & -3 \\
0 & -3 & -1 & -1 & -2 & -4 & 40 & -2 & -3 & -4 \\
-1 & -3 & -2 & -3 & -2 & -1 & -2 & 40 & -4 & -1 \\
-2 & -4 & -3 & -1 & -3 & -3 & -4 & 0 & 27 & -2 \\
-2 & -1 & 0 & -2 & -4 & -3 & -1 & 0 & -3 & 25
\end{pmatrix}.
$$

It is easy to see that A is a strictly diagonally dominant M-matrix. By calculations with Matlab 7.1, we have

$$
\begin{aligned}
&\|A^{-1}\|_\infty \le 100 \quad (\text{by (1)}), \qquad \|A^{-1}\|_\infty \le 11.2862 \quad (\text{by (2)}), \\
&\|A^{-1}\|_\infty \le 5.2305 \quad (\text{by (3)}), \qquad \|A^{-1}\|_\infty \le 1.0003 \quad (\text{by (7)}),
\end{aligned}
$$

respectively. It is obvious that the bound in (7) is the best result.

Example 2 Let

$$
A = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
-0.5 & -0.5 & 2
\end{pmatrix}.
$$

It is easy to see that A is a strictly α-diagonally dominant M-matrix by taking $\alpha = 0.5$, and A is not a strictly diagonally dominant matrix. Thus the bound of $\|A^{-1}\|_\infty$ cannot be estimated by (1), (2), and (3), but it can be estimated by (8). By (8), we get

$$
\|A^{-1}\|_\infty \le 8.0322.
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work. All authors read and approved the final manuscript.

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