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On the meromorphic solutions of certain class of nonlinear differential equations

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Dedicated to Professor George Csordas on the occasion of his retirement.

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Abstract

Let α be an entire function, $a_{n-1}, \ldots, a_1, a_0$, R be small functions of f, and let $n \geq 2$ be an integer. Then, for any positive integer k, the differential equation $f^n f^{(k)} + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = Re^{\alpha}$ has transcendental meromorphic solutions under appropriate conditions on the coefficients. In addition, for $n = 1$ and $k = 1$, we have extended some well-known and relevant results obtained by others, by using different arguments.

MSC: 34M10; 30D35

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1 Introduction and main results

In this paper, a meromorphic function means meromorphic in the whole complex plane. We shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions (see, *e.g.*, [1, 2]).

Given a meromorphic function f, recall that $\alpha \not\equiv 0, \infty$ is a small function with respect to f, if $T(r,\alpha) = S(r,f)$, where $S(r,f)$ denotes any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$, possibly outside a set of r of finite linear measure.

Theorem A *Let f be a transcendental meromorphic function, n (≥ 3) be an integer. Then $F = f^n f'$ assumes all finite values, except possibly zero, infinitely many times.*

The above theorem was derived by Hayman [3] in 1959. Later, he conjectured [4] that Theorem A remains valid even if $n = 1$ or $n = 2$. Mues [5] proved the result for $n = 2$ and the case $n = 1$ was proved by Bergweiler and Eremenko [6] and independently by Chen and Fang [7]. For entire functions and difference polynomials, similar results have been obtained by others earlier (see, *e.g.*, [8–11]).

Theorem B ([12]) *If f is a transcendental meromorphic function of finite order and a $(\not\equiv 0)$ is a polynomial, then $ff' - a$ has infinitely many zeros.*

Wang [13] obtained the following result.

Theorem C *Let f be a transcendental entire function and n, k be positive integers, and let $c(z)$ ($\not\equiv 0$) be a small function with respect to f. If $T(r,f) \neq \tau N_{1)}(r,1/f) + S(r,f)$, then $f^n(z)f^{(k)}(z) - c(z)$ has infinitely many zeros, where $\tau = 0$ if $n \geq 2$ or $k = 1$; $\tau = 1$ otherwise.*

In this paper, by using methods different from that were used by others (see, *e.g.*, [10, 14] and [15]), we shall extend and generalize the above results with $f^nf^{(k)}$ being replaced by a differential polynomial $P_{n+1}(f)$. Specifically, our main results can be stated as follows.

Theorem 1.1 *Let α be an entire function, R and a_i ($i = 0, 1, \ldots, n-1$) be small functions of f with $a_0 \not\equiv 0$. If, for $n \geq 2$, a transcendental meromorphic function f satisfies the differential equation*

$$f^nf^{(k)} + a_{n-1}f^{n-1} + \cdots + a_1f + a_0 = R\mathrm{e}^{\alpha}, \tag{1.1}$$

then, for any positive integer k, we have $f = g\exp(\alpha/(n+1)) - (n+1)\frac{a_0}{a_1}$ with $g^{n+1} = [\frac{(n+1)a_0}{a_1}]^{n+1}\frac{R}{a_0}$, and $(\frac{a_0}{a_1})^{(k)} + \frac{n}{n+1}(\frac{1}{n+1}\frac{a_1}{a_0})^n a_0 \equiv 0$.

Remark 1.1 Let a_0 and a_1 be non-zero constants in Theorem 1.1. Then (1.1) has no transcendental meromorphic solutions.

A meromorphic solution f of (1.1) is called admissible, if $T(r,\alpha_j) = S(r,f)$ holds for all coefficients α_j ($j = 0, \ldots, n-1$) and $T(r,R) = S(r,f)$.

Remark 1.2 If $a_0 \equiv 0$ and $n \geq 2$, $k \geq 1$, then the other coefficients $a_1, \ldots, a_{n-1}$ must be identically zero. In this case, (1.1) becomes $f^nf^{(k)} = R\mathrm{e}^{\alpha}$ and f has the form $f = u\exp(\alpha/(n+1))$ as the only possible admissible solution of (1.1), where u is a small function of f.

We have the following corollary by Theorem 1.1.

Corollary 1.1 *Let f be a transcendental meromorphic function with $N(r,f) = S(r,f)$, $n \geq 2$ be an integer. If $(\frac{a_0}{a_1})^{(k)} + \frac{n}{n+1}(\frac{1}{n+1}\frac{a_1}{a_0})^n a_0 \not\equiv 0$, then $F = f^nf^{(k)} + a_{n-1}f^{n-1} + \cdots + a_1f + a_0$ has infinitely many zeros, where a_i ($i = 0, 1, \ldots, n-1$) are small functions of f such that $a_0 \not\equiv 0$.*

Note that in Theorem 1.1, it is assumed that $n \geq 2$ and $k \geq 1$. However, for $n = 1$ and $k = 1$, we can derive the following result.

Theorem 1.2 *Let p, q, and R be non-zero polynomials, α be an entire function. Then the differential equation $pff' - q = R\mathrm{e}^{\alpha}$ has no transcendental meromorphic solutions, where p, q, and R are small functions of f with $pq \not\equiv 0$.*

Remark 1.3 From the proof of Theorem 1.2, we see that the restriction in Theorem 1.2 to p, q, and R may extend to small functions. In fact, it is easy to find that the conclusion is valid provided that p, q, and R are non-vanishing small functions of f. The following corollary arises directly from an immediate consequence of Theorem 1.2.

Corollary 1.2 *Let f be a transcendental meromorphic function with $N(r,f) = S(r,f)$, p and q be non-vanishing small functions of f. Then $F = pff' - q$ has infinitely many zeros.*

2 Some lemmas and proofs of theorems

In order to prove our conclusions, we need some lemmas. The following lemma is fundamental to Clunie's theorem [16].

Lemma 2.1 ([17, 18]) *Let f be a transcendental meromorphic solution of*

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients $\{a_\lambda | \lambda \in I\}$ such that $m(r,a_\lambda) = S(r,f)$ for all $r \in I$. If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is less than or equal to n, then $m(r,P(r,f)) = S(r,f)$.

The following lemma is crucial to the proof of our theorems.

Lemma 2.2 ([18, 19]) *Let f be a meromorphic solution of an algebraic equation*

$$P\bigl(z,f,f',\ldots,f^{(n)}\bigr) = 0, \tag{2.1}$$

where P is a polynomial in $f,f',\ldots,f^{(n)}$ with meromorphic coefficients small with respect to f. If a complex constant c does not satisfy (2.1), *then*

$$m\left(r,\frac{1}{f-c}\right) = S(r,f).$$

Proof of Theorem 1.1 Let f be a transcendental meromorphic function that satisfies (1.1). Then two cases are to be treated, namely case 1: $N(r,f) \neq S(r,f)$, and case 2: $N(r,f) = S(r,f)$. For case 1, it is impossible as α is an entire function and R, $a_1,\ldots,a_n$ are small functions of f.

To prove Theorem 1.1, we now suppose that $N(r,f) = S(r,f)$.

Denoting $\phi := f^n f^{(k)} + a_{n-1}f^{n-1} + \cdots + a_1 f$, and assuming that $T(r,\phi) = S(r,f)$, then by Lemma 2.1, we get $m(r,f^{(k)}) = S(r,f)$ and then $T(r,f^{(k)}) = S(r,f)$, since $N(r,f) = S(r,f)$ by the assumption. The contradiction $T(r,f) = S(r,f)$ now follows by the theorem in [20] and combining it with the proof of Proposition E in [21]. Thus, for any transcendental meromorphic function f under the condition: $N(r,f) = S(r,f)$,

$$T\bigl(r,f^n f^{(k)} + a_{n-1}f^{n-1} + \cdots + a_1 f\bigr) \neq S(r,f). \tag{2.2}$$

From (1.1) and the result of Milloux (see, *e.g.*, [1], Theorem 3.1), one can easily show that

$$T\bigl(r,\mathrm{e}^{\alpha}\bigr) \leq (n+1)T(r,f) + S(r,f),$$

which leads to $T(r,\alpha) + T(r,\alpha') = S(r,f)$.

By taking the logarithmic derivative on both sides of (1.1), we have

$$\frac{nf^{n-1}f'f^{(k)} + f^n f^{(k+1)} + a'_{n-1}f^{n-1} + \cdots + a'_1 f + a_1 f' + a'_0}{f^n f^{(k)} + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0} = \frac{R'}{R} + \alpha'. \tag{2.3}$$

It follows by (2.3) that

$$-\left(\frac{R'}{R}+\alpha'\right)f^nf^{(k)}+nf^{n-1}f'f^{(k)}+f^nf^{(k+1)}+\left\{a'_{n-1}-\left(\frac{R'}{R}+\alpha'\right)a_{n-1}\right\}f^{n-1}$$
$$+(n-1)a_{n-1}f^{n-2}f'+\cdots+\left\{a'_1-\left(\frac{R'}{R}+\alpha'\right)a_1\right\}f+a_1f'=\left(\frac{R'}{R}+\alpha'\right)a_0-a'_0. \tag{2.4}$$

If $(\frac{R'}{R}+\alpha')a_0-a'_0\equiv 0$, then $Aa_0=Re^{\alpha}$, where A is a non-zero constant. From (1.1), we get

$$f^nf^{(k)}+a_{n-1}f^{n-1}+\cdots+a_1f=(A-1)a_0. \tag{2.5}$$

If $A=1$, then from (2.5), we obtain

$$f^nf^{(k)}+a_{n-1}f^{n-1}+\cdots+a_1f\equiv 0,$$

which contradicts (2.2). However, if $A\neq 1$, then again from (2.5), we would derive

$$T\left(r,f^nf^{(k)}+a_{n-1}f^{n-1}+\cdots+a_1f\right)=S(r,f),$$

a contradiction.

Thus

$$\left(\frac{R'}{R}+\alpha'\right)a_0-a'_0:=\varphi\not\equiv 0.$$

In this case, from (2.4), we have

$$N_{(2}\left(r,\frac{1}{f}\right)\leq N\left(r,\frac{1}{\varphi}\right)+S(r,f)\leq T(r,\varphi)+S(r,f)=S(r,f),$$

where $N_{(2}(r,\frac{1}{f})$, as usually, denotes the counting function of zeros of f whose multiplicities are not less than 2, which implies that the zeros of f are mainly simple zeros. Again, from (2.4), the fact that α' is a small function of f and Lemma 2.2 (where $c=0$ is used), we conclude $m(r,\frac{1}{f})=S(r,f)$. This together with Nevanlinna's first theorem will result in

$$T(r,f)=N\left(r,\frac{1}{f}\right)+S(r,f)=N_{1)}\left(r,\frac{1}{f}\right)+S(r,f), \tag{2.6}$$

where in $N_{1)}(r,1/f)$ only the simple zeros of f are to be considered.

Assume that $a_1\equiv 0$. It follows by (2.4) and $n\geq 2$ that $N_{1)}(r,1/f)=S(r,f)$, which contradicts (2.6). Thus $a_1\not\equiv 0$. Let z_0 be a simple zero of f, and z_0 be not a pole of one of the coefficients a_i, $(\frac{R'}{R}+\alpha')a_i-a'_i$ $(i=1,2,\ldots,n-1)$. From (2.4), we see that z_0 is a zero of $a_1f'+a'_0-(\frac{R'}{R}+\alpha')a_0$. Set

$$h=\frac{a_1f'+a'_0-(\frac{R'}{R}+\alpha')a_0}{f}. \tag{2.7}$$

Then (2.7) gives $T(r,h)=S(r,f)$. We have

$$f'=\frac{1}{a_1}\left\{hf-a'_0+\left(\frac{R'}{R}+\alpha'\right)a_0\right\}:=\mu_1f+\nu_1. \tag{2.8}$$

Clearly, it follows from (2.6) and $T(r,\mu_1) + T(r,\nu_1) = S(r,f)$ that $\mu_1\nu_1 \not\equiv 0$. By (2.3), we obtain

$$f^{n-1}\psi = P_{n-1}(f), \tag{2.9}$$

where $\psi = -(\frac{R'}{R} + \alpha')ff^{(k)} + nf'f^{(k)} + ff^{(k+1)}$, $P_{n-1}(f) = (\frac{R'}{R} + \alpha')(a_{n-1}f^{n-1} + \cdots + a_1f + a_0) - (a_{n-1}f^{n-1} + \cdots + a_1f + a_0)'$. It follows by (2.2) that $P_{n-1}(f) \not\equiv 0$. Thus $\psi \not\equiv 0$. Moreover, by applying Lemma 2.1 to (2.9), we get $m(r,\psi) = S(r,f)$. It is easy to see by $N(r,f) = S(r,f)$ that $T(r,\psi) = S(r,f)$.

From (2.8) and induction, we have $f'' = (\mu_1' + \mu_1^2)f + \mu_1\nu_1 + \nu_1' := \mu_2 f + \nu_2$, and

$$f^{(k)} = \mu_k f + \nu_k, \tag{2.10}$$

where μ_k, ν_k are small functions of f. By the expression of ψ and (2.6), we get $\nu_k \not\equiv 0$. If $\mu_k \equiv 0$, then (2.10) gives $T(r,f^{(k)}) = S(r,f)$, which is impossible. Therefore, $\mu_k \not\equiv 0$.

By (2.10), (1.1) becomes

$$\mu_k f^{n+1} + \nu_k f^n + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0 = Re^{\alpha}. \tag{2.11}$$

By applying the Tumura-Clunie lemma (see, *e.g.*, [1], Theorem 3.9) to the left-hand side of (2.11), we have $\mu_k[f + \frac{\nu_k}{(n+1)\mu_k}]^{n+1} = Re^{\alpha}$ and $f = ge^{\alpha/(n+1)} - \frac{\nu_k}{(n+1)\mu_k}$ with $g^{n+1} = \frac{R}{\mu_k}$.

In view of (2.11), we have

$$\mu_k f^{n+1} + \nu_k f^n + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0 = \mu_k\left[f + \frac{\nu_k}{(n+1)\mu_k}\right]^{n+1}.$$

Thus, we have

$$\frac{1}{n+1}\frac{\nu_k}{\mu_k} = (n+1)\frac{a_0}{a_1} \quad \text{and} \quad \mu_k = \left(\frac{1}{n+1}\frac{a_1}{a_0}\right)^{n+1} a_0. \tag{2.12}$$

By (2.12), we obtain $\nu_k = (n+1)(\frac{1}{n+1}\frac{a_1}{a_0})^n a_0$ and $g^{n+1} = [\frac{(n+1)a_0}{a_1}]^{n+1}\frac{R}{a_0}$.

Set $(n+1)\gamma = \alpha$. It follows by (2.10) and $f = ge^{\gamma} - (n+1)\frac{a_0}{a_1}$ that

$$f^{(k)} = \left(\frac{1}{n+1}\frac{a_1}{a_0}\right)^{n+1} a_0\left[ge^{\gamma} - (n+1)\frac{a_0}{a_1}\right] + (n+1)\left(\frac{1}{n+1}\frac{a_1}{a_0}\right)^n a_0. \tag{2.13}$$

In addition, by $f = ge^{\gamma} - (n+1)\frac{a_0}{a_1}$ we get

$$f^{(k)} = Q\big(g,g',\ldots,g^{(k)}\big)e^{\gamma} - (n+1)\left(\frac{a_0}{a_1}\right)^{(k)}, \tag{2.14}$$

where $Q(g,g',\ldots,g^{(k)})$ is a differential polynomial of g.

Thus, (2.13) and (2.14) imply

$$Q\big(g,g',\ldots,g^{(k)}\big) = \left(\frac{1}{n+1}\frac{a_1}{a_0}\right)^{n+1} a_0 g$$

and

$$(n+1)\left(\frac{a_0}{a_1}\right)^{(k)} = \left(\frac{1}{n+1}\frac{a_1}{a_0}\right)^{n+1} a_0(n+1)\frac{a_0}{a_1} - (n+1)\left(\frac{1}{n+1}\frac{a_1}{a_0}\right)^{n} a_0. \tag{2.15}$$

It follows by (2.15) that

$$\left(\frac{a_0}{a_1}\right)^{(k)} + \frac{n}{n+1}\left(\frac{1}{n+1}\frac{a_1}{a_0}\right)^{n} a_0 = 0.$$

This completes the proof of Theorem 1.1. □

Proof of Remark 1.2 Let f be a transcendental meromorphic solution of (1.1). Since $a_0 \equiv 0$, we have $N(r,1/f) \le N(r,1/R) + S(r,f) = S(r,f)$. Obviously, $N(r,f) = S(r,f)$. In this case, there exist a meromorphic function u and an entire function v such that $f = ue^{v}$, and $N(r,1/u) + N(r,u) = S(r,f)$. Clearly, from the expressions of f and the Borel lemma (see, *e.g.*, [2], Theorem 1.52), all the a_j ($j = 1,2,\ldots,n-1$) must be identically zero. Thus, Remark 1.2 follows. □

Proof of Theorem 1.2 Now we proceed to prove the theorem by contradiction. Let f be a transcendental meromorphic function that satisfies $pff' - q = Re^{\alpha}$. Then two cases are to be retreated, namely $N(r,f) \neq S(r,f)$ and $N(r,f) = S(r,f)$. For $N(r,f) \neq S(r,f)$, this is impossible as α is an entire function and R, p, q are non-zero polynomials.

To prove Theorem 1.2, we now suppose that $N(r,f) = S(r,f)$. We differentiate $pff' - q = Re^{\alpha}$ and eliminate e^{α},

$$t_1 ff' + p\left(f'\right)^2 + pff'' = t_2, \tag{2.16}$$

where $t_1 = p' - (\frac{R'}{R} + \alpha')p$, $t_2 = q' - (\frac{R'}{R} + \alpha')q$.

If $t_2 \equiv 0$, then, by integrating the definition of t_2, α must be a constant, hence ff' is rational, and then, by Lemma 2.1, $m(r,f') = S(r,f)$. Hence $T(r,f') = S(r,f)$. This is a contradiction by Proposition E in [21]. Thus, $t_2 \not\equiv 0$, and then by (2.16), we get (2.6). By differentiating both sides of (2.16), we have

$$t_1' ff' + (t_1 + p')\left(f'\right)^2 + (t_1 + p')ff'' + 3pf'f'' + pff''' = t_2'. \tag{2.17}$$

Letting z_0 be a simple zero of f, (2.16) and (2.17) imply

$$\left(p\left(f'\right)^2 - t_2\right)(z_0) = 0 \tag{2.18}$$

and

$$\left\{(t_1 + p')\left(f'\right)^2 + 3pf'f'' - t_2'\right\}(z_0) = 0. \tag{2.19}$$

Let

$$g = \frac{3pt_2 f'' + [t_2(t_1 + p') - t_2' p]f'}{f}. \tag{2.20}$$

From (2.6), (2.18), and (2.19), we get

$$T(r,g) = S(r,f).$$

By (2.20), we obtain

$$f'' = \alpha_1 f + \beta_1 f', \tag{2.21}$$

where

$$\alpha_1 = \frac{g}{3pt_2}, \qquad \beta_1 = \frac{t_2' p - t_2(t_1 + p')}{3pt_2}$$

and

$$T(r,\alpha_1) = S(r,f), \qquad T(r,\beta_1) = S(r,f).$$

Substituting (2.21) into (2.16) yields

$$(t_1 + p\beta_1)ff' + p\bigl(f'\bigr)^2 + \alpha_1 pf^2 = t_2. \tag{2.22}$$

On the other hand, from (2.21), we have

$$f''' = \alpha_2 f + \beta_2 f', \tag{2.23}$$

where $\alpha_2 = \alpha_1' + \alpha_1\beta_1$, $\beta_2 = \alpha_1 + \beta_1' + \beta_1^2$, and

$$T(r,\alpha_2) = S(r,f), \qquad T(r,\beta_2) = S(r,f).$$

Substituting (2.23) into (2.17), we have

$$\bigl[t_1' + \beta_1\bigl(t_1 + p'\bigr) + 3p\alpha_1 + p\beta_2\bigr]ff' + \bigl(t_1 + p' + 3p\beta_1\bigr)\bigl(f'\bigr)^2 + \bigl[\alpha_1\bigl(t_1 + p'\bigr) + \alpha_2 p\bigr]f^2 = t_2'. \tag{2.24}$$

It follows by (2.22) and (2.24) that

$$\begin{aligned}&\bigl\{p\bigl[t_1' + \beta_1\bigl(t_1 + p'\bigr) + 3p\alpha_1 + p\beta_2\bigr] - \bigl(t_1 + p' + 3p\beta_1\bigr)(t_1 + p\beta_1)\bigr\}ff' \\ &\quad + \bigl\{p\bigl[\alpha_1\bigl(t_1 + p'\bigr) + \alpha_2 p\bigr] - \alpha_1 p\bigl(t_1 + p' + 3p\beta_1\bigr)\bigr\}f^2 = t_2' p - t_2\bigl(t_1 + p' + 3p\beta_1\bigr).\end{aligned} \tag{2.25}$$

From the definition of β_1, we now claim $t_2' p - t_2(t_1 + p' + 3p\beta_1) \equiv 0$. To show this, we assume the contrary, that is, $t_2' p - t_2(t_1 + p' + 3p\beta_1) \not\equiv 0$. Then from the fact that $t_2' p - t_2(t_1 + p' + 3p\beta_1)$ is a small function of f and (2.25), we get

$$\begin{aligned}N_{1)}\left(r, \frac{1}{f}\right) &\le N\left(r, \frac{1}{t_2' p - t_2(t_1 + p' + 3p\beta_1)}\right) \\ &\le T\bigl(r, t_2' p - t_2\bigl(t_1 + p' + 3p\beta_1\bigr)\bigr) + S(r,f) = S(r,f),\end{aligned}$$

and from this and (2.6) we deduce $T(r,f) = S(r,f)$, a contradiction. Thus, we have

$$t_2'p - t_2(t_1 + p' + 3p\beta_1) \equiv 0. \tag{2.26}$$

Now, (2.25) and (2.26) lead to

$$p[\alpha_1(t_1 + p') + \alpha_2 p] - \alpha_1 p(t_1 + p' + 3p\beta_1) \equiv 0. \tag{2.27}$$

From the definition of α_2 and (2.27), we deduce

$$\alpha_1' \equiv 2\alpha_1\beta_1. \tag{2.28}$$

It follows from (2.28) and the definitions of t_1, β_1 that

$$\alpha_1^3 p^4 \equiv t_2^2 e^{2\alpha}.$$

In the beginning of the proof it was already shown that $t_2 \not\equiv 0$. Hence, the contradiction here is immediate.

This also completes the proof of Theorem 1.2. □

3 Remarks and a conjecture

Remark 3.1 Corollary 1.1 or Corollary 1.2 can be strengthened to

$$N\left(r, \frac{1}{F}\right) \neq S(r,f).$$

Remark 3.2 What can be said if '$pff' - q$' is replaced by '$pff^{(k)} - q$', for any integer $k \geq 2$, in Theorem 1.2?

Remark 3.3 Taking $f(z) = e^z$, we have

$$N\left(r, \frac{1}{ff^{(k)} - a}\right) \sim 2T(r,f) + S(r,f),$$

where k is a positive integer, and a is a non-zero constant.

Finally, we present the following more general and quantitative conjecture.

Conjecture 3.1 *Let f be a transcendental entire function. Then for any integer* $k \geq 1$, *and any small function* a $(\not\equiv 0)$,

$$N\left(r, \frac{1}{ff^{(k)} - a}\right) \sim 2T(r,f) + S(r,f).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, and they read and approved the final manuscript.

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References

1. Hayman, WK: Meromorphic Functions. Clarendon, Oxford (1964)
2. Yang, CC, Yi, HX: Uniqueness Theory of Meromorphic Functions. Science Press, Beijing (2003)
3. Hayman, WK: Picard values of meromorphic functions and their derivatives. Ann. Math. (2) **70**, 9-42 (1959)
4. Hayman, WK: Research Problems in Function Theory. Athlone Press, London (1967)
5. Mues, E: Über ein Problem von Hayman. Math. Z. **164**, 239-259 (1979)
6. Bergweiler, W, Eremenko, A: On the singularities of the inverse to a meromorphic function of finite order. Rev. Mat. Iberoam. **11**, 355-373 (1995)
7. Chen, HH, Fang, ML: The value distribution of $f^n f'$. Sci. China Ser. A **38**, 789-798 (1995)
8. Liu, NN, Lü, WR, Shen, TT, Yang, CC: Entire solutions of certain type of difference equations. J. Inequal. Appl. **2014**, 63 (2014)
9. Liu, NN, Lü, WR, Yang, CC: On Picard exceptional values of difference polynomials. J. Differ. Equ. Appl. **20**, 1437-1443 (2014)
10. Yang, CC, Hu, PC: On the value distribution of $ff^{(k)}$. Kodai Math. J. **19**, 157-167 (1996)
11. Yang, CC, Yang, L, Wang, YF: On the zeros of $(f^{(k)})^n f - a$. Chin. Sci. Bull. **38**, 2125-2128 (1993)
12. Bergweiler, W: On the product of a meromorphic function and its derivative. Bull. Hong Kong Math. Soc. **1**, 97-101 (1997)
13. Wang, JP: On the zeros of $f^n(z)f^{(k)}(z) - c(z)$. Complex Var. Theory Appl. **48**, 695-703 (2003)
14. Hennekemper, W: Über die Wertverteilung von $(f^{k+1})^{(k)}$. Math. Z. **177**, 375-380 (1981)
15. Zhang, DQ: On the value distribution of $\varphi(z)f(z)f'(z)$. Acta Math. Sin. **37**, 91-98 (1994)
16. Clunie, J: On integral and meromorphic functions. J. Lond. Math. Soc. **37**, 17-27 (1962)
17. Bank, S, Laine, I: On the growth of meromorphic solutions of linear and algebraic differential equations. Math. Scand. **40**, 116-126 (1977)
18. Laine, I: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)
19. Mohon'ko, AZ, Mohon'ko, VD: Estimates of the Nevanlinna characteristics of certain classes of meromorphic functions, and their applications to differential equations. Sib. Mat. Zh. **15**, 1305-1322 (1974) (in Russian)
20. Hayman, WK, Miles, J: On the growth of a meromorphic function and its derivatives. Complex Var. Theory Appl. **12**, 245-260 (1989)
21. He, YZ, Laine, I: The Hayman-Miles theorem and the differential equation $(y')^n = R(z, y)$. Analysis **10**, 387-396 (1990)