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Critical curves for fast diffusion equations coupled via nonlinear boundary flux

Zhengqiu Ling*

*Correspondence:
lingzq00@163.com
Institute of Mathematics and Information Science, Yulin Normal University, Yulin, 537000, P.R. China

Abstract

This paper is concerned with fast diffusion equations for coupling via nonlinear boundary flux. By means of the theory of linear equations and constructing self-similar super-solutions and sub-solutions, we obtain a critical global existence curve. The critical curve of Fujita type is conjectured with the aid of some new results. In addition, we show that the constant ε_0 of the linear system

$$A(\alpha_1,\alpha_2,\dots,\alpha_s)^{\mathrm{T}} = (\varepsilon_0,\varepsilon_0,\dots,\varepsilon_0)^{\mathrm{T}}$$

plays an important role in our discussion.

MSC: 35B33; 35K50; 36K65

Keywords: diffusion equation; critical global existence curve; critical Fujita curve; Newtonian filtration equation; nonlinear boundary flux

1 Introduction

In this paper, we investigate the existence and non-existence of global weak solutions to the following porous medium equations:

$$(u_i)_t = \left(u_i^{m_i}\right)_{xx}, \quad i = 1,2,\dots,s, x > 0, 0 < t < T \tag{1.1}$$

coupled via nonlinear boundary flux

$$-\left(u_i^{m_i}\right)_x(0,t) = u_i^{p_i}(0,t)u_{i+1}^{q_i}(0,t), \quad i = 1,2,\dots,s, \qquad u_{s+1} := u_1, \quad 0 < t < T, \tag{1.2}$$

with continuous, nonnegative initial data

$$u_i(x,0) = u_{0i}(x), \quad i = 1,2,\dots,s, x > 0 \tag{1.3}$$

compactly supported in $\mathbb{R}_+$, where $s \geq 2$, $0 < m_i < 1$, $p_i \geq 0$, $q_i > 0$ ($i = 1,2,\dots,s$) are parameters.

The particular feature of equations (1.1) is their gradient-dependent diffusivity. Such equations can be used to provide a model for nonlinear heat propagation, they also appear in several branches of applied mathematics such as plasma physics, population dynamics, chemical reactions, and so on. At the same time, these equations are also called the Newtonian filtration equations, which have been intensively studied since the last century (see [1, 2] and the references therein). In addition, for the case $s \geq 2$, system (1.1)-(1.3) suggests that equations (1.1) are linked by the influx of energy input at the boundary $x = 0$. For instance, in the heat transfer process, $-(u_i^{m_i})_x$ represents the heat flux, and hence the boundary conditions represent a nonlinear radiation law at the boundary. These kinds of boundary conditions appear also in combustion problem when the reaction happens only at the boundary of the container, for example, because of the presence of a solid catalyzer, see [3] for justification.

In general, system (1.1)-(1.3) does not possess classical solutions. This is due to the fact that the equations in (1.1) are parabolic only where $u_{ix} > 0$, but degenerate where $u_{ix} = 0$. However, local in time existence of weak solution $(u_1, u_2, \dots, u_s)$ to problem (1.1)-(1.3), defined in the usual integral way, as well as a comparison principle can be easily established as, for instance, in [2, 4, 5]. Let T be the maximal existence time of a solution $(u_1, u_2, \dots, u_s)$, which may be finite or infinite. If $T < \infty$, then $\|u_1\|_\infty + \|u_2\|_\infty + \cdots + \|u_s\|_\infty$ becomes unbounded in finite time and we say that the solution blows up; while if $T = \infty$, we say that the solution is global. In particular, the problem of determining critical Fujita exponents is very interesting for various nonlinear parabolic equations of mathematical physics. See the book [6] and the surveys [7, 8], where a full list of references can be found. Here, we recall some known results on system (1.1)-(1.3)

In 2001, Quirós and Rossi [9] considered the following degenerate equations coupled via variational nonlinear boundary flux ($s = 2$):

$$\begin{cases} u_t = (u^m)_{xx}, \qquad v_t = (v^n)_{xx}, & x > 0, t > 0, \\ -(u^m)_x(0,t) = v^p(0,t), \qquad -(v^n)_x(0,t) = u^q(0,t), & t > 0, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), & x > 0 \end{cases} \tag{1.4}$$

with $m, n > 1$ and notations

$$\alpha_1 = \frac{1+n+2p}{(1+m)(1+n)-4pq}, \qquad \alpha_2 = \frac{1+m+2q}{(1+m)(1+n)-4pq},$$
$$\beta_1 = \frac{p(m-1-2q)+(1+n)m}{(1+m)(1+n)-4pq}, \qquad \beta_2 = \frac{q(n-1-2p)+(1+m)n}{(1+m)(1+n)-4pq}.$$

They obtained that the critical global existence curve of (1.4) is $pq = (\frac{1+m}{2})(\frac{1+n}{2})$ and the critical Fujita type curve is $\min\{\alpha_1 + \beta_1, \alpha_2 + \beta_2\} = 0$. Besides, it was Zheng *et al.* who dealt with the general system (1.1)-(1.3) for $s = 2$ with $m_1, m_2 > 1$ in [10], in which the authors proved that for $p_1 < \frac{1+m_1}{2}$, $p_2 < \frac{1+m_2}{2}$, the critical global existence curve is $q_1 q_2 = (\frac{1+m_1}{2} - p_1)(\frac{1+m_2}{2} - p_2)$ and the critical Fujita curve is $\min\{l_1 - k_1, l_2 - k_2\} = 0$, while if $p_1 > \frac{1+m_1}{2}$ or $p_2 > \frac{1+m_2}{2}$, then the solutions may blow up in a finite time.

To our knowledge, however, there are few works in the literature dealing with the heat conduction systems such as (1.1)-(1.3). Motivated by the above mentioned works, in this paper we have a purpose to extend the results of the slow diffusion case [10] to the fast diffusion case and s components, and the aim is twofold. Firstly, we construct the self-similar super-solution and sub-solution to obtain the critical global existence curve of system (1.1)-(1.3). Secondly, the critical curve of Fujita type is conjectured with the aid of some new results. A very interesting feature of our results is that the critical curves are determined by a matrix and a linear algebraic system. The fact that we are dealing with a general system instead of a single equation and with nonlinear diffusion forces us to develop some new techniques.

In order to state our results, we introduce some useful symbols and a lemma. Denote by

$$A = \begin{pmatrix} 1+m_1-2p_1 & -2q_1 & 0 & \cdots & 0 & 0 \\ 0 & 1+m_2-2p_2 & -2q_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1+m_{s-1}-2p_{s-1} & -2q_{s-1} \\ -2q_s & 0 & 0 & \cdots & 0 & 1+m_s-2p_s \end{pmatrix} \tag{1.5}$$

and let

$$m_{s+i} := m_i, \qquad p_{s+i} := p_i, \qquad q_{s+i} := q_i, \qquad k_{s+i} := k_i, \qquad l_{s+i} := l_i.$$

A series of standard computations yields

$$\det A = \prod_{i=1}^{s}(1+m_i-2p_i) - 2^s q_1 q_2 \cdots q_s.$$

Next, we shall see that $\det A = 0$ is the critical global existence curve. In addition, we give the following lemma which comes from linear algebra, and its proof is obtained by using the Cramer theorem.

Lemma 1 *For the matrix A which is defined by* (1.5) *and any constant ε_0, according to the Cramer principle, if $\det A \neq 0$, then the following linear system*

$$A(\alpha_1,\alpha_2,\ldots,\alpha_{s-1},\alpha_s)^{\mathrm{T}} = (\varepsilon_0,\varepsilon_0,\ldots,\varepsilon_0,\varepsilon_0)^{\mathrm{T}} \tag{1.6}$$

has a unique solution $(\alpha_1,\alpha_2,\ldots,\alpha_{s-1},\alpha_s)^{\mathrm{T}}$ which is given by

$$\begin{aligned} \alpha_i = {} & \frac{\varepsilon_0}{\prod_{j=1}^{s}(1+m_j-2p_j) - 2^s q_1 q_2 \cdots q_s} \Biggl(\prod_{j=1}^{s-1}(1+m_{i+j}-2p_{i+j}) \\ & + \sum_{j=1}^{s-2} \Biggl(\prod_{j_1=1}^{j}(2q_{i+j_1-1}) \prod_{j_2=j+1}^{s-1}(1+m_{i+j_2}-2p_{i+j_2}) \Biggr) + \prod_{j=1}^{s-1}(2q_{i+j-1}) \Biggr) \end{aligned} \tag{1.7}$$

and $2q_i\alpha_{i+1} + \varepsilon_0 = (1+m_i-2p_i)\alpha_i$, $\alpha_{s+1} := \alpha_1$, $i = 1,2,\ldots,s$.

Now we state the main results of this paper.

Theorem 1 *Assume $2p_i < 1+m_i$ ($i = 1,2,\ldots,s$). If $\prod_{i=1}^{s}(1+m_i-2p_i) \geq 2^s q_1 q_2 \cdots q_s$, i.e., $\det A \geq 0$, then every nonnegative solution of system* (1.1)-(1.3) *is global in time.*

Theorem 2 *Assume $2p_i \leq 1+m_i$ ($i = 1,2,\ldots,s$). If $\prod_{i=1}^{s}(1+m_i-2p_i) < 2^s q_1 q_2 \cdots q_s$, i.e., $\det A < 0$, then system* (1.1)-(1.3) *has a nonnegative solution blowing up in a finite time.*

Remark 1 From Theorems 1 and 2, we see that the critical global existence curve for system (1.1)-(1.3) is $\prod_{i=1}^{s}(1+m_i-2p_i) = 2^s q_1 q_2 \cdots q_s$, *i.e.*, $\det A = 0$.

Let $(k_1, k_2, \ldots, k_s)$ denote the unique positive solution of linear system (1.6) with constant $\varepsilon_0 = -1$, that is,

$$A(k_1, k_2, \ldots, k_{s-1}, k_s)^{\mathrm{T}} = (-1, -1, \ldots, -1, -1)^{\mathrm{T}} \tag{1.8}$$

and define

$$l_i = \frac{1 + k_i(1 - m_i)}{2}, \quad i = 1, 2, \ldots, s. \tag{1.9}$$

Then we have the following.

Theorem 3 *Assume* $2p_i \le 1 + m_i$ $(i = 1, 2, \ldots, s)$ *and* $\prod_{i=1}^{s}(1 + m_i - 2p_i) < 2^s q_1 q_2 \cdots q_s$, *i.e.*, $\det A < 0$.

1. *If* $\min_i\{l_i - k_i\} > 0$, *there exist nonnegative solutions with blow-up and nonnegative solutions that are global.*
2. *If* $\max_i\{l_i - k_i\} < 0$, *then every nonnegative, nontrivial solution of system* (1.1)-(1.3) *blows up in a finite time.*

Remark 2 From Theorem 3, we conjecture that the critical curve of Fujita type is $\min_i\{l_i - k_i\} = 0$ if $2p_i \le 1 + m_i$ $(i = 1, 2, \ldots, s)$.

For the case $2p_1 > 1 + m_1$ or $\ldots$ or $2p_s > 1 + m_s$, we have the following.

Theorem 4 *If there exists* $i = 1, 2, \ldots, s$ *such that* $2p_i > 1 + m_i$, *then every nonnegative solution of* (1.1)-(1.3) *will blow up in a finite time.*

Remark 3 By Theorem 4, it is seen that the critical global existence curve for system (1.1)-(1.3) is $2p_i = 1 + m_i$ $(i = 1, 2, \ldots, s)$ if $2^s q_1 q_2 \cdots q_s = \prod_{i=1}^{s}(1 + m_i - 2p_i)$.

The rest of this paper is organized as follows. In Section 2, we consider a critical global existence curve and prove Theorems 1 and 2. The proofs of Theorems 3 and 4 are given in Section 3.

2 Critical global existence curve

In this section, we characterize when all solutions to problem (1.1)-(1.3) are global in time or they blow up. Motivated by [10, 11], we base our methods on the construction of self-similar solutions and on the comparison arguments.

Throughout this paper, we always assume $(u_{0i}^{m_i})''(x) \ge 0$, $i = 1, 2, \ldots, s$. Now, let us prove Theorem 1 first.

Proof of Theorem 1 In order to prove that the solution $(u_1, u_2, \ldots, u_s)$ of (1.1)-(1.3) is global, we look for a globally defined in time strict super-solution of self-similar form

$$\bar{u}_i(x, t) = e^{K_i t}\left(M + e^{-L_i x e^{\frac{K_i(1-m_i)t}{2}}}\right)^{\frac{1}{m_i}}, \quad x \ge 0, t \ge 0, i = 1, 2, \ldots, s, \tag{2.1}$$

where $M \ge \max_i\{\|u_{0i}\|_\infty + 1\}$ and $K_i, L_i > 0$ are to be determined. It is easy to see that

$$\bar{u}_i(x, 0) \ge u_{0i}(x), \quad x \ge 0, i = 1, 2, \ldots, s.$$

By a direct computation, we obtain

$$(\bar{u}_1)_t = K_1 e^{K_1 t}\big(M + e^{-L_1 x e^{\frac{K_1(1-m_1)t}{2}}}\big)^{\frac{1}{m_1}} \\ - \frac{K_1 L_1(1-m_1)}{2m_1} e^{K_1 t}\big(M + e^{-L_1 x e^{\frac{K_1(1-m_1)t}{2}}}\big)^{\frac{1-m_1}{m_1}} e^{\frac{K_1(1-m_1)t}{2}} x e^{-L_1 x e^{\frac{K_1(1-m_1)t}{2}}}.$$

Note that the function $Z_1(x) = x e^{-L_1 x e^{\frac{K_1(1-m_1)t}{2}}}$ reaches its maximum $Z(x_0) = \frac{1}{eL_1} e^{-\frac{K_1(1-m_1)t}{2}}$ at the point $x_0 = \frac{1}{L_1} e^{-\frac{K_1(1-m_1)t}{2}}$. Then we have

$$(\bar{u}_1)_t \geq K_1 e^{K_1 t} M^{\frac{1}{m_1}} - \frac{K_1(1-m_1)}{2em_1} e^{K_1 t}(1+M)^{\frac{1-m_1}{m_1}}. \tag{2.2}$$

At the same time,

$$\big(\bar{u}_1^{m_1}\big)_{xx} = L_1^2 e^{K_1 t} e^{-L_1 x e^{\frac{K_1(1-m_1)t}{2}}} \leq L_1^2 e^{K_1 t}. \tag{2.3}$$

Thus $\bar{u}_1$ is a super-solution of Eq. (1.1) if

$$K_1\left(M^{\frac{1}{m_1}} - \frac{(1-m_1)}{2em_1}(1+M)^{\frac{1-m_1}{m_1}}\right) \geq L_1^2. \tag{2.4}$$

Similarly,

$$K_i\left(M^{\frac{1}{m_i}} - \frac{(1-m_i)}{2em_i}(1+M)^{\frac{1-m_i}{m_i}}\right) \geq L_i^2, \quad i = 2, \ldots, s. \tag{2.5}$$

Therefore, we can first take M large enough so that

$$M^{\frac{1}{m_i}} - \frac{(1-m_i)}{2em_i}(1+M)^{\frac{1-m_i}{m_i}} > 0, \quad i = 1, 2, \ldots, s. \tag{2.6}$$

On the other hand, it remains to verify the boundary conditions (1.2), a simple computation yields

$$-\big(\bar{u}_1^{m_1}\big)_x(0,t) = L_1 e^{\frac{K_1(1+m_1)t}{1}}, \qquad \bar{u}_1^{p_1}(0,t)\bar{u}_2^{q_1}(0,t) = e^{(p_1 K_1 + q_1 K_2)t}(1+M)^{\frac{p_1}{m_1}+\frac{q_1}{m_2}}.$$

Then we have $-(\bar{u}_1^{m_1})_x(0,t) \geq \bar{u}_1^{p_1}(0,t)\bar{u}_2^{q_1}(0,t)$, if we impose

$$L_1 e^{\frac{K_1(1+m_1)t}{2}} \geq e^{(p_1 K_1 + q_1 K_2)t}(1+M)^{\frac{p_1}{m_1}+\frac{q_1}{m_2}}. \tag{2.7}$$

Similarly,

$$L_i e^{\frac{K_i(1+m_i)t}{2}} \geq e^{(p_i K_i + q_i K_{i+1})t}(1+M)^{\frac{p_i}{m_i}+\frac{q_i}{m_{i+1}}}, \quad i = 2, \ldots, s. \tag{2.8}$$

Let

$$L_i = (1+M)^{\frac{p_i}{m_i}+\frac{q_i}{m_{i+1}}}, \quad i = 1, 2, \ldots, s.$$

Hence, from (2.7) and (2.8) we get

$$e^{\frac{K_i(1+m_i)t}{2}} \geq e^{(p_iK_i+q_iK_{i+1})t}, \quad i = 1, 2, \ldots, s. \tag{2.9}$$

Next, we divide the proof into two cases.

Case (i). If $\prod_{i=1}^{s}(1 + m_i - 2p_i) > 2^s q_1 q_2 \cdots q_s$, then we choose that $(K_1, K_2, \ldots, K_s)$ denotes the unique solution $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ of the linear system (1.6), *i.e.*, $(K_1, K_2, \ldots, K_s) = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ with

$$\varepsilon_0 = \max_i \left\{ \frac{L_i^2(\prod_{j=1}^{s}(1 + m_j - 2p_j) - 2^s q_1 q_2 \cdots q_s)}{\beta_i(M^{\frac{1}{m_i}} - \frac{1-m_i}{2em_i}(1+M)^{\frac{1-m_i}{m_i}})} \right\},$$

where

$$\beta_i = \prod_{j=1}^{s-1}(1 + m_{i+j} - 2p_{i+j}) + \sum_{j=1}^{s-2}\left(\prod_{j_1=1}^{j}(2q_{i+j_1-1}) \prod_{j_2=j+1}^{s-1}(1 + m_{i+j_2} - 2p_{i+j_2})\right) + \prod_{j=1}^{s-1}(2q_{i+j-1}).$$

Thus, these constants K_i and ε_0 can ensure that inequalities (2.4), (2.5) and (2.9) hold. Therefore, we have proved that $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_s)$ is a global super-solution of system (1.1)-(1.3).

Case (ii). If $\prod_{i=1}^{s}(1 + m_i - 2p_i) = 2^s q_1 q_2 \cdots q_s$, the linear system (1.6) with $\varepsilon_0 = 0$ has nonzero positive solutions. Let $(K_1, K_2, \ldots, K_s)$ be such a positive solution of (1.6) and satisfy

$$K_i \geq \frac{L_i^2}{M^{\frac{1}{m_i}} - \frac{1-m_i}{2em_i}(1+M)^{\frac{1-m_i}{m_i}}}, \quad i = 1, 2, \ldots, s,$$

which imply that (2.4), (2.5) and (2.9) hold. Thus, we have proved that $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_s)$ is a global super-solution of system (1.1)-(1.3).

Combining Cases (i) and (ii), the solutions of system (1.1)-(1.3) exist globally by comparison principle. The proof is complete. □

Proof of Theorem 2 To prove the non-existence of global solutions, we construct a blow-up self-similar sub-solution of system (1.1)-(1.3). Let k_i and l_i satisfy (1.8) and (1.9), respectively. Then we have

$$m_ik_i + l_i = p_ik_i + q_ik_{i+1}, \quad i = 1, 2, \ldots, s. \tag{2.10}$$

Consider the functions

$$\tilde{u}_i(x,t) = (T-t)^{-k_i} f_i(\xi_i), \quad \xi_i = x(T-t)^{-l_i}, i = 1, 2, \ldots, s, \tag{2.11}$$

with a positive constant T and compactly supported functions

$$f_i(\xi_i) = (A_i + B_i\xi_i)^{\frac{2}{m_i-1}}, \quad i = 1, 2, \ldots, s, \tag{2.12}$$

where A_i, B_i are positive constants to be determined. By a direct calculation, we have

$$\begin{cases} (\tilde{u}_i)_t = (T-t)^{-k_i-1}(k_i f_i(\xi_i) + l_i\xi_i f_i'(\xi_i)), \\ (\tilde{u}_i^{m_i})_{xx} = (T-t)^{-m_ik_i-2l_i}(f_i^{m_i})'', \\ (\tilde{u}_i^{m_i})_x(0,t) = (T-t)^{-m_ik_i-l_i}(f_i^{m_i})'(0), \\ \tilde{u}_i^{p_i}(0,t)\tilde{u}_{i+1}^{q_i}(0,t) = (T-t)^{-p_ik_i-q_ik_{i+1}}f_i^{p_i}(0)f_{i+1}^{q_i}(0). \end{cases}$$

Thus, $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_s)$ is a sub-solution of (1.1) and (1.2) provided that

$$k_i f_i(\xi_i) + l_i\xi_i f_i'(\xi_i) \le \left(f_i^{m_i}\right)''(\xi_i), \quad i = 1,2,\dots,s \tag{2.13}$$

and

$$-\left(f_i^{m_i}\right)'(0) \le f_i^{p_i}(0)f_{i+1}^{q_i}(0), \quad i = 1,2,\dots,s, \qquad f_{s+1} := f_1. \tag{2.14}$$

Clearly,

$$\begin{aligned} &\left(f_i^{m_i}\right)''(\xi_i) = \frac{2m_i(m_i+1)B_i^2}{(m_i-1)^2}(A_i + B_i\xi_i)^{\frac{2}{m_i-1}}, \\ &k_i f_i(\xi_i) + l_i\xi_i f_i'(\xi_i) = k_i(A_i + B_i\xi_i)^{\frac{2}{m_i-1}} + \frac{2l_iB_i}{m_i-1}\xi_i(A_i + B_i\xi_i)^{\frac{3-m_i}{m_i-1}} \le k_i(A_i + B_i\xi_i)^{\frac{2}{m_i-1}}. \end{aligned}$$

Hence, inequalities (2.13) are satisfied if we choose the constants $B_1, \dots, B_s$ such that

$$B_i^2 = \frac{k_i(1-m_i)^2}{2m_i(1+m_i)}, \quad i = 1,2,\dots,s. \tag{2.15}$$

On the other hand, the boundary conditions in (2.14) are satisfied provided that

$$\frac{2m_i}{1-m_i}B_i \le A_i^{\frac{1+m_i-2p_i}{1-m_i}} A_{i+1}^{\frac{2q_i}{m_{i+1}-1}}, \qquad A_{s+1} := A_1, \quad i = 1,2,\dots,s. \tag{2.16}$$

Substituting (2.15) into (2.16), we obtain

$$\left(\frac{2k_im_i}{1+m_i}\right)^{\frac{1-m_{i+1}}{2(1+m_i-2p_i)}} A_{i+1}^{\frac{2q_i}{1+m_i-2p_i}} \le A_i^{\frac{1-m_{i+1}}{1-m_i}}, \quad i = 1,2,\dots,s. \tag{2.17}$$

For $i = 1,2,\dots,s$, $\prod_{i=1}^s(1+m_i-2p_i) < 2^s q_1q_2\cdots q_s$ implies that

$$\frac{2q_i}{1+m_i-2p_i} > \prod_{j=1}^{s-1}\frac{1+m_{i+j}-2p_{i+j}}{2q_{i+j}}.$$

At the same time, by (2.17) we get

$$A_{i+1}^{\frac{2q_i}{1+m_i-2p_i}} \le C_i A_{i+1}^{\prod_{j=1}^{s-1}\frac{1+m_{i+j}-2p_{i+j}}{2q_{i+j}}}, \tag{2.18}$$

where

$$C_i = \delta_i^{-1}\prod_{j=1}^{s-1}\left(\delta_{i+j}^{-\frac{1-m_{i+1}}{1-m_{i+j+1}}\prod_{j_1=j}^{s-1}\frac{1+m_{i+j_1}-2p_{i+j_1}}{2q_{i+j_1}}}\right), \qquad \delta_i = \left(\frac{2k_im_i}{1+m_i}\right)^{\frac{1-m_{i+1}}{2(1+m_i-2p_i)}}, \qquad \delta_{s+i} := \delta_i.$$

Therefore, there exists A_{i+1} small enough such that inequality (2.18) is valid. Thus, if the initial data $u_{01}(x), u_{02}(x), \dots, u_{0s}(x)$ are large enough so that

$$\tilde{u}_i(x,0) \le u_{0i}(x), \quad i = 1,2,\dots,s,$$

then $(\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_s)$ is a sub-solution of system (1.1)-(1.3) by the comparison principle, which implies that the solutions of system (1.1)-(1.3) with large initial data blow up in a finite time. The proof is complete. □

3 Critical curve of Fujita type

Now we turn out attention to the results of Fujita type. This is, we shall show when all solutions of system (1.1)-(1.3) blow up in a finite time or both global and non-global solutions exist.

Proof of Theorem 3 (1) In order to prove the conclusion, we only need to show that the solutions of system (1.1)-(1.3) with small enough initial data have global existence, which will be proved by constructing kinds of self-similar global super-solutions

$$\bar{u}_i(x,t) = (\tau + t)^{-k_i} g_i(\xi_i), \quad \xi_i = x(\tau + t)^{-l_i}, i = 1,2,\dots,s, \tag{3.1}$$

where $\tau > 0$ is a positive constant, k_i and l_i are defined by (1.8) and (1.9), respectively.

A direct computation together with (1.9) and (2.10), the function $(\bar{u}_1, \dots, \bar{u}_s)$ is a super-solution of system (1.1)-(1.3) provided that the nonnegative functions $g_1(\xi_1), \dots, g_s(\xi_s)$ satisfy

$$\big(g_i^{m_i}\big)''(\xi_i) + l_i \xi_i g_i'(\xi_i) + k_i g_i(\xi_i) \le 0, \quad i = 1,2,\dots,s, \tag{3.2}$$

$$-\big(g_i^{m_i}\big)'(0) \ge g_i^{p_i}(0) g_{i+1}^{q_i}(0), \quad i = 1,2,\dots,s, \qquad g_{s+1} := g_1. \tag{3.3}$$

We claim that (3.2) and (3.3) admit a solution of the form

$$g_i(\xi_i) = D_i \big(d_i^2 a_i^2 + (\xi_i + a_i)^2\big)^{\frac{1}{m_i - 1}}, \quad i = 1,2,\dots,s. \tag{3.4}$$

Next, we will show that there exist suitable positive constants D_i, d_i, a_i $(i = 1,2,\dots,s)$ such that inequalities (3.2) and (3.3) are satisfied. In fact, for $i = 1,2,\dots,s$, substituting g_i and g_i' into (3.2), we have

$$\left(\frac{2m_i D_i^{m_i}}{m_i - 1} + k_i D_i\right)\big(d_i^2 a_i^2 + (\xi_i + a_i)^2\big) + \frac{4m_i D_i^{m_i}}{(m_i - 1)^2}(\xi_i + a_i)^2 + \frac{2l_i D_i}{m_i - 1}\xi_i(\xi_i + a_i) \le 0.$$

That is,

$$\begin{aligned} &-\left(\frac{2m_i D_i^{m_i - 1}}{1 - m_i} - k_i\right)\big(d_i^2 a_i^2 + (\xi_i + a_i)^2\big) \\ &\quad - \frac{2(\xi_i + a_i)}{1 - m_i}\left(\left(l_i - \frac{2m_i D_i^{m_i - 1}}{1 - m_i}\right)(\xi_i + a_i) - l_i a_i\right) \le 0. \end{aligned} \tag{3.5}$$

Therefore, according to that $k_i < l_i$, we may first take the constant D_i such that

$$k_i < \frac{2m_i D_i^{m_i-1}}{1-m_i} < l_i.$$

Secondly, setting $y_i = \xi_i + a_i$, then the inequality in (3.5) can be written as

$$\left(\frac{2m_i(1+m_i)}{1-m_i}D_i^{m_i-1} - 1\right)y_i^2 + 2l_i a_i y_i - (1-m_i)\left(\frac{2m_i D_i^{m_i-1}}{1-m_i} - k_i\right)d_i^2 a_i^2 \leq 0. \tag{3.6}$$

For simplicity, we define the function h_i as

$$h_i(y_i) = \left(\frac{2m_i(1+m_i)}{1-m_i}D_i^{m_i-1} - 1\right)y_i^2 + 2l_i a_i y_i - (1-m_i)\left(\frac{2m_i D_i^{m_i-1}}{1-m_i} - k_i\right)d_i^2 a_i^2, \quad y_i \geq a_i.$$

And then $h_i(y_i)$ reaches its maximum at the point

$$y_i^* = \frac{l_i a_i}{1-\gamma_i(1+m_i)}, \quad \gamma_i = \frac{2m_i D_i^{m_i-1}}{1-m_i}.$$

Hence, we only need that $h_i(y_i^*) \leq 0$, it follows from the following:

$$d_i \geq l_i\big((1-m_i)(\gamma_i - k_i)\big(1-(1+m_i)\gamma_i\big)\big)^{-\frac{1}{2}}.$$

On the other hand, for the above constants D_i, d_i ($i = 1,2,\ldots,s$), the boundary conditions (3.3) are satisfied if we have

$$a_i^{\frac{1+m_i-2p_i}{2q_i}} \leq R_i a_{i+1}^{\frac{1-m_i}{1-m_{i+1}}}, \quad i = 1,2,\ldots,s, \tag{3.7}$$

where

$$\begin{cases} R_i = D_{i+1}^{\frac{m_i-1}{2}}\left(\frac{2m_i D_i^{m_i-p_i}}{1-m_i}\right)^{\frac{1-m_i}{2q_i}}(d_{i+1}^2+1)^{\frac{1-m_i}{2(1-m_{i+1})}}(d_i^2+1)^{\frac{p_i-1}{2q_i}}, \\ D_{s+1} := D_1, \qquad a_{s+1} := a_1, \qquad d_{s+1} := d_1. \end{cases}$$

Similar to the analysis of the proof in Theorem 2, for $i = 1,2,\ldots,s$, $\prod_{i=1}^s(1+m_i-2p_i) < 2^s q_1 q_2 \cdots q_s$ implies that

$$\frac{1+m_i-2p_i}{2q_i} < \prod_{j=1}^{s-1}\frac{2q_{i+j}}{1+m_{i+j}-2p_{i+j}}.$$

In addition, by (3.7), there exists a positive constant $\bar{C}_i$ as

$$\bar{C}_i = R_i \prod_{j=1}^{s-1}\left(R_{i+j}^{\frac{1-m_i}{1-m_{i+j}}\prod_{j_1=1}^{j}\frac{2q_{i+j_1}}{1+m_{i+j_1}-2p_{i+j_1}}}\right), \qquad R_{s+i} := R_i$$

such that

$$a_i^{\frac{1+m_i-2p_i}{2q_i}} \leq \bar{C}_i a_i^{\prod_{j=1}^{s-1}\frac{2q_{i+j}}{1+m_{i+j}-2p_{i+j}}}, \quad i = 1,2,\ldots,s. \tag{3.8}$$

Thus, we can choose $a_1, a_2, \ldots, a_s$ large enough for the above inequalities (3.8) to hold.

Therefore, it follows from the comparison principle that $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_s)$ given by (3.1) is a super-solution of system (1.1)-(1.3) with $\bar{u}_1(x,0) \geq u_{01}(x), \ldots, \bar{u}_s(x,0) \geq u_{0s}(x)$, which means that the solutions of (1.1)-(1.3) with small initial data have global existence.

(2) We construct the self-similar sub-solution of Eq. (1.1) in the following form:

$$\hat{u}_i(x,t) = (\tau + t)^{-\frac{1}{m_i+1}} g_i(\xi_i), \quad \xi_i = x(\tau + t)^{-\frac{1}{m_i+1}}, i = 1, 2, \ldots, s \tag{3.9}$$

with

$$g_i(\xi_i) = D_i\bigl(c^2 + \xi_i^2\bigr)^{\frac{1}{m_i-1}},$$

where positive constants τ and c are to be determined. If we take

$$D_i = \left(\frac{1-m_i}{2m_i(m_i+1)}\right)^{\frac{1}{m_i-1}}, \quad i = 1, 2, \ldots, s,$$

then it is easy to check that g_i satisfies

$$\bigl(g_i^{m_i}(\xi_i)\bigr)'' + \frac{1}{m_i+1}\xi_i g_i'(\xi_i) + \frac{1}{m_i+1} g_i(\xi_i) = 0, \qquad g_i'(0) = 0, \quad i = 1, 2, \ldots, s, \tag{3.10}$$

which implies that

$$\begin{cases} (\hat{u}_i)_t = (\hat{u}_i^{m_i})_{xx}, & x > 0, t > 0, \\ -(\hat{u}_i^{m_i})_x(0,t) = 0, & t > 0, \end{cases} \quad i = 1, 2, \ldots, s.$$

Since $u_i(x,t)$ $(i = 1, 2, \ldots, s)$ are nonnegative, nontrivial functions, we see that $u_i(0,t_0) > 0$ for some $t_0 > 0$. It is well known that $u_i(x,t_0) > 0$ $(i = 1, 2, \ldots, s)$ are continuous (see [2, 4]). Then we can choose τ and c large enough so that

$$u_i(x,t_0) \geq \hat{u}_i(x,t_0), \quad x > 0, i = 1, 2, \ldots, s.$$

Thus, the self-similar solution $(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_s)$ is a sub-solution of (1.1)-(1.3) in $(0,\infty) \times (t_0, T)$. The comparison principle follows

$$u_i(x,t) \geq \hat{u}_i(x,t), \quad x > 0, t \geq t_0, i = 1, 2, \ldots, s.$$

Recalling that $\max_i\{l_i - k_i\} < 0$, we see for large T that $T^{l_i} \ll T^{k_i}$ $(i = 1, 2, \ldots, s)$. So there exists $t^* > t_0$ such that

$$T^{l_i} \ll \bigl(\tau + t^*\bigr)^{\frac{1}{m_i+1}} \ll T^{k_i}, \quad i = 1, 2, \ldots, s. \tag{3.11}$$

Let $\tilde{u}_i(x,t)$ be defined by (2.11) and (2.12) in the proof of Theorem 2, for any $x > 0$, the inequalities (3.11) imply that

$$\tilde{u}_i(x,0) \leq \hat{u}_i\bigl(x,t^*\bigr) \leq u_i\bigl(x,t^*\bigr).$$

By the comparison principle again, every nonnegative and nontrivial solution $(u_1, u_2, \ldots, u_s)$ of system (1.1)-(1.3) blows up in a finite time. The proof is complete. □

Proof of Theorem 4 Without loss of generality, we may assume that $2p_1 > 1 + m_1$ and $(u_{0i}^{m_i})'' \geq 0$ $(i = 1, 2, \ldots, s)$. Then $u_{1t}, u_{2t}, \ldots, u_{st} \geq 0$ for $x > 0$, $t > 0$ (see [9, 10]). Furthermore, we have $u_1^{p_1}(0,t)u_2^{q_1}(0,t) \geq u_1^{p_1}(0,t)u_{02}^{q_1}(0)$. Consider the following single-equation problem:

$$\begin{cases} w_t = (w^{m_1})_{xx}, & x > 0, 0 < t < T, \\ -(w^{m_1})_x(0,t) = w^{p_1}(0,t)u_{02}^{q_1}(0), & 0 < t < T, \\ w(x,0) = u_{01}(x), & x > 0. \end{cases} \tag{3.12}$$

It is easy to verify that $(w, u_{02}, \ldots, u_{0s})$ is a sub-solution of system (1.1)-(1.3). According to the results of [11], we know that the solutions of (3.12) with large initial data blow up in a finite time, and so the solutions of (1.1)-(1.3) do too. The proof is complete. □

Competing interests
The author declares that they have no competing interests.

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