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Necessary and sufficient conditions for functions involving the psi function to be completely monotonic

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Abstract

We present the necessary and sufficient conditions such that the functions involving $R(x) = \psi(x + 1/2) - \ln x$ with a parameter are completely monotonic on $(0, \infty)$, find three new sequences which are fast convergence toward the Euler-Mascheroni constant, and give a positive answer to the conjecture proposed by Chen (J. Math. Inequal. 3(1):79-91, 2009), where ψ is the digamma function.

MSC: 33B15; 26D15

Keywords: psi function; completely monotone function; Euler-Mascheroni constant

1 Introduction

A real-valued function f is said to be completely monotonic on the interval I if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1.1}$$

for all $x \in I$ and $n = 0, 1, 2, \ldots$. f is said to be strictly completely monotonic on I if inequality (1.1) is strict.

It is well known that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt}\, d\mu(t),$$

where μ is a nonnegative measure on $[0, \infty)$ such that the integral is convergent for all $x > 0$ (see [1], p.161).

Let $x > 0$, then the classical Euler gamma function Γ and psi (digamma) function ψ are, respectively, defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.2}$$

The derivatives $\psi', \psi'', \psi''', \ldots$ are known as polygamma functions. Recently, the gamma and polygamma functions have attracted the attention of many researchers since they play important roles in many branches, such as mathematical physics, probability, statistics, and engineering.

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ be the harmonic number and $D_n = H_n - \ln n$. Then the well-known Euler-Mascheroni constant $\gamma = 0.577215664\ldots$ can be expressed as $\gamma = H_n - \psi(n+1)$ or $\gamma = \lim_{n\to\infty} D_n$, and the double inequality

$$\frac{1}{2(n+1)} < D_n - \gamma < \frac{1}{2n}$$

holds for all $n \in \mathbb{N}$ (see [2, 3]). Therefore, the convergence rate of D_n is very slowly. Recently, many results involving the quicker convergence toward the Euler-Mascheroni constant can be found in the literature [4–27].

In 1993, DeTemple [7] introduced the DeTemple sequence

$$R_n = \sum_{k=1}^{n} \frac{1}{k} - \ln\left(n + \frac{1}{2}\right) \tag{1.3}$$

and found that it satisfies the double inequalities

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2} \tag{1.4}$$

and

$$\frac{7}{960}\frac{1}{(n+1)^4} < R_n - \gamma - \frac{1}{24n^2} < \frac{7}{960}\frac{1}{n^4} \tag{1.5}$$

for all $n \in \mathbb{N}$.

Villarino ([14], Theorem 1.7) proved that the double inequality

$$\frac{1}{24(n+1/2)^2 + 21/5} < R_n - \gamma < \frac{1}{24(n+1/2)^2 + 1/(1 - \ln 3 + \ln 2 - \gamma) - 54} \tag{1.6}$$

holds for all $n \in \mathbb{N}$ with the best possible constants $21/5$ and $1/(1 - \ln 3 + \ln 2 - \gamma) - 54 = 3.739\ldots$.

In [18], Chen proved that the double inequality

$$\frac{1}{24}(n+\lambda)^{-2} < R_n - \gamma < \frac{1}{24}\left(n + \frac{1}{2}\right)^{-2} \tag{1.7}$$

holds for all $n \in \mathbb{N}$ with the best possible constants

$$\lambda = \frac{1}{2\sqrt{6(1 - \gamma - \ln 3 + \ln 2)}} - 1 = 0.551\ldots$$

and $1/2$.

Mortici ([28], Theorem 2.1) presented the bounds for $R_n - \gamma$ as follows:

$$\frac{1}{24}\left(n + \frac{1}{2} + \frac{7}{80n}\right)^{-2} < R_n - \gamma < \frac{1}{24}\left(n + \frac{1}{2}\right)^{-2}. \tag{1.8}$$

In [29–31], the authors established the inequality

$$\gamma + \ln\left(n + \frac{1}{2}\right) < \sum_{k=1}^{n} \frac{1}{k} \le \gamma + \ln\left(n + e^{1-\gamma} - 1\right),$$

which is equivalent to

$$0 < R_n - \gamma \le \ln \frac{n + e^{1-\gamma} - 1}{n + 1/2}.$$

Karatsuba [32] proved that the sequence

$$H(n) = (R_n - \gamma)n^2 = \left(\psi(n+1) - \ln\left(n + \frac{1}{2}\right)\right)n^2 \tag{1.9}$$

is strictly increasing with respect to all $n \in \mathbb{N}$.

In [33], the authors pointed out that $(1 + 1/n)^2 H(n)$ is a strictly decreasing and convex sequence by use of computer experiments. Chen ([15], Theorem 2) proved that both $H(n)$ and $((n+1/2)/n)^2 H(n)$ are strictly increasing and concave sequences, while $((n+1)/n)^2 H(n)$ is a strictly decreasing and convex sequence, and conjectured that:

(i) The two functions $H(x) = [\psi(x+1) - \ln(x+1/2)]x^2$ and $[(x+1/2)/x]^2 H(x)$ are so-called Bernstein functions on $(0,\infty)$. That is,

$$H(x) > 0, \quad (-1)^n\big[H(x)\big]^{(n+1)} > 0,$$
$$\big((x+1/2)/x\big)^2 H(x) > 0, \quad (-1)^n\Big[\big((x+1/2)/x\big)^2 H(x)\Big]^{(n+1)} > 0$$

for $x > 0$ and $n \in \mathbb{N}$.

(ii) The function $((x+1)/x)^2 H(x)$ is strictly completely monotonic on $(0,\infty)$.

It is not difficult to verify that

$$-H''(0^+) = 2\ln 2 - 2\gamma = -0.2318\ldots < 0,$$
$$-H''(1/2) = 2\gamma + 4\ln 2 + \frac{7}{2}\sum_{n=1}^{\infty}\frac{1}{n^3} - \pi^2 + \frac{7}{4} = 0.01461\ldots > 0.$$

Therefore, the function $H(x)$ is not a Bernstein function on $(0,\infty)$.

The main purpose of this paper is to give a positive answer to the conjecture (ii) and present several necessary and sufficient conditions such that the functions involving

$$R(x) = \psi(x+1/2) - \ln x \tag{1.10}$$

with a parameter are strictly completely monotone on $(0,\infty)$.

2 Lemmas

In order to prove our results we need several lemmas, which we present in this section.

Lemma 1 *Let* $R(x)$ *be defined by* (1.10) *and* $Q(t)$ *be defined on* $(0,\infty)$ *by*

$$Q(t) = \frac{1}{t} - \frac{1}{2\sinh\frac{t}{2}}. \tag{2.1}$$

Then the following identities are valid:

$$R(x) = \int_0^\infty e^{-xt} Q(t)\,dt, \tag{2.2}$$

$$xR(x) = \int_0^\infty e^{-xt} Q'(t)\,dt, \tag{2.3}$$

$$x^2 R(x) = \frac{1}{24} + \int_0^\infty e^{-xt} Q''(t)\,dt, \tag{2.4}$$

$$x^3 R(x) = \frac{1}{24}x + \int_0^\infty e^{-xt} Q'''(t)\,dt, \tag{2.5}$$

$$x^4 R(x) = \frac{1}{24}x^2 - \frac{7}{960} + \int_0^\infty e^{-xt} Q^{(4)}(t)\,dt. \tag{2.6}$$

Proof Making use of the integral representations [34], p.259

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right) dt \quad \text{and} \quad \ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t}\,dt,$$

we get

$$\begin{aligned} R(x) &= \psi\left(x + \frac{1}{2}\right) - \ln x = \int_0^\infty \left(\frac{e^{-xt}}{t} - \frac{e^{-(x+1/2)t}}{1-e^{-t}}\right) dt \\ &= \int_0^\infty e^{-xt}\left(\frac{1}{t} - \frac{1}{2\sinh\frac{t}{2}}\right) dt = \int_0^\infty e^{-xt} Q(t)\,dt. \end{aligned}$$

Integration by parts leads to

$$\begin{aligned} xR(x) &= x\int_0^\infty e^{-xt} Q(t)\,dt = -\int_0^\infty Q(t)\,de^{-xt} \\ &= -e^{-xt}Q(t)\big|_0^\infty + \int_0^\infty e^{-xt} Q'(t)\,dt = \int_0^\infty e^{-xt} Q'(t)\,dt, \end{aligned}$$

where the last equality holds due to $\lim_{t\to\infty}(e^{-xt}Q(t)) = \lim_{t\to 0}(e^{-xt}Q(t)) = 0$.

Integration by parts again together with

$$\lim_{t\to\infty}\left(e^{-xt}Q'(t)\right) = 0 \quad \text{and} \quad \lim_{t\to 0}\left(e^{-xt}Q'(t)\right) = -\frac{1}{24}$$

leads to

$$\begin{aligned} x^2R(x) &= x\int_0^\infty e^{-xt} Q'(t)\,dt = -e^{-xt}Q'(t)\big|_0^\infty + \int_0^\infty e^{-xt} Q''(t)\,dt \\ &= \frac{1}{24} + \int_0^\infty e^{-xt} Q''(t)\,dt. \end{aligned}$$

Similarly, we have

$$\begin{aligned} x^3R(x) &= \frac{1}{24}x + x\int_0^\infty e^{-xt} Q''(t)\,dt = \frac{1}{24}x - e^{-xt}Q''(t)\big|_0^\infty + \int_0^\infty e^{-xt} Q'''(t)\,dt \\ &= \frac{1}{24}x + \int_0^\infty e^{-xt} Q'''(t)\,dt \end{aligned}$$

due to $\lim_{t\to\infty}(e^{-xt}Q''(t)) = \lim_{t\to 0}(e^{-xt}Q''(t)) = 0$, and

$$x^4R(x) = \frac{1}{24}x^2 + x\int_0^\infty e^{-xt}Q'''(t)\,dt = \frac{1}{24}x^2 - e^{-xt}Q'''(t)|_0^\infty + \int_0^\infty e^{-xt}Q^{(4)}(t)\,dt$$
$$= \frac{1}{24}x^2 - \frac{7}{960} + \int_0^\infty e^{-xt}Q^{(4)}(t)\,dt$$

due to

$$\lim_{t\to\infty}\left(e^{-xt}Q'''(t)\right) = 0 \quad \text{and} \quad \lim_{t\to 0}\left(e^{-xt}Q'''(t)\right) = -\frac{7}{960}. \qquad \square$$

Lemma 2 ([35], Lemma 7) *Let $P(t)$ be a power series which is convergent on $(0,\infty)$ defined by*

$$P(t) = \sum_{i=m+1}^{\infty} a_i t^i - \sum_{i=0}^{m} a_i t^i,$$

where $a_i \geq 0$ and $a_j \geq 0$ for $i \geq m+1$ and $0 \leq j \leq m-1$, $a_m > 0$, and $\sum_{i=m+1}^{\infty} a_i > 0$. Then there exists $t_0 \in (0,\infty)$ such that $P(t_0) = 0$, $P(t) < 0$ for $t \in (0,t_0)$ and $P(t) > 0$ for $t \in (t_0,\infty)$.

Lemma 3 *Let $Q(t)$ be defined by* (2.1). *Then $Q'(t) \geq c_0 Q(t)$ for $t > 0$, where*

$$c_0 = \min_{t>0}\left(\frac{Q'(t)}{Q(t)}\right) = -0.06187\ldots. \tag{2.7}$$

Proof Simple computations lead to

$$\frac{Q'(t)}{Q(t)} = \frac{\frac{1}{4}\frac{\cosh\frac{t}{2}}{\sinh^2\frac{t}{2}} - \frac{1}{t^2}}{\frac{1}{t} - \frac{1}{2\sinh\frac{t}{2}}} = \frac{t^2\cosh\frac{t}{2} - 4\sinh^2\frac{t}{2}}{4t\sinh^2\frac{t}{2} - 2t^2\sinh\frac{t}{2}},$$
$$\left(\frac{Q'(t)}{Q(t)}\right)' = \frac{1}{4t^2(t-2\sinh\frac{t}{2})^2\sinh^2\frac{t}{2}}\Big(-16t\sinh^3\frac{t}{2} + t^4\cosh^2\frac{t}{2} + 2t^3\sinh^3\frac{t}{2}$$
$$- t^4\sinh^2\frac{t}{2} + 16\sinh^4\frac{t}{2} + 8t^2\cosh\frac{t}{2}\sinh^2\frac{t}{2} - 4t^3\cosh^2\frac{t}{2}\sinh\frac{t}{2}\Big)$$
$$:= \frac{p(\frac{t}{2})}{4t^2(t-2\sinh\frac{t}{2})^2\sinh^2\frac{t}{2}},$$

where

$$p(t) = 16\big(t^4\cosh^2 t + t^3\sinh^3 t - t^4\sinh^2 t + \sinh^4 t - 2t\sinh^3 t$$
$$+ 2t^2\cosh t\sinh^2 t - 2t^3\cosh^2 t\sinh t\big).$$

Using the 'product into sum' formulas and Taylor expansion we get

$$\frac{1}{2}p(t) = \cosh 4t + 4t^2\cosh 3t - 2t^3\sinh 3t - 4t\sinh 3t - 4\cosh 2t$$
$$- 4t^2\cosh t - 10t^3\sinh t + 12t\sinh t + 8t^4 + 3$$

$$\begin{aligned}&= \sum_{n=0}^{\infty} \frac{4^{2n}t^{2n}}{(2n)!} + 4\sum_{n=1}^{\infty} \frac{3^{2n-2}t^{2n}}{(2n-2)!} - 2\sum_{n=2}^{\infty} \frac{3^{2n-3}t^{2n}}{(2n-3)!} - 4\sum_{n=1}^{\infty} \frac{3^{2n-1}t^{2n}}{(2n-1)!} \\ &\quad - 4\sum_{n=0}^{\infty} \frac{2^{2n}t^{2n}}{(2n)!} - 4\sum_{n=1}^{\infty} \frac{t^{2n}}{(2n-2)!} - 10\sum_{n=2}^{\infty} \frac{t^{2n}}{(2n-3)!} + 12\sum_{n=1}^{\infty} \frac{t^{2n}}{(2n-1)!} + 8t^4 + 3 \\ &:= \sum_{n=3}^{\infty} \frac{u_n}{(2n)!} t^{2n},\end{aligned}$$

where

$$u_n = 4^{2n} - 8n(2n^2 - 9n + 13)3^{2n-3} - 2^{2n+2} - 8n(10n^2 - 13n + 1).$$

It is not difficult to verify that $u_3 = 0$, $u_n < 0$ for $4 \leq n \leq 10$ and $u_{11} = 1{,}636{,}643{,}754{,}240 > 0$. Note that

$$\begin{aligned}u_{n+1} - 16u_n &= 8(14n^3 - 117n^2 + 199n - 54)3^{2n-3} + 48 \times 2^{2n} \\ &\quad + 1{,}200n^3 - 1{,}800n^2 + 88n + 16 > 0\end{aligned}$$

for $n \geq 11$. Therefore, $u_n \geq 0$ for $n \geq 11$.

From Lemma 2 we clearly see that there exists $t_0 \in (0, \infty)$ such that $Q'(t)/Q(t)$ is strictly decreasing on $(0, t_0)$ and strictly increasing on (t_0, ∞). Therefore, Lemma 3 follows from the piecewise monotonicity of Q'/Q and the numerical computations results $t_0 = 15.4015\ldots$ and $Q'(t_0)/Q(t_0) = -0.06187\ldots$. □

Lemma 4 *The inequalities*

$$\frac{\sinh t}{t} > 3\frac{2\cosh t + 3}{\cosh t + 14}, \tag{2.8}$$

$$\frac{\sinh t}{t} > 15\frac{2\cosh^2 t + 10\cosh t + 9}{2\cosh^2 t + 101\cosh t + 212}, \tag{2.9}$$

$$\frac{\sinh t}{t} < 15\frac{18\cosh^2 t + 160\cosh t + 179}{1{,}159\cosh^2 t + 4{,}192\cosh t + 4}\cosh t \tag{2.10}$$

hold for $t > 0$.

Proof Inequality (2.8) can be found in [36], Theorem 18.

To prove (2.9), it suffices to show that for $t > 0$,

$$p_1(t) := \frac{2\cosh^2 t + 101\cosh t + 212}{2\cosh^2 t + 10\cosh t + 9}\sinh t - 15t > 0.$$

Simple computations lead to

$$\begin{aligned}p_1'(t) &= \frac{2\cosh^2 t + 101\cosh t + 212}{2\cosh^2 t + 10\cosh t + 9}\cosh t - 7\frac{26\cosh^2 t + 116\cosh t + 173}{(2\cosh^2 t + 10\cosh t + 9)^2}\sinh^2 t - 15 \\ &= \frac{4(\cosh t - 1)^5}{(2\cosh^2 t + 10\cosh t + 9)^2} > 0,\end{aligned}$$

which implies that $p_1(t) > p_1(0) = 0$.

Similarly, inequality (2.10) is equivalent to

$$p_2(t) := \frac{1{,}159\cosh^2 t + 4{,}192\cosh t + 4}{(18\cosh^2 t + 160\cosh t + 179)\cosh t}\sinh t - 15t < 0.$$

Differentiating $p_2(t)$ yields

$$\begin{aligned} p_2'(t) &= \frac{1{,}159\cosh^2 t + 4{,}192\cosh t + 4}{(18\cosh^2 t + 160\cosh t + 179)\cosh t}\cosh t \\ &\quad + \left(\sinh^2 t\right)\frac{d}{dx}\left(\frac{1{,}159x^2 + 4{,}192x + 4}{(18x^2 + 160x + 179)x}\right) - 15 \\ &= -\frac{4(1{,}215x + 179)(x-1)^5}{x^2(18x^2 + 160x + 179)^2} < 0, \end{aligned}$$

where $x = \cosh t > 1$. Therefore, $p_2(t) < p_2(0) = 0$. □

Lemma 5 *Let* $Q(t)$ *be defined by* (2.1). *Then the inequality*

$$q_1(t) := Q''(t) + \frac{7}{40}Q(t) > 0$$

holds for all $t > 0$.

Proof Simple computations lead to

$$\begin{aligned} Q''(t) &= \frac{1}{8\sinh\frac{t}{2}} - \frac{1}{4}\frac{\cosh^2\frac{t}{2}}{\sinh^3\frac{t}{2}} + \frac{2}{t^3}, \qquad (2.11) \\ q_1(t) &= -\frac{1}{80}\frac{20t^3\cosh^2\frac{t}{2} - 14t^2\sinh^3\frac{t}{2} - 3t^3\sinh^2\frac{t}{2} - 160\sinh^3\frac{t}{2}}{t^3\sinh^3\frac{t}{2}}. \end{aligned}$$

Making use of inequality (2.8) we get

$$\begin{aligned} \left(80\sinh^3 t\right)q_1(2t) &= 20\left(\frac{\sinh t}{t}\right)^3 + 7\left(\sinh^2 t\right)\frac{\sinh t}{t} \\ &\quad + 3\sinh^2 t - 20\cosh^2 t \\ &> 20\left(3\frac{2\cosh t + 3}{\cosh t + 14}\right)^3 + 7\left(\cosh^2 t - 1\right)\left(3\frac{2\cosh t + 3}{\cosh t + 14}\right) \\ &\quad + 3\left(\cosh^2 t - 1\right) - 20\cosh^2 t \\ &= 25\left(\cosh^2 t + 24\cosh t + 240\right)\frac{(\cosh t - 1)^3}{(\cosh t + 14)^3} > 0. \end{aligned}$$

□

Lemma 6 *Let* $Q(t)$ *be defined by* (2.1). *Then*

$$q_2(t) := Q^{(4)}(t) - \frac{31}{336}Q(t) < 0$$

for all $t > 0$.

Proof Simple computations lead to

$$Q^{(4)}(t) = \frac{7}{8}\frac{\cosh^2\frac{1}{2}t}{\sinh^3\frac{1}{2}t} - \frac{3}{4}\frac{\cosh^4\frac{1}{2}t}{\sinh^5\frac{1}{2}t} - \frac{5}{32\sinh\frac{1}{2}t} + \frac{24}{t^5}, \tag{2.12}$$

$$q_2(t) = -\frac{1}{336} \times \frac{252t^5\cosh^4\frac{t}{2} + 31t^4\sinh^5\frac{t}{2} + 37t^5\sinh^4\frac{t}{2} - 8{,}064\sinh^5\frac{t}{2} - 294t^5\cosh^2\frac{t}{2}\sinh^2\frac{t}{2}}{t^5\sinh^5\frac{t}{2}},$$

$$-\left(672\sinh^5 t\right)q_2(2t) = -504\left(\frac{\sinh t}{t}\right)^5 + 31\left(\sinh^4 t\right)\frac{\sinh t}{t} + 504\cosh^4 t - 588\cosh^2 t\sinh^2 t + 74\sinh^4 t.$$

Let

$$U(y) = -504y^5 + 31\left(\sinh^4 t\right)y + 504\cosh^4 t - 588\cosh^2 t\sinh^2 t + 74\sinh^4 t.$$

Then it suffices to prove that $U((\sinh t)/t) > 0$ for $t > 0$.

It follows from $U'(y) = 31\sinh^4 t - 2{,}520y^4$ that U is strictly increasing with respect to y on $(1, \sqrt[4]{31/2{,}520}\sinh t]$ and strictly decreasing with respect to y on $[\sqrt[4]{31/2{,}520}\sinh t, \infty)$. We divide the proof into two cases.

Case 1: $t \in (\sqrt[4]{2{,}520/31}, \infty)$. Then inequality (2.8) leads to

$$1 < 3\frac{2\cosh t + 3}{\cosh t + 14} < \frac{\sinh t}{t} < \sqrt[4]{\frac{31}{2{,}520}}\sinh t,$$

that is,

$$3\frac{2\cosh t + 3}{\cosh t + 14}, \frac{\sinh t}{t} \in \left(1, \sqrt[4]{31/2{,}520}\sinh t\right),$$

and so

$$\begin{aligned} U\left(\frac{\sinh t}{t}\right) &> U\left(3\frac{2\cosh t + 3}{\cosh t + 14}\right) \\ &= \left(504\cosh^4 t - 588\cosh^2 t\sinh^2 t + 74\sinh^4 t\right) \\ &\quad + 31\left(\sinh^4 t\right) \times 3\frac{2\cosh t + 3}{\cosh t + 14} - 504\left(3\frac{2\cosh t + 3}{\cosh t + 14}\right)^5. \end{aligned}$$

Let $\cosh t = x$, then $\sinh^2 t = x^2 - 1$, and

$$U\left(\frac{\sinh t}{t}\right) > \frac{(x-1)^3}{(x+14)^5}U_1(x),$$

where

$$\begin{aligned} U_1(x) &= 176x^6 + 10{,}523x^5 + 245{,}869x^4 + 2{,}810{,}864x^3 \\ &\quad + 12{,}467{,}224x^2 + 12{,}511{,}688x - 20{,}756{,}344. \end{aligned}$$

It is not difficult to verify that $U_1(x) > U_1(1) = 7{,}290{,}000 > 0$, which implies that $U((\sinh t)/t) > 0$ for $t \in (\sqrt[4]{2{,}520/31}, \infty)$.

Case 2: $t \in (0, \sqrt[4]{2{,}520/31}]$. Then it follows from (2.10) and the piecewise monotonicity of U that

$$\infty > 15\frac{18\cosh^2 t + 160\cosh t + 179}{1{,}159\cosh^2 t + 4{,}192\cosh t + 4}\cosh t > \frac{\sinh t}{t} > \sqrt[4]{\frac{31}{2{,}520}}\sinh t,$$

$$U\left(\frac{\sinh t}{t}\right) > U\left(15\frac{18\cosh^2 t + 160\cosh t + 179}{1{,}159\cosh^2 t + 4{,}192\cosh t + 4}\cosh t\right).$$

Let $\cosh t = x$, then

$$U\left(\frac{\sinh t}{t}\right) > U\left(15\frac{18x^2 + 160x + 179}{1{,}159x^2 + 4{,}192x + 4}x\right) = \frac{(x-1)^4}{(1{,}159x^2 + 4{,}192x + 4)^5}U_2(x),$$

where

$$\begin{aligned} U_2(x) = {}& 14{,}379{,}675{,}269{,}523{,}570x^{11} + 357{,}214{,}567{,}270{,}415{,}330x^{10} \\ &+ 3{,}604{,}910{,}878{,}299{,}956{,}955x^9 + 19{,}027{,}526{,}850{,}473{,}930{,}600x^8 \\ &+ 55{,}570{,}610{,}110{,}726{,}848{,}080x^7 + 85{,}295{,}682{,}448{,}077{,}545{,}696x^6 \\ &+ 54{,}079{,}668{,}524{,}631{,}977{,}864x^5 + 560{,}130{,}320{,}580{,}220{,}160x^4 \\ &+ 1{,}016{,}873{,}963{,}329{,}280x^3 + 923{,}378{,}178{,}560x^2 \\ &+ 418{,}677{,}504x + 75{,}776 > 0. \end{aligned}$$

□

Lemma 7 *Let $Q(t)$ be defined by* (2.1). *Then*

$$q_3(t) := Q^{(4)}(t) + \frac{11{,}165}{8{,}284}Q''(t) + \frac{199{,}849}{1{,}391{,}712}Q(t) > 0$$

for all $t > 0$.

Proof It follows from (2.1), (2.11), and (2.12) that

$$\begin{aligned} q_3(t) = {}& \frac{1}{2{,}783{,}424t^5\sinh^5\frac{t}{2}}\left(-2{,}087{,}568t^5\cosh^4\frac{t}{2} + 1{,}497{,}636t^5\cosh^2\frac{t}{2}\sinh^2\frac{t}{2}\right. \\ &- 165{,}829t^5\sinh^4\frac{t}{2} + 399{,}698t^4\sinh^5\frac{t}{2} \\ &\left. + 7{,}502{,}880t^2\sinh^5\frac{t}{2} + 66{,}802{,}176\sinh^5\frac{t}{2}\right), \end{aligned}$$

$$\begin{aligned} &\left(2{,}783{,}424\sinh^5 t\right)q_3(2t) \\ &\quad = -\left(2{,}087{,}568\cosh^4 t + 165{,}829\sinh^4 t - 1{,}497{,}636\cosh^2 t\sinh^2 t\right) \\ &\qquad + 199{,}849\left(\sinh^4 t\right)\frac{\sinh t}{t} + 937{,}860\left(\sinh^2 t\right)\left(\frac{\sinh t}{t}\right)^3 \\ &\qquad + 2{,}087{,}568\left(\frac{\sinh t}{t}\right)^5. \end{aligned}$$

Let $\cosh t = x > 1$, then (2.9) leads to

$$\begin{aligned}
\left(2{,}783{,}424\sinh^5 t\right)q_3(2t) &> -\left(2{,}087{,}568x^4 + 165{,}829\left(x^2-1\right)^2 - 1{,}497{,}636x^2\left(x^2-1\right)\right) \\
&\quad + 199{,}849\left(x^2-1\right)^2\left(15\frac{2x^2+10x+9}{2x^2+101x+212}\right) \\
&\quad + 937{,}860\left(x^2-1\right)\left(15\frac{2x^2+10x+9}{2x^2+101x+212}\right)^3 \\
&\quad + 2{,}087{,}568\left(15\frac{2x^2+10x+9}{2x^2+101x+212}\right)^5 \\
&= \frac{7(x-1)^5}{(2x^2+101x+212)^5}q_4(x) > 0,
\end{aligned}$$

where the last inequality holds due to

$$\begin{aligned}
q_4(x) &= 10{,}249{,}024x^9 + 2{,}015{,}594{,}800x^8 + 163{,}876{,}520{,}192x^7 \\
&\quad + 6{,}681{,}271{,}280{,}040x^6 + 136{,}012{,}433{,}414{,}956x^5 \\
&\quad + 1{,}069{,}481{,}086{,}377{,}851x^4 + 4{,}121{,}483{,}475{,}973{,}500x^3 \\
&\quad + 8{,}450{,}810{,}874{,}059{,}188x^2 + 8{,}899{,}895{,}239{,}232{,}240x \\
&\quad + 3{,}802{,}278{,}457{,}617{,}584.
\end{aligned}$$

□

3 Main results

Theorem 1 *Let $R(x)$ be defined on $(0,\infty)$ by* (1.10). *Then the function*

$$h_a(x) = (x+a)^2R(x)$$

is strictly completely monotonic on $(0,\infty)$ if $a \ge a_0 = \sqrt{c_0^2 + 7/40} - c_0 = 0.4847\ldots$, where $c_0 = -0.06187\ldots$ is given by (2.7).

Proof It follows from (2.2)-(2.4) that

$$\begin{aligned}
h_a(x) &= (x+a)^2R(x) = x^2R(x) + 2axR(x) + a^2R(x) \\
&= \frac{1}{24} + \int_0^\infty e^{-xt}Q''(t)\,dt + 2a\int_0^\infty e^{-xt}Q'(t)\,dt + a^2\int_0^\infty e^{-xt}Q(t)\,dt \\
&= \frac{1}{24} + \int_0^\infty e^{-xt}\left(Q''(t) + 2aQ'(t) + a^2Q(t)\right)dt \\
&\triangleq \frac{1}{24} + \int_0^\infty e^{-xt}Q(t)\delta_a(t)\,dt.
\end{aligned}$$

We clearly see that $Q(t) > 0$ for $t > 0$ and Lemmas 3 and 5 imply that

$$\begin{aligned}
\delta_a(t) &= \frac{Q''(t)}{Q(t)} + 2a\frac{Q'(t)}{Q(t)} + a^2 \ge a^2 + 2ac_0 - \frac{7}{40} \\
&= \left(a + c_0 + \sqrt{c_0^2 + \frac{7}{40}}\right)\left(a + c_0 - \sqrt{c_0^2 + \frac{7}{40}}\right) \ge 0
\end{aligned}$$

if $a \ge \sqrt{c_0^2 + 7/40} - c_0$. □

Taking $a = 1/2$ and replacing x by $(x + 1/2)$ in Theorem 1, we have the following.

Corollary 1 *The function* $((x+1)/x)^2 H(x)$ *is strictly completely monotonic on* $(-1/2, \infty)$.

Remark 1 Corollary 1 gives a positive answer to the conjecture (ii) posed by Chen in [15].

Theorem 2 *Let* $R(x)$ *be defined on* $(0, \infty)$ *by* (1.10). *Then the function*

$$x \mapsto F_a(x) = 24\big(x^2 + a\big)R(x) - 1$$

is strictly completely monotonic on $(0, \infty)$ *if and only if* $a \geq a_1 = 7/40$.

Proof The necessity follows from

$$\lim_{x\to\infty} \frac{F_a(x)}{x^{-2}} = \lim_{x\to\infty} \frac{24(x^2 + a)(\psi(x + 1/2) - \ln x) - 1}{x^{-2}} = a - \frac{7}{40} \geq 0.$$

It follows from (2.2) and (2.4) that

$$\begin{aligned} F_a(x) &= 24\big(x^2 + a\big)R(x) - 1 = 24x^2 R(x) + 24aR(x) - 1 \\ &= 24\left(\frac{1}{24} + \int_0^\infty e^{-xt} Q''(t)\,dt\right) + 24a\int_0^\infty e^{-xt} Q(t)\,dt - 1 \\ &= 24\int_0^\infty e^{-xt}\big(Q''(t) + aQ(t)\big)\,dt. \end{aligned}$$

From Lemma 5 we clearly see that

$$Q''(t) + aQ(t) \geq Q''(t) + \frac{7}{40}Q(t) > 0$$

for $t > 0$ if $a \geq 7/40$. □

Note that

$$F_{7/40}\left(n + \frac{1}{2}\right) = 24\big((n + 1/2)^2 + 7/40\big)R_n - 1,$$

$$F_{7/40}(3/2) = \frac{286}{5} - \frac{291}{5}\ln\frac{3}{2} - \frac{291}{5}\gamma = 0.00797\ldots, \qquad F_{7/40}(\infty) = 0.$$

Therefore, we have the following.

Corollary 2 *Let* R_n *be defined by* (1.3). *Then the double inequality*

$$\frac{1}{24((n + 1/2)^2 + 7/40)} < R_n - \gamma < \frac{1 + \lambda_1}{24((n + 1/2)^2 + 7/40)} \tag{3.1}$$

holds for $n \in \mathbb{N}$ *with the best possible constants* $\lambda_1 = F_{7/40}(3/2) = 0.00797\ldots$.

Theorem 3 *Let* $R(x)$ *be defined on* $(0, \infty)$ *by* (1.10). *Then the function*

$$x \mapsto f_a(x) = -24\big(x^4 + a\big)R(x) + x^2 - \frac{7}{40}$$

is strictly completely monotonic on $(0, \infty)$ *if and only if* $a \leq a_2 = -31/336$.

Proof The necessity can be deduced by

$$\lim_{x\to\infty}\frac{f_a(x)}{x^{-2}}=\lim_{x\to\infty}\frac{-24(x^4+a)(\psi(x+1/2)-\ln x)+x^2-\frac{7}{40}}{x^{-2}}=-a-\frac{31}{336}\geq 0.$$

It follows from (2.2), (2.4), and (2.6) that

$$\begin{aligned}f_a(x)&=-24x^4R(x)-24aR(x)+x^2-\frac{7}{40}\\&=-24\left(\frac{1}{24}x^2-\frac{7}{960}+\int_0^\infty e^{-xt}Q^{(4)}(t)\,dt\right)-24a\int_0^\infty e^{-xt}Q(t)\,dt+x^2-\frac{7}{40}\\&=24\int_0^\infty e^{-xt}\bigl(-Q^{(4)}(t)-aQ(t)\bigr)\,dt.\end{aligned}$$

From Lemma 6 we clearly see that

$$-Q^{(4)}(t)-aQ(t)\geq -Q^{(4)}(t)+\frac{31}{336}Q(t)>0$$

if $a\leq a_2=-31/336$. □

Making use of the monotonicity of f_{a_2} and the facts that

$$f_{a_2}\left(\frac{3}{2}\right)=\frac{835}{7}\gamma+\frac{835}{7}\ln\frac{3}{2}-\frac{32{,}819}{280}=0.009063\ldots,\qquad f_{a_2}(\infty)=0$$

we have the following.

Corollary 3 *Let R_n be defined by* (1.3). *Then the double inequality*

$$\frac{1}{24}\frac{(n+\frac{1}{2})^2-\frac{7}{40}-\lambda_2}{(n+\frac{1}{2})^4-\frac{31}{336}}<R_n-\gamma<\frac{1}{24}\frac{(n+\frac{1}{2})^2-\frac{7}{40}}{(n+\frac{1}{2})^4-\frac{31}{336}}\tag{3.2}$$

holds for $n\in\mathbb{N}$ with the best possible constant $\lambda_2=f_{a_2}(3/2)=0.009063\ldots$.

Remark 2 The upper bound for $R_n-\gamma$ given in (3.2) is better than that given in (1.5). Indeed, simple computations show that

$$\begin{aligned}&\frac{1}{24}\frac{(n+\frac{1}{2})^2-\frac{7}{40}}{(n+\frac{1}{2})^4-\frac{31}{336}}-\left(\frac{1}{24n^2}+\frac{7}{960}\frac{1}{n^4}\right)\\&\quad=-\frac{6{,}720n^5+10{,}752n^4+5{,}712n^3+1{,}564n^2+588n-35}{960n^4(168n^4+336n^3+252n^2+84n-5)}<0\end{aligned}$$

for all $n\in\mathbb{N}$.

Theorem 4 *Let $R(x)$ be defined on $(0,\infty)$ by* (1.10). *Then the function*

$$x\mapsto G_a(x)=24\left(x^4+ax^2+\frac{7}{40}a-\frac{31}{336}\right)R(x)-\left(x^2-\frac{7}{40}+a\right)\tag{3.3}$$

is strictly completely monotonic on $(0,\infty)$ if and only if $a\geq a_3=11{,}165/8{,}284$.

Proof The necessity can be derived from

$$\begin{aligned}\lim_{x\to\infty}\frac{G_a(x)}{x^{-4}} &= \lim_{x\to\infty}\frac{24(x^4+ax^2+\frac{7}{40}a-\frac{31}{336})(\psi(x+1/2)-\ln x)-(x^2-\frac{7}{40}+a)}{x^{-4}}\\ &= \frac{2{,}071}{33{,}600}\left(a-\frac{11{,}165}{8{,}284}\right)\ge 0.\end{aligned}$$

It follows from (2.2), (2.4), and (2.6) that

$$\begin{aligned}G_a(x) &= 24x^4R(x)+24ax^2R(x)+24\left(\frac{7}{40}a-\frac{31}{336}\right)R(x)-\left(x^2-\frac{7}{40}+a\right)\\ &= 24\left(\frac{1}{24}x^2-\frac{7}{960}+\int_0^\infty e^{-xt}Q^{(4)}(t)\,dt\right)+24a\left(\frac{1}{24}+\int_0^\infty e^{-xt}Q''(t)\,dt\right)\\ &\quad+24\left(\frac{7}{40}a-\frac{31}{336}\right)\int_0^\infty e^{-xt}Q(t)\,dt-\left(x^2-\frac{7}{40}+a\right)\\ &= 24\int_0^\infty e^{-xt}\left(Q^{(4)}(t)+aQ''(t)+\left(\frac{7}{40}a-\frac{31}{336}\right)Q(t)\right)dt\\ &\triangleq 24\int_0^\infty e^{-xt}g_a(t)\,dt.\end{aligned}$$

From Lemmas 5 and 7 we clearly see that

$$\begin{aligned}g_a(t) &= Q^{(4)}(t)-\frac{31}{336}Q(t)+a\left(Q''(t)+\frac{7}{40}Q(t)\right)\\ &\ge Q^{(4)}(t)-\frac{31}{336}Q(t)+\frac{11{,}165}{8{,}284}\left(Q''(t)+\frac{7}{40}Q(t)\right)\\ &= Q^{(4)}(t)+\frac{11{,}165}{8{,}284}Q''(t)+\frac{199{,}849}{1{,}391{,}712}Q(t)>0\end{aligned}$$

if $a\ge a_3=11{,}165/8{,}284$. □

The monotonicity of G_{a_3} and the facts that

$$\begin{aligned}&G_{a_3}\left(\frac{3}{2}\right)=\frac{112{,}672{,}809}{579{,}880}-\frac{11{,}465{,}761}{57{,}988}\ln\frac{3}{2}-\frac{11{,}465{,}761}{57{,}988}\gamma=0.001690\ldots,\\ &G_{a_3}(\infty)=0\end{aligned}$$

lead to the following.

Corollary 4 *Let R_n be defined by* (1.3). *Then the double inequality*

$$\begin{aligned}&\frac{1}{24}\frac{(n+\frac{1}{2})^2+\frac{97{,}153}{82{,}840}}{(n+\frac{1}{2})^4+\frac{11{,}165}{8{,}284}(n+\frac{1}{2})^2+\frac{199{,}849}{1{,}391{,}712}}\\ &\quad<R_n-\gamma<\frac{1}{24}\frac{(n+\frac{1}{2})^2+\frac{97{,}153}{82{,}840}+\lambda_3}{(n+\frac{1}{2})^4+\frac{11{,}165}{8{,}284}(n+\frac{1}{2})^2+\frac{199{,}849}{1{,}391{,}712}}\end{aligned}\tag{3.4}$$

holds for all $n\in\mathbb{N}$ with the best possible constant $\lambda_3=G_{a_3}(3/2)=0.001690\ldots$.

4 Remarks

Remark 3 The function G_a defined by (3.3) can be rewritten as

$$G_a(x) = a \times f_{7/40}(x) - F_{-31/336}(x) = f_{7/40}(x) \times \left(a - \frac{F_{-31/336}(x)}{f_{7/40}(x)}\right). \tag{4.1}$$

Theorem 4 leads to the conclusion that

$$\frac{F_{-31/336}(x)}{f_{7/40}(x)} \leq \lim_{x\to\infty} \frac{F_{-31/336}(x)}{f_{7/40}(x)} = \lim_{x\to\infty} \frac{-24(x^4 - \frac{31}{336})R(x) + x^2 - \frac{7}{40}}{24(x^2 + \frac{7}{40})R(x) - 1} \leq \frac{11{,}165}{8{,}284}. \tag{4.2}$$

Moreover, we can prove that

$$\frac{F_{-31/336}(x)}{f_{7/40}(x)} \geq \lim_{x\to 0^+} \frac{F_{-31/336}(x)}{f_{7/40}(x)} = \lim_{x\to 0^+} \frac{-24(x^4 - \frac{31}{336})R(x) + x^2 - \frac{7}{40}}{24(x^2 + \frac{7}{40})R(x) - 1} \geq \frac{155}{294}. \tag{4.3}$$

It suffices to prove the function

$$x \mapsto V(x) = \psi(x + 1/2) - \ln x - \frac{1}{24} \frac{x^2 + \frac{2{,}071}{5{,}880}}{x^2(x^2 + \frac{155}{294})}$$

is increasing on $(0, \infty)$. Differentiation gives

$$V'(x) = \psi'(x + 1/2) - \frac{1}{x} + \frac{1{,}728{,}720x^4 + 1{,}217{,}748x^2 + 321{,}005}{10x^3(294x^2 + 155)^2}.$$

From $\psi'(x + 1) - \psi'(x) = -1/x^2$ we get

$$\begin{aligned} &V'(x + 1) - V'(x) \\ &\quad = -\frac{1}{(x + 1/2)^2} - \frac{1}{x + 1} + \frac{1{,}728{,}720(x + 1)^4 + 1{,}217{,}748(x + 1)^2 + 321{,}005}{10(x + 1)^3(294(x + 1)^2 + 155)^2} \\ &\qquad + \frac{1}{x} - \frac{1{,}728{,}720x^4 + 1{,}217{,}748x^2 + 321{,}005}{10x^3(294x^2 + 155)^2} \\ &\quad = -\frac{V_1(x)}{10x^3(2x + 1)^2(294x^2 + 155)^2(x + 1)^3(294x^2 + 588x + 449)^2}, \end{aligned}$$

where

$$\begin{aligned} V_1(x) = {}& 1{,}718{,}371{,}882{,}080x^{12} + 10{,}310{,}231{,}292{,}480x^{11} \\ &+ 29{,}399{,}355{,}669{,}600x^{10} + 52{,}486{,}324{,}833{,}600x^9 \\ &+ 66{,}690{,}983{,}696{,}400x^8 + 65{,}258{,}530{,}001{,}280x^7 \\ &+ 51{,}909{,}045{,}513{,}612x^6 + 34{,}352{,}301{,}620{,}196x^5 \\ &+ 18{,}881{,}999{,}450{,}054x^4 + 8{,}378{,}736{,}976{,}048x^3 \\ &+ 2{,}808{,}871{,}359{,}013x^2 + 622{,}502{,}847{,}155x \\ &+ 64{,}714{,}929{,}005 > 0 \end{aligned}$$

for $x > 0$. Therefore,

$$V'(x) > V'(x+1) > V'(x+2) > \cdots > \lim_{n\to\infty} V'(x+n) = 0$$

for all $x > 0$.

In addition, (4.1) implies that the necessary condition such that the function $-G_a$ is completely monotone on $(0,\infty)$ is

$$a \le \lim_{x\to 0+} \frac{F_{-31/336}(x)}{f_{7/40}(x)} = \lim_{x\to 0+} \frac{-24(x^4 - \frac{31}{336})R(x) + x^2 - \frac{7}{40}}{24(x^2 + \frac{7}{40})R(x) - 1} = \frac{155}{294}.$$

Motivated by inequalities (4.2) and (4.3) we propose two conjectures.

Conjecture 1 *Let $R(x)$ be defined on $(0,\infty)$ by (1.10). Then we conjecture that*

(i) *the function*

$$x \mapsto \frac{-24(x^4 - \frac{31}{336})R(x) + x^2 - \frac{7}{40}}{24(x^2 + \frac{7}{40})R(x) - 1}$$

is increasing on $(0,\infty)$;

(ii) *the function $-G_a$ is completely monotone on $(0,\infty)$ if and only if $a \le 155/294$.*

Remark 4 The monotonicity of the function V proved in Remark 3 and the facts that

$$V\left(\frac{3}{2}\right) = \frac{866{,}519}{881{,}820} - \ln\frac{3}{2} - \gamma = -0.00003238\ldots, \qquad V(\infty) = 0$$

lead to the conclusion that the double inequality

$$\frac{1}{24}\frac{(n+\frac{1}{2})^2 + \frac{2{,}071}{5{,}880}}{(n+\frac{1}{2})^2((n+\frac{1}{2})^2 + \frac{155}{294})} + \lambda_4 < R_n - \gamma < \frac{1}{24}\frac{(n+\frac{1}{2})^2 + \frac{2{,}071}{5{,}880}}{(n+\frac{1}{2})^2((n+\frac{1}{2})^2 + \frac{155}{294})} \tag{4.4}$$

holds with the best possible constant $\lambda_4 = -0.00003238\ldots$.

The upper bound for $R_n - \gamma$ in (4.4) is better than that in (3.2) because of

$$\begin{aligned}&\frac{1}{24}\frac{(n+\frac{1}{2})^2 + \frac{2{,}071}{5{,}880}}{(n+\frac{1}{2})^2((n+\frac{1}{2})^2 + \frac{155}{294})} - \frac{1}{24}\frac{(n+\frac{1}{2})^2 - \frac{7}{40}}{(n+\frac{1}{2})^4 - \frac{31}{336}}\\ &\quad= -\frac{64{,}201}{120(2n+1)^2(588n^2+588n+457)(168n^4+336n^3+252n^2+84n-5)} < 0.\end{aligned}$$

Remark 5 Let

$$\begin{aligned} w_n &= \sum_{k=1}^{n}\frac{1}{k} - \ln(n+1/2) - \frac{1}{24}\frac{(n+\frac{1}{2})^2 - \frac{7}{40}}{(n+\frac{1}{2})^4 - \frac{31}{336}},\\ y_n &= \sum_{k=1}^{n}\frac{1}{k} - \ln(n+1/2) - \frac{1}{24}\frac{(n+\frac{1}{2})^2 + \frac{97{,}153}{82{,}840}}{(n+\frac{1}{2})^4 + \frac{11{,}165}{8{,}284}(n+\frac{1}{2})^2 + \frac{199{,}849}{1{,}391{,}712}},\\ z_n &= \sum_{k=1}^{n}\frac{1}{k} - \ln(n+1/2) - \frac{1}{24}\frac{(n+\frac{1}{2})^2 + \frac{2{,}071}{5{,}880}}{(n+\frac{1}{2})^2((n+\frac{1}{2})^2 + \frac{155}{294})}.\end{aligned}$$

Then Theorems 3 and 4 together with Remark 4 lead to

$$w_n < z_n < \gamma < y_n,$$

and simple computations show that

$$\lim_{n\to\infty} n^8(w_n - \gamma) = -\frac{319}{92{,}160},$$

$$\lim_{n\to\infty} n^{10}(y_n - \gamma) = \frac{627{,}404{,}761}{246{,}900{,}842{,}496},$$

$$\lim_{n\to\infty} n^8(z_n - \gamma) = -\frac{199{,}849}{94{,}832{,}640}.$$

Lastly, inspired by Theorems 2-4, we propose an open problem as follows.

Problem 1 We wonder what the sequences $\{a_k\}$ and $\{b_k\}$ are such that the function

$$x \mapsto R(x)\sum_{k=0}^{n+1} a_k x^{2k} - \sum_{k=0}^{n} b_k x^{2k}$$

is completely monotone on $(0,\infty)$ and

$$\lim_{x\to\infty} \frac{R(x)\sum_{k=0}^{n+1} a_k x^{2k} - \sum_{k=0}^{n} b_k x^{2k}}{x^{-2n-4}} = c \neq 0, \pm\infty.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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