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Well-posedness for lexicographic vector quasiequilibrium problems with lexicographic equilibrium constraints

Rabian Wangkeeree[1,2*], Thanatporn Bantaojai[1] and Panu Yimmuang[1]

*Correspondence: rabianw@nu.ac.th
[1]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok, 65000, Thailand
[2]Research Center for Academic Excellence in Mathematics, Naresuan University, Phitsanulok, 65000, Thailand

Abstract
We consider the well-posedness for lexicographic vector equilibrium problems and optimization problems with lexicographic equilibrium constraints in metric spaces. Sufficient conditions for a family of such problems to be (uniquely) well-posed at the reference point are established. Numerous examples are provided to explain that all the assumptions we impose are very relaxed and cannot be dropped.

Keywords: lexicographic vector equilibrium problems; optimization problems; lexicographic equilibrium constraints; well-posedness

1 Introduction

Equilibrium problems first considered by Blum and Oettli [1] have been playing an important role in optimization theory with many striking applications, particularly in transportation, mechanics, economics, *etc.* Equilibrium models incorporate many other important problems, such as optimization problems, variational inequalities, complementarity problems, saddle point/minimax problems, and fixed points. Equilibrium problems with scalar and vector objective functions have been widely studied. The crucial issue of solvability (the existence of solutions) has attracted the most considerable attention of researchers; see, *e.g.*, [2, 3]. A relatively new but rapidly growing topic is the stability of solutions, including semicontinuity properties in the sense of Berge and Hausdorff; see, *e.g.*, [4, 5] and the Hölder/Lipschitz continuity of solution mappings; see, *e.g.*, [6–10].

On the other hand, well-posedness of optimization-related problems can be defined in two ways. The first and oldest is Hadamard well-posedness [11], which means existence, uniqueness, and continuous dependence of the optimal solution and optimal value from perturbed data. The second is Tikhonov well-posedness [12], which means the existence and uniqueness of the solution and convergence of each minimizing sequence to the solution. Well-posedness properties have been intensively studied and the two classical well-posedness notions have been extended and blended. Recently, the Tikhonov notion has been more interested. The major reason is its vital role in numerical methods. Any algorithm can generate only an approximating sequence of solutions. Hence, this sequence is applicable only if the problem under consideration is well-posed. For parametric problems, well-posedness is closely related to stability. Up to now, there have been many works dealing with well-posedness of optimization-related problems as mathematical programming [13, 14], constrained minimization [15, 16] variational inequalities [17–19], Nash equilibria [20], and equilibrium problems [21].

On the other hand, many papers appeared dealing with bilevel problems such as mathematical programming with equilibrium constraints [22], optimization problems with variational inequality constraints [20], optimization problems with Nash equilibrium constraints [20], optimization problems with equilibrium constraints [23, 24], *etc.* The increasing importance of these bilevel problems in mathematical applications in engineering and economics is recognized. For instance, the multileader-follower game in economics is a bilevel problem, since each leader has to solve a Stackelberg game formulated as a mathematical program with equilibrium constraints. Recently, Anh *et al.* in [25] considered the bilevel equilibrium and optimization problems with equilibrium constraints. They proposed a relaxed level closedness and use it together with pseudocontinuity assumptions to establish sufficient conditions for the well-posedness and unique well-posedness.

With regard to vector equilibrium problems, most of the existing results correspond to the case when the order is induced by a closed convex cone in a vector space. Thus, they cannot be applied to lexicographic cones, which are neither closed nor open. These cones have been extensively investigated in the framework of vector optimization; see, *e.g.*, [26–30]. For instance, Chadli *et al.* in [31] obtained conditions for the existence of solutions of a sequential equilibrium problem via a viscosity argument under quite strong conditions. Bianchi *et al.* in [32] analyzed lexicographic equilibrium problems on a topological Hausdorff vector space, and their relationship with some other vector equilibrium problems. They obtained the existence results for the tangled lexicographic problem via the study of a related sequential problem. However, for equilibrium problems, the main emphasis has been on the issue of solvability/existence. To the best of our knowledge, very recently, Anh *et al.* in [26] studied the well-posedness for lexicographic vector equilibrium problems in metric spaces and gave the sufficient conditions for a family of such problems to be well-posed and uniquely well-posed at the considered point. Furthermore, they derived several results on well-posedness for a class of variational inequalities.

Motivated by the work reported above, this paper aims to consider the lexicographic vector equilibrium problems and optimization problems with lexicographic equilibrium constraints in metric spaces and establishes necessary and/or sufficient conditions for such problems to be well-posed and uniquely well-posed at the considered point assumed always that the mentioned solutions exist.

The layout of the paper is as follows. In Section 2, we propose the lexicographic vector equilibrium problems and optimization problems with lexicographic equilibrium constraints in metric spaces under our consideration and recall notions and preliminaries needed in the sequel. In Section 3, we study the well-posedness of the lexicographic vector equilibrium problems with lexicographic equilibrium constraints in metric spaces. Section 4 is devoted to the well-posedness of optimization problems with lexicographic equilibrium constraints.

2 Preliminaries

We first recall the concept of lexicographic cone in finite dimensional spaces and models of equilibrium problems with the order induced by such a cone. The lexicographic cone of $\mathbb{R}^n$, denoted C_l, is the collection of zero and all vectors in $\mathbb{R}^n$ with the first nonzero coordinate being positive, *i.e.*,

$$C_l := \{0\} \cup \{x \in \mathbb{R}^n | \exists i \in \{1, 2, \dots, n\} : x_i > 0 \text{ and } x_j = 0, \forall j < i\}.$$

This cone is convex and pointed, and it induces the total order as follows:

$$x \geq_l y \quad \Leftrightarrow \quad x - y \in C_l.$$

We also observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n | x_1 \geq 0\}$, we see that $\operatorname{int} C_1 \subsetneq C_l \subsetneq C_1$, while

$$\operatorname{int} C_l = \operatorname{int} C_1 \quad \text{and} \quad \operatorname{cl} C_l = C_1.$$

Throughout this paper, if not otherwise specified, X and Λ denote the metric spaces. Let $f := (f_1, f_2, \dots, f_n) : X \times X \times \Lambda \to \mathbb{R}^n$ and $K_i : X \times \Lambda \to 2^X$, $i = 1, 2$. The *lexicographic vector quasiequilibrium problem* consists of, for each $\lambda \in \Lambda$,

(LQEP$_\lambda$) finding $\bar{x} \in K_1(\bar{x}, \lambda)$ such that

$$f(\bar{x}, y, \lambda) \geq_l 0, \quad \forall y \in K_2(\bar{x}, \lambda).$$

Remark 2.1

(i) When $f := f_1 : X \times X \times \Lambda \to \mathbb{R}$, the (LQEP$_\lambda$) collapses to the parametric quasiequilibrium problem (QEP) considered by Anh *et al.* [25].

(ii) When $K_i(\bar{x}, \lambda) = K(\lambda)$, for all $i = 1, 2$, that is, K_i does not depend on $\bar{x}$, the (LQEP$_\lambda$) reduces to the lexicographic vector equilibrium problem (LEP$_\lambda$) considered by Anh *et al.* [26].

Instead of writing $\{(\text{LQEP}_\lambda) | \lambda \in \Lambda\}$ for the family of lexicographic vector quasiequilibrium problem, *i.e.*, the lexicographic parametric problem, we will simply write (LQEP) in the sequel. Let $S_f : \Lambda \to 2^X$ be the solution map of (LQEP).

Following the line of investigating ε-solutions to vector optimization problems initiated by Loridan [33], we consider the following *approximate problem*: for each $\varepsilon \in [0, \infty)$,

(LQEP$_{\lambda,\varepsilon}$) find $\bar{x} \in K_1(\bar{x}, \lambda)$ such that

$$f(\bar{x}, y, \lambda) + \varepsilon e \geq_l 0, \quad \forall y \in K_2(\bar{x}, \lambda),$$

where $e = (0, 0, \dots, 0, 1) \in \mathbb{R}^n$. The solution set of (LQEP$_{\lambda,\varepsilon}$) is denoted by $\tilde{S}_f(\lambda, \varepsilon)$.

Let $Y = X \times \Lambda$ and $F =: (F_1, F_2, \dots, F_n) : Y \times Y \to \mathbb{R}^n$ be given. The *lexicographic vector equilibrium problem with lexicographic equilibrium constraints* under question is

(LVQEPLEC) finding $\bar{\mathbf{y}} \in \operatorname{gr} S_f$ such that

$$F(\bar{\mathbf{y}}, \mathbf{y}) \geq_l 0, \quad \forall \mathbf{y} \in \operatorname{gr} S_f,$$

where $\operatorname{gr} S_f$ denotes the graph of S_f, *i.e.*, $\operatorname{gr} S_f := \{(x, \lambda) | x \in S_f(\lambda)\}$. We denote the solution set of (LVQEPLEC) by $\mathbf{S}_F$. Next we consider for each $\xi \in [0, \infty)$, the following approximate problem of (LVQEPLEC):

(LVQEPLEC$_\xi$) find $\bar{\mathbf{y}} \in \operatorname{gr} S_f$ such that

$$F(\bar{\mathbf{y}}, \mathbf{y}) + \xi e \geq_l 0, \quad \forall \mathbf{y} \in \operatorname{gr} S_f.$$

For the function $g : X \times \Lambda \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = (-\infty, \infty]$, the *optimization problem with lexicographic equilibrium constraints* is the problem of, for each $\lambda \in \Lambda$,

(OPLEC) finding $\bar{\mathbf{x}} := (\bar{x}, \lambda) \in \operatorname{gr} S_f$ such that

$$g(\bar{\mathbf{x}}) = \min\{g(\mathbf{x}) | \mathbf{x} := (x, \lambda) \in \operatorname{gr} S_f\}.$$

Let $\mathrm{S}_g : \Lambda \to 2^{X \times \Lambda}$ be the solution map for (OPLEC); that is,

$$\mathrm{S}_g(\lambda) = \Big\{\bar{\mathbf{x}} := (\bar{x}, \lambda) \in \operatorname{gr} S_f \Big| g(\bar{\mathbf{x}}) = \min_{\mathbf{x} := (x,\lambda) \in \operatorname{gr} S_f} g(\mathbf{x})\Big\}.$$

Remark 2.2 When $f := f_1 : X \times X \times \Lambda \to \mathbb{R}$, the (OPLEC) collapses to the optimization problem with equilibrium constraints (OPEC) considered by Anh *et al.* [25].

We next give the concept of an approximating sequence, well-posedness, and unique well-posedness for (LQEP), (LVQEPLEC), and (OPLEC).

Definition 2.3 A sequence $\{x_n\}$ is an *approximating sequence* of (LQEP) corresponding to a sequence $\{\lambda_n\} \subset \Lambda$ converging to $\bar{\lambda}$ if there is a sequence $\{\varepsilon_n\} \subset (0, \infty)$ converging to 0 such that $x_n \in \tilde{S}_f(\lambda_n, \varepsilon_n)$ for all n.

Definition 2.4 A sequence $\{\mathbf{x}_n\} := \{(x_n, \lambda_n)\} \subseteq Y := X \times \Lambda$ is termed an *approximating sequence* for (LVQEPLEC) iff there exists $\varepsilon_n \downarrow 0$ such that

(i) $F(\mathbf{x}_n, \mathbf{y}) + \varepsilon_n e \geq_l 0$, for all $\mathbf{y} := (y, \lambda) \in S_f(\lambda) \times \Lambda$;
(ii) $\{x_n\}$ is an approximating sequence for (LQEP) corresponding to $\{\lambda_n\}$.

Definition 2.5 A sequence $\{\mathbf{x}_n\} := \{(x_n, \lambda_n)\} \subseteq Y := X \times \Lambda$ is called an *approximating* (*or minimizing*) sequence for (OPLEC) iff there exists $\varepsilon_n \downarrow 0$ such that

(i) $g(\mathbf{x}_n) \leq g(\mathbf{y}) + \varepsilon_n$, for all $\mathbf{y} := (y, \lambda) \in S_f(\lambda) \times \Lambda$;
(ii) $\{x_n\}$ is an approximating sequence for (LQEP) corresponding to $\{\lambda_n\}$.

Definition 2.6 Problem (LVQEPLEC) or (OPLEC) is called *well-posed* at $\bar{\lambda}$ iff

(i) it has solutions;
(ii) for any approximating sequence $\{\mathbf{x}_n\} := \{(x_n, \lambda_n)\}$ for (LVQEPLEC), where $\lambda_n \to \bar{\lambda}$, has a subsequence converging to a solution.

Definition 2.7 Problem (LVQEPLEC) or (OPLEC) is called *uniquely well-posed* at $\bar{\lambda}$ iff

(i) it has a unique solution $\bar{\mathbf{x}} := (\bar{x}, \bar{\lambda})$;
(ii) every approximating sequence $\{\mathbf{x}_n\} := \{(x_n, \lambda_n)\}$ for (LVQEPLEC) or (OPLEC), where $\lambda_n \to \bar{\lambda}$, converges to $\bar{\mathbf{x}}$.

Now we recall the continuity-like properties which will be used for our analysis.

Definition 2.8 [34] Let $Q : X \rightrightarrows Y$ be a set-valued mapping between two metric spaces.

(i) Q is *upper semicontinuous* (usc) at $\bar{x}$ if, for any open set $U \supseteq Q(\bar{x})$, there is a neighborhood N of $\bar{x}$ such that $Q(N) \subseteq U$.
(ii) Q is *lower semicontinuous* (lsc) at $\bar{x}$ if, for any open subset U of Y with $Q(\bar{x}) \cap U \neq \emptyset$, there is a neighborhood N of $\bar{x}$ such that $Q(x) \cap U \neq \emptyset$ for all $x \in N$.
(iii) Q is *closed* at $\bar{x}$ if, for any sequences $\{x_k\}$ and $\{y_k\}$ with $x_k \to \bar{x}$ and $y_k \to \bar{y}$ and $y_k \in Q(x_k)$, we have $\bar{y} \in Q(\bar{x})$.

Lemma 2.9 [34, 35]
(i) *If Q is usc at $\bar{x}$ and $Q(\bar{x})$ is compact, then for any sequence $\{x_n\}$ converging to $\bar{x}$, every sequence $\{y_n\}$ with $y_n \in Q(x_n)$ has a subsequence converging to some point in $Q(\bar{x})$. If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton, then such a sequence $\{y_n\}$ must converge to $\bar{y}$.*
(ii) *Q is lsc at $\bar{x}$ if and only if, for any sequence $\{x_n\}$ with $x_n \to \bar{x}$ and any point $y \in Q(\bar{x})$, there is a sequence $\{y_n\}$ with $y_n \in Q(x_n)$ converging to y.*

Definition 2.10 [25, 26] Let g be an extended real-valued function on a metric space X and ε be a real number.
(i) g is *upper ε-level closed* at $\bar{x} \in X$ if, for any sequence $\{x_n\}$ satisfying

$$x_n \to \bar{x} \quad \text{and} \quad g(x_n) \geq \varepsilon \quad \text{for all } n,$$

$g(\bar{x}) \geq \varepsilon$.
(ii) g is *strongly upper ε-level closed* at $\bar{x} \in X$ if, for any sequences $\{x_n\}$ in X and $\{r_n\} \subset [0, \infty)$ satisfying

$$x_n \to \bar{x}, \qquad r_n \to 0 \quad \text{and} \quad g(x_n) + r_n \geq \varepsilon \quad \text{for all } n,$$

$g(\bar{x}) \geq \varepsilon$.

Definition 2.11 [25, 26] Let X be a topological space and $f : X \to \bar{\mathbb{R}}$.
(i) f is called *upper pseudocontinuous* at $x_0 \in X$ iff for any point x and sequence $\{x_n\}$ in X such that

$$f(x_0) < f(x) \quad \text{and} \quad x_n \to x_0,$$

$\limsup_{n\to\infty} f(x_n) < f(x)$.
(ii) f is called *lower pseudocontinuous* at $x_0 \in X$ iff for any point x and sequence $\{x_n\}$ in X such that

$$f(x) < f(x_0) \quad \text{and} \quad x_n \to x_0,$$

$f(x) < \liminf_{n\to\infty} f(x_n)$.
(iii) f is *termed pseudocontinuous* at $x_0 \in X$ iff it is both lower and upper pseudocontinuous at this point.

Remark 2.12 The class of the upper pseudocontinuous functions strictly contains that of the usc functions; see [16].

Let A, B be two subsets of a metric space X. The Hausdorff distance between A and B is defined as follows:

$$H(A,B) = \max\{H^*(A,B), H^*(B,A)\},$$

where $H^*(A,B) = \sup_{a\in A} d(a,B)$, and $d(x,A) = \inf_{y\in A} d(x,y)$.

3 Lexicographic vector equilibrium problems with lexicographic equilibrium constraints (LVQEPLEC)

In this section, we shall establish necessary and/or sufficient conditions for (LVQEPLEC) to be (uniquely) well-posed at the reference point $\bar{\lambda} \in \Lambda$. To simplify the presentation, in the sequel, the results will be formulated for the case $n = 2$.

For any positive numbers ϵ and ξ, as above, $\tilde{S}_f(\lambda,\varepsilon)$ and $\tilde{S}_F(\xi)$ are defined by the solution sets of ($\mathrm{LQEP}_{\lambda,\epsilon}$) and ($\mathrm{LVQEPLEC}_{\xi}$), respectively; that is,

$$\tilde{S}_f(\lambda,\varepsilon) = \{x \in K_1(x,\lambda) | f(x,y,\lambda) + \varepsilon e \geq_l 0, \forall y \in K_2(x,\lambda)\}$$

and

$$\tilde{S}_F(\xi) = \{\bar{y} \in \operatorname{gr} S_f | F(\bar{\mathbf{y}},\mathbf{y}) + \xi e \geq_l 0, \forall \mathbf{y} \in \operatorname{gr} S_f\}.$$

For positive ξ and ε, the corresponding approximate solution set of (LVQEPLEC) is defined by

$$\Gamma(\xi,\varepsilon) = \left\{\begin{array}{l} \mathbf{x} := (x,\lambda) \in K_1(x,\lambda) \times \Lambda \text{ s.t.} \\ F(\mathbf{x},\mathbf{y}) + \varepsilon e \geq_l 0, \forall \mathbf{y} \in \operatorname{gr} S_f, \\ f(x,y,\lambda) + \xi e \geq_l 0, \forall y \in K_2(x,\lambda) \end{array}\right\}.$$

The set-valued mapping $Z_f : X \times \Lambda \to 2^X$ next defined will play an important role our analysis

$$Z_f(x,\lambda) = \begin{cases} \{z \in K_2(x,\lambda) | f_1(x,z,\lambda) = 0\} & \text{if } (x,\lambda) \in \operatorname{gr} Z_{1,f}; \\ X & \text{otherwise,} \end{cases}$$

where $Z_{1,f} : \Lambda \to 2^X$ denotes the solution mapping of the scalar equilibrium problem determined by the real-valued function f_1; that is,

$$Z_{1,f}(\lambda) = \{x \in K_1(x,\lambda) | f_1(x,y,\lambda) \geq 0, \forall y \in K_2(x,\lambda)\}.$$

Then the problem ($\mathrm{LQEP}_{\lambda,\varepsilon}$) can be equivalently stated as follows:

($\mathrm{LQEP}_{\lambda,\varepsilon}$) find $\bar{x} \in K_1(\bar{x},\lambda)$ such that

$$\begin{cases} f_1(\bar{x},y,\lambda) \geq 0, & \forall y \in K_2(\bar{x},\lambda); \\ f_2(\bar{x},z,\lambda) + \varepsilon \geq 0, & \forall z \in Z_f(\bar{x},\lambda). \end{cases}$$

Next, let the set-valued map $Z_F : X \times \Lambda \to 2^X$ be defined by

$$Z_F(\bar{\mathbf{y}}) = \begin{cases} \{\mathbf{y} \in \operatorname{gr} S_f | F_1(\bar{\mathbf{y}}, \mathbf{y}) = 0\} & \text{if } \bar{\mathbf{y}} \in \operatorname{gr} Z_{1,F}; \\ X & \text{otherwise,} \end{cases}$$

where $Z_{1,F}(\lambda) := \{\bar{\mathbf{y}} = (\bar{y}, \bar{\lambda}) \in \operatorname{gr} S_f | F_1(\bar{\mathbf{y}}, \mathbf{y}') \geq 0, \forall \mathbf{y}' \in \operatorname{gr} S_f\}$. Then the problem (LVQEPLEC$_\xi$) can be equivalently stated as follows:

(LVQEPLEC$_\xi$) find $\bar{\mathbf{y}} \in \operatorname{gr} S_f$ such that

$$\begin{cases} F_1(\bar{\mathbf{y}}, \mathbf{y}) \geq 0, & \forall \mathbf{y} \in \operatorname{gr} S_f; \\ F_2(\bar{\mathbf{y}}, \mathbf{y}') + \varepsilon \geq 0, & \forall \mathbf{y}' \in Z_F(\bar{\mathbf{y}}). \end{cases}$$

Thus, for any positive numbers ξ and ε, $\Gamma(\xi, \varepsilon)$ is equivalent to

$$\Gamma(\xi, \varepsilon) = \left\{ \begin{array}{l} \mathbf{x} := (x, \lambda) \in K_1(x, \lambda) \times \Lambda \text{ s.t.} \\ F_1(\mathbf{x}, \mathbf{y}) \geq 0, \forall \mathbf{y} \in \operatorname{gr} S_f, \\ F_2(\mathbf{x}, \mathbf{y}') + \xi \geq 0, \forall \mathbf{y}' \in Z_F(\mathbf{x}), \\ f_1(x, y, \lambda) \geq 0, \forall y \in K_2(x, \lambda), \\ f_2(x, z, \lambda) + \varepsilon \geq 0, \forall z \in Z_f(x, \lambda) \end{array} \right\}.$$

Lemma 3.1 *Let $\{x_n\}$ converging to $\bar{x} \in Z_{1,f}(\bar{\lambda})$ be an approximating sequence of* (LQEP$_{\bar{\lambda}}$) *corresponding to a sequence $\lambda_n \to \bar{\lambda}$ and assume that Z_f is lsc at $(\bar{x}, \bar{\lambda})$ and f_2 is strongly upper* 0*-level closed on $\{\bar{x}\} \times Z_f(\bar{x}, \bar{\lambda}) \times \{\bar{\lambda}\}$. Then $\bar{x} \in S_f(\bar{\lambda})$.*

Proof Suppose to the contrary that $\bar{x} \notin S_f(\bar{\lambda})$. Then there exists $\bar{z} \in Z_f(\bar{x}, \bar{\lambda})$ such that $f_2(\bar{x}, \bar{z}, \bar{\lambda}) < 0$. For each n, we conclude with the lower semicontinuity of Z_f at $(\bar{x}, \bar{\lambda})$ and Lemma 2.9(ii) there exists $z_n \in Z_f(x_n, \lambda_n)$ such that $z_n \to \bar{z}$. Since $\{x_n\}$ is an approximating sequence of (LQEP$_{\bar{\lambda}}$) corresponding to a sequence λ_n, there is a sequence $\{\varepsilon_n\} \subset (0, \infty)$ converging to 0 such that $x_n \in \tilde{S}_f(\lambda_n, \varepsilon_n)$ for all n. This implies that

$$f_2(x_n, z_n, \lambda_n) + \varepsilon_n \geq 0 \quad \text{for all } n. \tag{3.1}$$

This together with the strongly upper 0-level closedness of f_2 at $(\bar{x}, \bar{z}, \bar{\lambda})$ implies that

$$f_2(\bar{x}, \bar{z}, \bar{\lambda}) \geq 0.$$

This yields a contradiction; we have $\bar{x} \in S_f(\bar{\lambda}) = \tilde{S}_f(\bar{\lambda}, 0)$. □

Theorem 3.2 *Assume that X be compact and*

(i) *in $X \times \Lambda$, K_1 is closed and K_2 is lsc;*
(ii) *Z_f is lsc on $Z_{1,f}(\bar{\lambda}) \times \{\bar{\lambda}\}$;*
(iii) *f_1 is upper* 0*-level closed on $K_1(\bar{x}, \bar{\lambda}) \times K_2(\bar{x}, \bar{\lambda}) \times \{\bar{\lambda}\}$;*
(iv) *f_2 is strongly upper* 0*-level closed on $K_1(\bar{x}, \bar{\lambda}) \times K_2(\bar{x}, \bar{\lambda}) \times \{\bar{\lambda}\}$;*
(v) *$F_1(\cdot, \mathbf{y})$ is upper* 0*-level closed at $(\bar{x}, \bar{\lambda})$, for all $\mathbf{y} \in X \times \Lambda$;*
(vi) *$F_2(\cdot, \mathbf{y})$ is strongly upper* 0*-level closed at $(\bar{x}, \bar{\lambda})$, for all $\mathbf{y} \in X \times \Lambda$.*

Then (LVQEPLEC) *is well-posed at $\bar{\lambda}$. Furthermore, if $S_f : \Lambda \to X$ is single-valued and* (LVQEPLEC) *admits a unique solution $\bar{\mathbf{x}}$, then* (LVQEPLEC) *is uniquely well-posed.*

Proof *Step* I: We first prove that $Z_{1,f}$ is closed at $\bar{\lambda}$. Suppose to the contrary that there are two sequences $\{\lambda_n\}$ and $\{x_n\}$ satisfying $\lambda_n \to \bar{\lambda}$ and $x_n \to \bar{x}$ with $x_n \in Z_{1,f}(\lambda_n)$ and $\bar{x} \notin Z_{1,f}(\bar{\lambda})$. Since K_1 is closed in $X \times \Lambda$ and $x_n \in K_1(x_n, \lambda_n)$ for all n, we conclude that $\bar{x} \in K_1(\bar{x}, \bar{\lambda})$. Then there exists $\bar{y} \in K_2(\bar{x}, \bar{\lambda})$ satisfying $f_1(\bar{x}, \bar{y}, \bar{\lambda}) < 0$. The lower semicontinuity of K_2 at $(\bar{x}, \bar{\lambda})$ ensures that, for each n, there is $y_n \in K_2(x_n, \lambda_n)$ such that $y_n \to \bar{y}$ as $n \to \infty$. Since $x_n \in Z_{1,f}(\lambda_n)$, it follows that

$$f_1(x_n, y_n, \lambda_n) \geq 0, \quad \forall n.$$

This together with the upper 0-level closedness of f_1 implies that

$$f_1(\bar{x}, \bar{y}, \bar{\lambda}) \geq 0,$$

which yields a contradiction and, hence, $Z_{1,f}$ is closed at $\bar{\lambda}$.

Step II: Next, we show that $\tilde{S}_f(\cdot, \cdot)$ is usc at $(\bar{\lambda}, 0)$. Indeed, if it were otherwise, then there is an open set $U \supseteq \tilde{S}_f(\bar{\lambda}, 0)$ such that for all neighborhood $N(\bar{\lambda}, 0)$ of $(\bar{\lambda}, 0)$,

$$\tilde{S}_f\bigl(N(\bar{\lambda}, 0)\bigr) \nsubseteq U.$$

In particular, for each $\{\lambda_n\}$ and $\{\epsilon_n\}$ satisfying $\lambda_n \to \bar{\lambda}$ and $\epsilon_n \to 0$, there exists $x_n \in \tilde{S}_f(\lambda_n, \epsilon_n)$ such that $x_n \notin U$ for all n. Since X is compact, we can assume that $\{x_n\}$ converges to some $\bar{x} \notin U$. By the closedness of $Z_{1,f}$ at $\bar{\lambda}$, one has $\bar{x} \in Z_{1,f}(\bar{\lambda})$. Applying Lemma 3.1, we conclude that

$$\bar{x} \in S_f(\bar{\lambda}) = \tilde{S}_f(\bar{\lambda}, 0),$$

which gives $\bar{x} \in U$. This yields a contradiction. Therefore the map $\tilde{S}_f$ is usc at $(\bar{\lambda}, 0)$.

Step III: We have to prove that $\tilde{S}_f(\bar{\lambda}, 0)$ is compact by checking its closedness. Take an arbitrary sequence $\{x_n\}$ in $S(\bar{\lambda}) = \tilde{S}_f(\bar{\lambda}, 0)$ converging to $\bar{x}$. Setting $\lambda_n := \bar{\lambda}$ for all n, we have $\lambda_n \to \bar{\lambda}$ and $x_n \in Z_{1,f}(\lambda_n)$ for all n. This together with the closedness of $Z_{1,f}$ at $\bar{\lambda}$ implies that $\bar{x} \in Z_{1,f}(\bar{\lambda})$. Note that $\{x_n\}$ is, of course, an approximating sequence of ($\mathrm{LQEP}_{\bar{\lambda}}$) corresponding to $\{\lambda_n\}$. Then Lemma 3.1 again implies that $\bar{x} \in S_f(\bar{\lambda}) = \tilde{S}_f(\bar{\lambda}, 0)$, and hence $S_f(\bar{\lambda})$ is compact; that is, $\tilde{S}_f(\bar{\lambda}, 0)$ is compact.

Step IV: Finally, we prove that (LVQEPLEC) is well-posed at $\bar{\lambda}$. To this end, let $\{\mathbf{x}_n\} := \{(x_n, \lambda_n)\}$, where $\lambda_n \to \bar{\lambda}$, be any approximating sequence for (LVQEPLEC). Hence, by Definition 2.4, $\{x_n\}$ is an approximating sequence for (LQEP) corresponding to $\{\lambda_n\}$. Then there exists a real sequence $\{\varepsilon_n\} \downarrow 0$ such that

$$x_n \in \tilde{S}_f(\lambda_n, \varepsilon_n) \quad \text{for all } n \in \mathbb{N}.$$

Applying Lemma 2.9(i), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to some $\bar{x} \in \tilde{S}_f(\bar{\lambda}, 0)$, and hence

$$\mathbf{x}_{n_k} := (x_{n_k}, \lambda_{n_k}) \to (\bar{x}, \bar{\lambda}) \quad \text{as } k \to \infty.$$

Now we check that $\bar{\mathbf{x}} := (\bar{x}, \bar{\lambda})$ is a solution of (LVQEPLEC). Since $\{\mathbf{x}_n\}$ is an approximating sequence, there exists $\{\varepsilon_n\} \downarrow 0$ such that $F_1(\mathbf{x}_n, \mathbf{y}) \geq 0$ and $F_2(\mathbf{x}_n, \mathbf{y}) + \varepsilon_n \geq 0$ for all $\mathbf{y} \in \operatorname{gr} S_f$.

The upper 0-level closedness of F_1 and the strongly upper 0-level closedness of F_2 implies that $F_1(\bar{\mathbf{x}}, \mathbf{y}) \geq 0$ and $F_2(\bar{\mathbf{x}}, \mathbf{y}) \geq 0$ for all $\mathbf{y} \in \operatorname{gr} S_f$, *i.e.*, $\bar{\mathbf{x}}$ is a solution. Thus, (LVQEPLEC) is well-posed at $\bar{\lambda}$.

Furthermore, suppose that $S_f : \Lambda \to X$ is single-valued and (LVQEPLEC) admits a unique solution $\bar{\mathbf{x}}$. We have to show that (LVQEPLEC) is uniquely well-posed. Let $\{\mathbf{x}_n\}$ be an approximating sequence for (LVQEPLEC). By the same argument as in the preceding part, there is a subsequence converging to $\bar{\mathbf{x}}$. If $\{\mathbf{x}_n\}$ did not converge to $\bar{\mathbf{x}}$, there would be an open set U containing $\bar{\mathbf{x}}$ such that some subsequence was outside U. By the above argument, this subsequence has a subsequence convergent to $\bar{\mathbf{x}}$, an impossibility. □

The following examples show that none of the assumptions in Theorem 3.2 can be dropped.

Example 3.3 (The compactness of X cannot be dropped) Let $X = \mathbb{R}$, $\Lambda = [0,1]$, $K_1(x,\lambda) = K_2(x,\lambda) = [\lambda, +\infty)$,

$$f(x,y,\lambda) = \bigl((\lambda x - 1)xy, 0\bigr)$$

and

$$F\bigl((x,\lambda_1),(y,\lambda_2)\bigr) = \bigl(2^{x+y}, 0\bigr) \geq_l 0.$$

It is clear that in $X \times \Lambda$, K_1 is closed and K_2 is lsc. One can check that $Z_{1,f}(\lambda) = [\frac{1}{\lambda}, +\infty)$. Thus Z_f is lsc. Furthermore, (iii)-(vi) hold as f and F are continuous in $X \times X \times \Lambda$ and $(X \times \Lambda) \times (X \times \Lambda)$, respectively. The solution set of (LVQEPLEC) is $\operatorname{gr} S_f$. But $S_f(0) = \{0\}$ and $S_f(\lambda) = [\frac{1}{\lambda}, \infty)$ for all $\lambda \in (0,1]$, $\operatorname{gr} S_f = \{(0,0)\} \cup \{([\frac{1}{\lambda}, \infty), \lambda) | \lambda \in (0,1]\}$. Hence, (LVQEPLEC) is not well-posed. Indeed, let $x_n = n$, $\lambda_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We see that $\mathbf{x}_n := (x_n, \lambda_n)$ is a solution of (LVQEPLEC). It is clear that $\{\mathbf{x}_n\}$ has no convergent subsequence. The reason is that X is not compact. We note further that $S_f(\cdot)$ is neither usc nor lsc at 0, even under the continuity assumptions of K_1, K_2, and f.

Example 3.4 (The closedness of K_1 is essential) Let $X = [-1,1]$, $\Lambda = [0,1]$, $K_1(x,\lambda) = K_2(x,\lambda) = (0,1]$,

$$f(x,y,\lambda) = (0,\lambda)$$

and

$$F\bigl((x,\lambda_1),(y,\lambda_2)\bigr) = (1,0).$$

It is not hard to see that X is compact, K_2 is lsc in $X \times \Lambda$. One can check that $Z_{1,f}(\lambda) = (0,1]$ and

$$\begin{aligned} Z_f(x,\lambda) &= \bigl\{z \in (0,1] | f_1(x,z,\lambda) = 0\bigr\}, \quad \forall (x,\lambda) \in \operatorname{gr} Z_{1,f} \\ &= (0,1]. \end{aligned}$$

Thus Z_f is lsc, (ii)-(vi) are satisfied (by the continuity of f and F). We see also that the solution set of (LVQEPLEC) is $\operatorname{gr} S_f$. But $S_f(\lambda) = (0,1]$ for all $\lambda \in [0,1]$, *i.e.*, $\operatorname{gr} S_f = \{(x,\lambda) | x \in (0,1], \lambda \in [0,1]\}$.

Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = \frac{1}{n}$, $\lambda_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\mathbf{x}_n := (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\{\mathbf{x}_n\}$ converges to $\mathbf{x} := (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Example 3.5 (The lower semicontinuity of K_2 cannot be dispensed) Let $X = \Lambda = [0,2]$,

$$K_1(x,\lambda) = K_2(x,\lambda) = \begin{cases} [0,1] & \text{if } \lambda \neq 0; \\ [0,2] & \text{if } \lambda = 0, \end{cases}$$

$$f(x,y,\lambda) = (x-y,\lambda)$$

and

$$F\bigl((x,\lambda_1),(y,\lambda_2)\bigr) = \bigl(2^{\lambda_1+\lambda_2}, 0\bigr).$$

One can check that K_1 is closed but K_2 is not lsc at $\bar{\lambda} = 0$ and

$$Z_{1,f}(\lambda) = \begin{cases} \{1\} & \text{if } \lambda \neq 0; \\ \{2\} & \text{if } \lambda = 0. \end{cases}$$

Thus (ii)-(vi) hold. One can check that

$$Z_f(x,\lambda) = \{x\}, \quad \forall (x,\lambda) \in \operatorname{gr} Z_{1,f}.$$

Moreover, the solution set of (LVQEPLEC) coincides with $\operatorname{gr} S_f$. But

$$S_f(\lambda) = \begin{cases} \{1\} & \text{if } \lambda \neq 0; \\ \{2\} & \text{if } \lambda = 0, \end{cases}$$

i.e., $\operatorname{gr} S_f := (2,0) \cup \{(1,\lambda) | \lambda \in (0,2]\}$. Hence, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 1$, $\lambda_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We see that $\mathbf{x}_n := (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\{\mathbf{x}_n\}$ converges to $\mathbf{x} := (1,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Example 3.6 (The lower semicontinuity of Z_f cannot be dropped) Let $X = \Lambda = [0,1]$ (compact), $K_1(x,\lambda) = [0,1]$ closed, $K_2 = [0,1]$ lsc,

$$f(x,y,\lambda) = \bigl(\lambda x(x-y), y-x\bigr)$$

and

$$F\bigl((x,\lambda_1),(x,\lambda_2)\bigr) = (1,0).$$

One can check that

$$Z_{1,f}(\lambda) = \begin{cases} [0,1] & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0, \end{cases}$$

and, for each $(x, \lambda) \in \operatorname{gr} S_{1,f}$,

$$Z_f(x,\lambda) = \begin{cases} [0,1] & \text{if } \lambda = 0 \text{ or } x = 0; \\ \{1\} & \text{if } \lambda \neq 0 \text{ and } x \neq 0. \end{cases}$$

Z_f is not lsc at $(0,1)$ because by taking $\{(\lambda_n = \frac{1}{n}, x_n = 1)\} \to (0,1)$, we have $Z_f(x_n,\lambda_n) = \{1\}$ for all n, while $Z_f(0,1) = [0,1]$. Assumptions (i), (iii)-(vi) are obviously satisfied. Finally, we observe that (LVQEPLEC) is not well-posed at $\bar{\lambda}$ by calculating the solution mapping S explicitly as follows:

$$S_f(\lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0, \end{cases}$$

i.e., $\operatorname{gr} S_f := (0,0) \cup \{(x,\lambda) | x = 0,1, \lambda \in (0,1]\}$. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 1$, $\lambda_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We see that $\mathbf{x}_n := (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\{\mathbf{x}_n\}$ converges to $\mathbf{x} := (1,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Example 3.7 (Upper 0-level closedness of f_1) Let $X = \Lambda = [0,1]$ (compact), $K_1(x,\lambda) = K_2(x,\lambda) = [0,1]$ (continuous and closed), $\bar{\lambda} = 0$,

$$f(x,y,\lambda) = \begin{cases} (x-y,\lambda) & \text{if } \lambda = 0; \\ (y-x,\lambda) & \text{if } \lambda \neq 0 \end{cases}$$

and

$$F\big((x,\lambda_1),(x,\lambda_2)\big) = \left(\frac{1}{2}, 0\right).$$

One can check that

$$S(\lambda) = Z_{1,f}(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0\} & \text{if } \lambda \neq 0, \end{cases}$$

$$Z_f(x,\lambda) = \{x\}, \quad \forall (x,\lambda) \in \operatorname{gr} S_1,$$

i.e., $\operatorname{gr} S_f := (1,0) \cup \{(0,\lambda) | \lambda \in (0,1]\}$. Hence, all the assumptions except (iii) hold true. However, (LVQEPLEC) is not well-posed at $\bar{\lambda}$. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 0$, $\lambda_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We see that $\mathbf{x}_n := (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\{\mathbf{x}_n\}$ converges to $\mathbf{x} := (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC). Finally, we show that assumption (iii) is not satisfied. Indeed, take $\{x_n\}$ and $\{\lambda_n\}$ as above and $\{y_n = 1\}$, we have $(x_n, y_n, \lambda_n) \to (0,1,0)$ and $f_1(x_n,y_n,\lambda_n) = 1 > 0$ for all n, while $f_1(0,1,0) = -1 < 0$.

Example 3.8 (Strong upper 0-level closedness of f_2) Let X, Λ, K_1, K_2, $\bar{\lambda}$, and F be as in Example 3.7,

$$f(x,y,\lambda) = \begin{cases} (0, x-y) & \text{if } \lambda = 0; \\ (0, x(x-y)) & \text{if } \lambda \neq 0. \end{cases}$$

One can check that

$$Z_{1,f}(\lambda) = Z(\lambda, x) = [0,1], \quad \forall x, \lambda \in [0,1],$$

$$S_f(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0, \end{cases}$$

i.e., $\operatorname{gr} S_f := (1,0) \cup \{(x,\lambda) | x = 0,1, \lambda \in (0,1]\}$. We can conclude that all the assumptions of Theorem 3.2 except (iv) are satisfied. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 0$, $\lambda_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We see that $\mathbf{x}_n := (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\mathbf{x}_n$ converges to $\mathbf{x} := (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC). Finally, we show that assumption (iv) is not satisfied. Indeed, take sequences $x_n = 0$, $y_n = 1$, $\lambda_n = \frac{1}{n}$ and $\varepsilon_n = \frac{1}{n}$, we have $\{(x_n, y_n, \lambda_n, \varepsilon_n)\}$ and $f_2(x_n, y_n, \lambda_n) + \varepsilon_n > 0$ for all n, while $f_2(0,1,0) = -1 < 0$.

Example 3.9 (Upper 0-level closedness of F_1) Let $X = \Lambda = [0,1]$, $K_1(x,\lambda) = [0,1]$ closed, $K_2(x,\lambda) = [0,1]$ lsc,

$$f(x,y,\lambda) = (0,\lambda)$$

and

$$F\big((x,\lambda_1),(y,\lambda_2)\big) = \begin{cases} (x-y,0) & \text{if } \lambda_1 = 0; \\ (y-x,0) & \text{otherwise.} \end{cases}$$

Then assumptions (i)-(vi) and (vi) are satisfied. We have $\operatorname{gr} S_f := [0,1]$, $\lambda \in [0,1]$. The solution set of (LVQEPLEC) is $(1,0) \cup \{(x,\lambda) | x = 0,1, \lambda \in (0,1]\}$. We can conclude that all the assumptions of Theorem 3.2 except (v) are satisfied. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 0$, $\lambda_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We see that $\mathbf{x}_n := (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\mathbf{x}_n$ converges to $\mathbf{x} := (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Example 3.10 (Strong upper 0-level closedness of F_2) Let X, Λ, K_1, K_2, $\bar{\lambda}$, and f be as in Example 3.9 and

$$F\big((x,\lambda_1),(y,\lambda_2)\big) = \begin{cases} (0,x-y) & \text{if } \lambda_1 = 0; \\ (0,x(x-y)) & \text{otherwise.} \end{cases}$$

One can check that

$$Z_{1,f}(\lambda) = Z_f(x,\lambda) = S_f(\lambda) = [0,1], \quad \forall x, \lambda \in [0,1]$$

i.e., $\operatorname{gr} S_f := \{(x,\lambda) | x \in [0,1], \lambda \in [0,1]\}$. The solution set of (LVQEPLEC) is $(1,0) \cup \{(x,\lambda) | x = 0,1, \lambda \in (0,1]\}$. We can conclude that all the assumptions of Theorem 3.2 except (vi) are satisfied. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 0$, $\lambda_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We see that $\mathbf{x}_n := (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\mathbf{x}_n$ converges to $\mathbf{x} := (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Theorem 3.11 *Let X and Λ be two metric spaces. Then:*

(i) *If* (LVQEPLEC) *is uniquely well-posed at $\bar{\lambda}$, then* $\operatorname{diam} \Gamma(\xi,\varepsilon) \downarrow 0$ *as* $(\xi,\varepsilon) \downarrow (0,0)$.

(ii) *Conversely, suppose that* X *and* Λ *are complete, assumptions* (i)-(vi) *in Theorem* 3.2 *hold and* $\operatorname{diam}\Gamma(\xi,\varepsilon)\downarrow 0$ *as* $\xi\downarrow 0$ *and* $\varepsilon\downarrow 0$. *Then* (LVQEPLEC) *is uniquely well-posed at* $\bar{\lambda}$.

Proof (1) Suppose that (LVQEPLEC) be uniquely well-posed at $\bar{\lambda}$. Then (LVQEPLEC) has a unique solution $\bar{\mathbf{x}} := (\bar{x},\bar{\lambda})$ for some $\bar{x}\in X$. Assume to the contrary that $\operatorname{diam}\Gamma(\xi_n,\varepsilon_n)$ does not converge to 0 as $n\to\infty$. This lead to the existence of a number $r>0$ such that for any $k\in\mathbb{N}$, there exists $n_k\geq k$ with

$$\operatorname{diam}\Gamma(\xi_{n_k},\varepsilon_{n_k})>r.$$

This implies that, for each k, there exist $(x^1_{n_k},\lambda^1_{n_k}),(x^2_{n_k},\lambda^2_{n_k})\in\Gamma(\xi_{n_k},\varepsilon_{n_k})$ such that

$$d\bigl((x^1_{n_k},\lambda^1_{n_k}),(x^2_{n_k},\lambda^2_{n_k})\bigr)>\frac{r}{2}. \tag{3.2}$$

Since $\{(x^1_{n_k},\lambda^1_{n_k})\}$ and $\{(x^2_{n_k},\lambda^2_{n_k})\}$ are approximating sequences for (LVQEPLEC), it follows from (3.2) that $0=d(\bar{\mathbf{x}},\bar{\mathbf{x}})>r/2$. Then we arrive at a contradiction.

(2) Let $\{\mathbf{x}_n\}:=\{(x_n,\lambda_n)\}$ be an approximating sequence of (LVQEPLEC) with $\lambda_n\to\bar{\lambda}$ as $n\to\infty$. Then there exists $\xi_n\downarrow 0$ such that

$$F(\mathbf{x}_n,\mathbf{y})+\xi_n e\geq_l 0\quad\text{for all }\mathbf{y}:=(y,\lambda)\in S(\lambda)\times\Lambda. \tag{3.3}$$

Furthermore, there is a sequence $\varepsilon_n\downarrow 0$ such that

$$x_n\in\tilde{S}_f(\lambda_n,\varepsilon_n)\quad\text{for all }n\in\mathbb{N}.$$

Hence we have $\mathbf{x}_n\in\Gamma(\xi_n,\varepsilon_n)$ for all n. By choosing subsequences if necessary, we can assume that both sequences $\{\xi_n\}$ and $\{\varepsilon_n\}$ are nonincreasing. Thus,

$$\Gamma(\xi_n,\varepsilon_n)\supseteq\Gamma(\xi_m,\varepsilon_m)\quad\text{whenever }n\leq m.$$

From this observation and $\operatorname{diam}\Gamma(\xi_n,\varepsilon_n)\downarrow 0$ as $n\to\infty$, one can directly check that $\{\mathbf{x}_n\}$ is a Cauchy sequence in $X\times\Lambda$. The completeness of $X\times\Lambda$ implies that $\mathbf{x}_n\to\bar{\mathbf{x}}:=(\bar{x},\bar{\lambda})$ as $n\to\infty$. By (3.3), we have

$$F_1(\mathbf{x}_n,\mathbf{y})\geq 0\quad\text{and}\quad F_2(\mathbf{x}_n,\mathbf{y})+\varepsilon_n\geq 0$$

for all $\mathbf{y}:=(y,\lambda)\in\operatorname{gr}S_f$. This together with the upper 0-level closedness of F_1 and the strongly upper 0-level closedness of F_2 implies that

$$F_1(\bar{\mathbf{x}},\mathbf{y})\geq 0\quad\text{and}\quad F_2(\bar{\mathbf{x}},\mathbf{y})\geq 0\quad\text{for all }\mathbf{y}\in\operatorname{gr}S_f,$$

i.e., $\bar{\mathbf{x}}$ is a solution of (LVQEPLEC). Finally, we show that $\bar{\mathbf{x}}:=(\bar{x},\bar{\lambda})$ is the only solution to (LVQEPLEC). Suppose to the contrary that $\mathbf{x}'$ is another solution to (LVQEPLEC), *i.e.*, $\mathbf{x}'\neq\bar{\mathbf{x}}$. It is clear that they both belong to $\Gamma(\xi,\varepsilon)$ for any $\xi,\varepsilon>0$. Then it follows that

$$0<d\bigl(\bar{\mathbf{x}},\bar{\mathbf{x}}'\bigr)\leq\operatorname{diam}\Gamma(\xi,\varepsilon)\downarrow 0\quad\text{as }\xi\downarrow 0\text{ and }\varepsilon\downarrow 0,$$

which gives a contradiction. Thus, (LVQEPLEC) is uniquely well-posed at $\bar{\lambda}$. □

To weaken the assumption of unique well-posedness in Theorem 3.11, we are going to use the notions of measures of noncompactness in a metric space X. We recall that a subset A of a metric space X is ε-discrete iff $d(x,y) \geq \varepsilon$ for all $x, y \in A$ with $x \neq y$.

Definition 3.12 Let M be a nonempty subset of a metric space X.

(i) The *Kuratowski measure* of M is

$$\mu(M) = \inf\left\{\varepsilon > 0 \,\middle|\, M \subseteq \bigcup_{k=1}^{n} M_k \text{ and } \operatorname{diam} M_k \leq \varepsilon, k = 1, \ldots, n, \exists n \in \mathbb{N}\right\}.$$

(ii) The *Hausdorff measure* of M is

$$\eta(M) = \inf\left\{\varepsilon > 0 \,\middle|\, M \subseteq \bigcup_{k=1}^{n} B(x_k, \varepsilon), x_k \in X \text{ for some } n \in \mathbb{N}\right\}.$$

(iii) The *Istrătescu measure* of M is

$$\iota(M) = \inf\{\varepsilon > 0 | M \text{ have no infinite } \varepsilon\text{-discrete subset}\}.$$

Daneš [36] obtained the following inequalities:

$$\eta(M) \leq \iota(M) \leq \mu(M) \leq 2\eta(M). \tag{3.4}$$

The measures μ, η, and ι share many properties and we will use γ in the sequel to denote either one of them. γ is a regular measure (see [37, 38]), *i.e.*, it enjoys the following properties:

(i) $\gamma(M) = +\infty$ if and only if the set M is unbounded;
(ii) $\gamma(M) = \gamma(\operatorname{cl} M)$;
(iii) from $\gamma(M) = 0$ it follows that M is a totally bounded set;
(iv) if X is a complete space and if $\{A_n\}$ is a sequence of closed subsets of X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$ and $\lim_{n\to+\infty} \gamma(A_n) = 0$, then $K := \bigcap_{n\in\mathbb{N}} A_n$ is a nonempty compact set and

$$\lim_{n\to+\infty} H(A_n, K) = 0,$$

where H is the Hausdorff metric;

(v) from $M \subseteq N$ it follows that $\gamma(M) \leq \gamma(N)$.

In terms of a measure $\gamma \in \{\mu, \eta, \iota\}$ of noncompactness we have the following result.

Theorem 3.13 *Let X and Λ be metric spaces.*

(i) *If* (LVQEPLEC) *is well-posed at $\bar{\lambda}$, then $\gamma(\Gamma(\xi,\varepsilon)) \downarrow 0$ as $\xi \downarrow 0$ and $\varepsilon \downarrow 0$.*

(ii) *Conversely, suppose that $\gamma(\Gamma(\xi,\varepsilon)) \downarrow 0$ as $\xi \downarrow 0$ and $\varepsilon \downarrow 0$, and the following conditions hold*:

(a) *X and Λ are complete*;
(b) *K_1 is closed and K_2 is lsc at $(\bar{x}, \bar{\lambda})$*;
(c) *Z_f is lsc on $(X \times \Lambda) \cap \operatorname{gr} Z_{1,f}$*;
(d) *f_1 is upper 0-level closed on $K_1(\bar{x}, \bar{\lambda}) \times K_2(\bar{x}, \bar{\lambda}) \times \{\bar{\lambda}\}$*;
(e) *f_2 is upper a-level closed on $K_1(\bar{x}, \bar{\lambda}) \times K_2(\bar{x}, \bar{\lambda}) \times \{\bar{\lambda}\}$ and $a < 0$*;

(f) $F_1(\cdot,\mathbf{y})$ *is upper* 0*-level closed at* $(\bar{x},\bar{\lambda})$, *for all* $\mathbf{y}\in X\times\Lambda$;

(g) $F_2(\cdot,\mathbf{y})$ *is upper* b*-level closed at* $(\bar{x},\bar{\lambda})$, *for all* $\mathbf{y}\in X\times\Lambda$ *and* $b<0$.

Then (LVQEPLEC) *is well-posed at* $\bar{\lambda}$.

Proof By (3.4) the proof is similar for the three mentioned measures of noncompactness. We discuss only the case $\gamma=\mu$, the Kuratowski measure.

(1) Suppose that (LVQEPLEC) be well-posed. For each $\xi>0$ and $\varepsilon>0$, the solution set $\mathbf{S}$ of (LVQEPLEC) clearly satisfies the relation $\mathbf{S}\subseteq\Gamma(\xi,\varepsilon)$. Hence,

$$H\big(\Gamma(\xi,\varepsilon),\mathbf{S}\big)=H^*\big(\Gamma(\xi,\varepsilon),\mathbf{S}\big).$$

Let $\{\mathbf{x}_n\}:=\{(x_n,\lambda_n)\}$ be arbitrary sequence in $\mathbf{S}$. Then, of course, $\{\mathbf{x}_n\}$ is an approximating sequence of (LVQEPLEC). Thus, it has a subsequence converging to a point in $\mathbf{S}$. Therefore, $\mathbf{S}$ is compact, and hence $\mu(\mathbf{S})=0$. Now for any $\delta>0$, there are $M_1^\delta,M_2^\delta,\dots,M_n^\delta$ for some $n\in\mathbb{N}$ such that

$$\mathbf{S}\subseteq\bigcup_{k=1}^{n}M_k^\delta\quad\text{and}\quad\operatorname{diam}M_k^\delta\le\delta\quad\text{for all }k=1,\dots,n.$$

Next, for each $k=1,\dots,n$, define the set

$$N_k^\delta=\big\{\mathbf{y}\in Y:=X\times\Lambda\,|\,d\big(\mathbf{y},M_k^\delta\big)\le H\big(\Gamma(\xi,\varepsilon),\mathbf{S}\big)\big\}.$$

Now, we show that $\Gamma(\xi,\varepsilon)\subseteq\bigcup_{k=1}^n N_k^\delta$. Let $\mathbf{x}\in\Gamma(\xi,\varepsilon)$. Due to $\mathbf{S}\subseteq\bigcup_{k=1}^n M_k^\delta$, one has

$$d\left(\mathbf{x},\bigcup_{k=1}^{n}M_k^\delta\right)\le d(\mathbf{y},\mathbf{S})\le H\big(\Gamma(\xi,\varepsilon),\mathbf{S}\big).$$

Then there is $\bar{k}\in\{1,2,\dots,n\}$ such that

$$d\big(\mathbf{x},M_{\bar{k}}^\delta\big)\le H\big(\Gamma(\xi,\varepsilon),\mathbf{S}\big),$$

which gives $\mathbf{x}\in N_{\bar{k}}^\delta$. Therefore, we obtain the claim

$$\Gamma(\xi,\varepsilon)\subseteq\bigcup_{k=1}^{n}N_k^\delta.$$

Furthermore, we see that $\operatorname{diam}N_k^\delta\le\delta+2H(\Gamma(\xi,\varepsilon),\mathbf{S})$. Indeed, for any $\mathbf{y},\mathbf{y}'\in N_k^\delta$, $\mathbf{m},\mathbf{m}'\in M_k^\delta$,

$$d\big(\mathbf{y},\mathbf{y}'\big)\le d(\mathbf{y},\mathbf{m})+d\big(\mathbf{m},\mathbf{m}'\big)+d\big(\mathbf{m}',\mathbf{y}'\big),$$

which gives

$$\begin{aligned}d\big(\mathbf{y},\mathbf{y}'\big)&\le\inf_{\mathbf{m}\in M_k^\delta}d(\mathbf{y},\mathbf{m})+\inf_{\mathbf{m},\mathbf{m}'\in M_k^\delta}d\big(\mathbf{m},\mathbf{m}'\big)+\inf_{\mathbf{m}'\in M_k^\delta}d\big(\mathbf{m}',\mathbf{y}'\big)\\&=d\big(\mathbf{y},M_k^\delta\big)+\inf_{\mathbf{m},\mathbf{m}'\in M_k^\delta}d\big(\mathbf{m},\mathbf{m}'\big)+d\big(\mathbf{y}',M_k^\delta\big)\le\delta+2H\big(\Gamma(\xi,\varepsilon),\mathbf{S}\big),\end{aligned}$$

and we arrive at $\operatorname{diam} N_k^\delta \leq \delta + 2H(\Gamma(\xi,\varepsilon),\mathbf{S})$. The definition of μ implies that

$$\mu\bigl(\Gamma(\xi,\varepsilon)\bigr) \leq 2H\bigl(\Gamma(\xi,\varepsilon),\mathbf{S}\bigr) + \delta \quad \text{for all } \delta > 0.$$

Therefore, we can conclude that

$$\mu\bigl(\Gamma(\xi,\varepsilon)\bigr) \leq 2H\bigl(\Gamma(\xi,\varepsilon),\mathbf{S}\bigr).$$

To check that $H(\Gamma(\xi,\varepsilon),\mathbf{S}) \downarrow 0$ as $(\xi,\varepsilon) \downarrow (0,0)$ by contradiction, assume the existence of $\rho > 0$, $(\xi_n,\varepsilon_n) \downarrow (0,0)$, and $\mathbf{x}_n \in \Gamma(\xi_n,\varepsilon_n)$ such that $d(\mathbf{x}_n,\mathbf{S}) \geq \rho$ for all $n \in \mathbb{N}$. Since $\{\mathbf{x}_n\}$ is an approximating sequence, one has a subsequence convergent to some point of $\mathbf{S}$, which is impossible. Hence $\mu(\Gamma(\xi,\varepsilon)) \downarrow 0$ as $\xi \downarrow 0$ and $\varepsilon \downarrow 0$.

(2) Assume that $\mu(\Gamma(\xi,\varepsilon)) \downarrow 0$ as $\xi \downarrow 0$ and $\varepsilon \downarrow 0$. We claim that $\Gamma(\xi,\varepsilon)$ is closed for all $\xi,\epsilon > 0$. Let the sequence $\{\mathbf{x}_n\} := \{(x_n,\lambda_n)\}$ in $\Gamma(\xi,\varepsilon)$ with $\mathbf{x}_n \to \mathbf{x} := (x,\lambda)$. Then, for all $\mathbf{y} \in \operatorname{gr} S_f$, $\mathbf{y}' \in Z_F(\mathbf{x}_n)$, $y_n \in K_2(x_n,\lambda_n)$, and all $z_n \in Z_f(x_n,\lambda_n)$, we have

$$F_1(\mathbf{x}_n,\mathbf{y}) \geq 0 \quad \text{and} \quad F_2\bigl(\mathbf{x}_n,\mathbf{y}'\bigr) + \xi \geq 0$$

and

$$f_1(x_n,y_n,\lambda_n) \geq 0, \qquad f_2(x_n,z_n,\lambda_n) + \varepsilon \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

As K_1 is closed at (x,λ), one has $x \in K_1(x,\lambda)$. By the upper 0-level closedness of F_1 and upper-ξ-level closedness of F_2, one obtains

$$F_1(\mathbf{x},\mathbf{y}) \geq 0 \quad \text{and} \quad F_2\bigl(\mathbf{x},\mathbf{y}'\bigr) + \xi \geq 0, \quad \forall \mathbf{y} \in \operatorname{gr} S_f, \mathbf{y}' \in Z_F(\mathbf{x}).$$

Next, we show by a contrapositive argument that

$$f_1(x,y,\lambda) \geq 0 \quad \text{and} \quad f_2(x,z,\lambda) + \varepsilon \geq 0, \quad \forall y \in K_2(x,\lambda), z \in Z_f(x,\lambda).$$

Suppose that there exist $y \in K_2(x,\lambda)$ and $z \in Z_f(x,\lambda)$ such that

$$f_1(x,y,\lambda) < 0 \quad \text{or} \quad f_2(x,z,\lambda) + \varepsilon < 0.$$

Since K_2 is lsc at (x,λ) and Z_f is lsc at (x,λ), there are two sequences $\{y_n\}$ and $\{z_n\}$ such that $y_n \in K_2(x_n,\lambda_n)$ and $z_n \in Z_f(x_n,\lambda_n)$ and

$$y_n \to y \quad \text{and} \quad z_n \to z \quad \text{as } n \to \infty.$$

By (d) and (e), there is $n_0 \in \mathbb{N}$ such that

$$f_1(x_n,y_n,\lambda_n) < 0 \quad \text{or} \quad f_2(x_n,z_n,\lambda_n) < -\varepsilon \quad \text{for all } n \geq n_0,$$

which leads to a contradiction. As a result, $\mathbf{x} \in \Gamma(\xi,\varepsilon)$ and this set is closed. Next, we observe further that

$$\mathbf{S} = \bigcap_{\xi>0,\epsilon>0} \Gamma(\xi,\varepsilon) \quad \text{and} \quad \mu\bigl(\Gamma(\xi,\varepsilon)\bigr) \downarrow 0 \quad \text{as } (\xi,\epsilon) \downarrow (0,0).$$

The properties of μ implies that $\mathbf{S}$ is compact and $H(\Gamma(\xi,\varepsilon),\mathbf{S}) \downarrow 0$ as $(\xi,\epsilon) \downarrow (0,0)$. Let $\{\mathbf{x}_n\} := \{(x_n,\lambda_n)\}$ be an approximating sequence. There is $(\xi_n,\epsilon_n) \downarrow (0,0)$ such that, for all $\mathbf{y} \in \operatorname{gr} S_f$, $y' \in Z_F(\mathbf{x}_n)$, $y_n \in K_2(x_n,\lambda_n)$, and all $z_n \in Z_f(x_n\lambda_n)$,

$$F_1(\mathbf{x}_n,\mathbf{y}) \geq 0, \qquad F_2(\mathbf{x}_n,y') + \xi_n \geq 0,$$
$$f_1(x_n,y_n,\lambda_n) \geq 0, \qquad f_2(x_n,z_n,\lambda_n) + \varepsilon_n \geq 0.$$

Therefore $(x_n,\lambda_n) \in \Gamma(\xi_n,\epsilon_n)$. Consequently,

$$d(x_n,\mathbf{S}) \leq H\bigl(\Gamma(\xi_n,\epsilon_n),\mathbf{S}\bigr) \downarrow 0.$$

By the compactness of $\mathbf{S}$, there is a subsequence of $\{\mathbf{x}_n\}$ converging to a point of $\mathbf{S}$. Hence (LVQEPLEC) is well-posed. This completes the proof. □

The following examples show that all assumptions of Theorem 3.13(ii) are essential.

Example 3.14 (The closedness of K_1 is essential) Let X, Λ, K_1, K_2, f, and F be as in Example 3.4. It is easy to check that Z_f is lsc and X is complete and K_2 is lsc in $X \times \Lambda$. Assumptions (ii)(c)-(ii)(f) are fulfilled since f and F are continuous in $X \times X \times \Lambda$ and $(X \times \Lambda) \times (X,\Lambda)$, respectively. Moreover, $\Gamma(\xi,\varepsilon) \subseteq [-1,1] \times [0,1]$, and hence $\gamma(\Gamma(\xi,\varepsilon)) \leq \gamma([-1,1] \times [0,1]) = 0$. It is easy to see that the solution set of (LVQEPLEC) coincides with $\operatorname{gr} S_f$. But $S_f(\lambda) = (0,1]$ for all $\lambda \in [0,1]$, *i.e.*, $\operatorname{gr} S_f = \{(x,\lambda) | x \in (0,1], \lambda \in [0,1]\}$. With the same arguments as in Example 3.4, (LVQEPLEC) is not well-posed. The reason is that K_1 is not closed at $(0,0)$.

Example 3.15 Let X, Λ, K_1, K_2, f, and F be as in Example 3.5. Then X is complete, K_1 is closed in $X \times \Lambda$, and (ii)(b) and (ii)(f) hold. $\Gamma(\xi,\varepsilon) \subseteq [0,2] \times [0,2]$, and hence $\gamma(\Gamma(\xi,\varepsilon)) = 0$. Furthermore, the solution set of (LVQEPLEC) is $\operatorname{gr} S_f$. But

$$S_f(\lambda) = \begin{cases} \{1\} & \text{if } \lambda \neq 0; \\ \{2\} & \text{if } \lambda = 0, \end{cases}$$

i.e., $\operatorname{gr} S_f := (2,0) \cup \{(1,\lambda) | \lambda \in (0,2]\}$. Thus, (LVQEPLEC) is not well-posed. The reason is that K_2 is not lsc in $X \times \Lambda$.

Example 3.16 (The lower semicontinuity of Z_f cannot be dropped) Let X, Λ, K_1, K_2, f, and F be as in Example 3.6. One can check that

$$Z_{1,f}(\lambda) = \begin{cases} [0,1] & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0, \end{cases}$$

and, for each $(x,\lambda) \in \operatorname{gr} S_{1,f}$,

$$Z_f(x,\lambda) = \begin{cases} [0,1] & \text{if } \lambda = 0 \text{ or } x = 0; \\ \{1\} & \text{if } \lambda \neq 0 \text{ and } x \neq 0. \end{cases}$$

Z_f is not lsc at $(0,1)$ because by taking $(\lambda_n = \frac{1}{n}, x_n = 1) \to (0,1)$, we have $Z_f(x_n,\lambda_n) = \{1\}$ for all n, while $Z_f(0,1) = [0,1]$. Assumptions (i), (iii)-(iv) are obviously satisfied. Furthermore, $\Gamma(\xi,\varepsilon) \subseteq [0,1] \times [0,1]$, and hence $\gamma(\Gamma(\xi,\varepsilon)) = 0$. Finally, we observe that (LVQEPLEC) is not well-posed at $\bar{\lambda}$ by calculating the solution mapping S_f explicitly as follows:

$$S_f(\lambda) = \begin{cases} \{0\} & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0, \end{cases}$$

i.e., $\operatorname{gr} S_f := (0,0) \cup \{(x,\lambda) | x = 0,1, \lambda \in (0,1]\}$. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 1$, $\lambda_n = \frac{1}{n}$. We see that $\mathbf{x}_n = (x_n,\lambda_n)$ is a solution of (LVQEPLEC) and $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (1,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Example 3.17 (Upper 0-level closedness of f_1) Let X, Λ, K_1, K_2, f, and F be as in Example 3.7. One can check that

$$S_f(\lambda) = Z_{1,f}(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0\} & \text{if } \lambda \neq 0, \end{cases}$$

$$Z_f(x,\lambda) = \{x\}, \quad \forall (\lambda,x) \in \operatorname{gr} S_{1,f},$$

i.e., $\operatorname{gr} S_f := (1,0) \cup \{(0,\lambda) | \lambda \in (0,1]\}$ Hence, all the assumptions except (iii) hold true. Moreover, $\Gamma(\xi,\varepsilon) \subseteq [0,1] \times [0,1]$, and hence $\gamma(\Gamma(\xi,\varepsilon)) = 0$. However, (LVQEPLEC) is not well-posed at $\bar{\lambda}$. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 0$, $\lambda_n = \frac{1}{n}$. We see that $\mathbf{x}_n = (x_n,\lambda_n)$ is a solution of (LVQEPLEC) and $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Finally, we show that assumption (iii) is not satisfied. Indeed, take $\{x_n\}$ and $\{\lambda_n\}$ as above and $y_n = 1$, we have $(x_n,y_n,\lambda_n) \to (0,1,0)$ and $f_1(x_n,y_n,\lambda_n) = 1 > 0$ for all n, while $f_1(0,1,0) = -1 < 0$.

Example 3.18 (Strong upper 0-level closedness of f_2) Let X, Λ, K_1, K_2, f, and F be as in Example 3.8. One can check that

$$Z_{1,f}(\lambda) = Z(\lambda,x) = [0,1], \quad \forall x,\lambda \in [0,1],$$

$$S_f(\lambda) = \begin{cases} \{1\} & \text{if } \lambda = 0; \\ \{0,1\} & \text{if } \lambda \neq 0, \end{cases}$$

i.e., $\operatorname{gr} S_f := (1,0) \cup \{(x,\lambda) | x = 0,1, \lambda \in (0,1]\}$. We can conclude that all the assumptions of Theorem 3.2 except (iv) are satisfied. In addition, $\Gamma(\xi,\varepsilon) \subseteq [0,1] \times [0,1]$, and hence $\gamma(\Gamma(\xi,\varepsilon)) = 0$. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 0$, $\lambda_n = \frac{1}{n}$. We see that $\mathbf{x}_n = (x_n,\lambda_n)$ is a solution of (LVQEPLEC) and $\mathbf{x}_n$ converges to $\mathbf{x} = (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Finally, we show that assumption (iv) is not satisfied. Indeed, take sequences $x_n = 0$, $y_n = 1$, $\lambda_n = \frac{1}{n}$, and $\varepsilon_n = \frac{1}{n}$, we have $\{(x_n,y_n,\lambda_n,\varepsilon_n)\}$ and $f_2(x_n,y_n,\lambda_n) + \varepsilon_n > 0$ for all n, while $f_2(0,1,0) = -1 < 0$.

Example 3.19 (Upper 0-level closedness of F_1) Let X, Λ, K_1, K_2, f, and F be as in Example 3.9. Then assumptions (i)-(vi) and (vi) are satisfied. Moreover, $\Gamma(\xi,\varepsilon) \subseteq [0,1] \times [0,1]$, and hence $\gamma(\Gamma(\xi,\varepsilon)) = 0$. We have $\operatorname{gr} S_f := [0,1], \lambda \in [0,1]$. The solution set of (LVQEPLEC) is $(1,0) \cup \{(x,\lambda) | x = 0,1, \lambda \in (0,1]\}$. We can conclude that all the assumptions of Theorem 3.2 except (vii) are satisfied. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 0$, $\lambda_n = \frac{1}{n}$. We see that $\mathbf{x}_n = (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\mathbf{x}_n$ converges to $\mathbf{x} = (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

Example 3.20 (Strong upper 0-level closedness of F_2) Let X, Λ, K_1, K_2, f, and F be as in Example 3.10. One can check that

$$Z_{1,f}(\lambda) = Z_f(x,\lambda) = S_f(\lambda) = [0,1], \quad \forall x, \lambda \in [0,1],$$

i.e., $\operatorname{gr} S_f := \{(x,\lambda) | x \in [0,1], \lambda \in [0,1]\}$. The solution set of (LVQEPLEC) is $(1,0) \cup \{(x,\lambda) | x = 0,1, \lambda \in (0,1]\}$. We can conclude that all the assumptions of Theorem 3.2 except (vii) are satisfied. By the way, $\Gamma(\xi,\varepsilon) \subseteq [0,1] \times [0,1]$, and hence $\gamma(\Gamma(\xi,\varepsilon)) = 0$. Therefore, (LVQEPLEC) is not well-posed. Indeed, let $x_n = 0$, $\lambda_n = \frac{1}{n}$. We see that $\mathbf{x}_n = (x_n, \lambda_n)$ is a solution of (LVQEPLEC) and $\mathbf{x}_n$ converges to $\mathbf{x} = (0,0)$. But $\mathbf{x}$ does not belong to the solution set of (LVQEPLEC).

4 Optimization problem with lexicographic equilibrium constraints (OPLEC)

We prove first a sufficient condition for the well-posedness in topological settings.

Theorem 4.1 *Assume that X is compact and*

(i) *in $X \times \Lambda$, K_1 is closed and K_2 is lsc*;
(ii) *Z_f is lsc on $Z_{1,f}(\bar{\lambda}) \times \{\bar{\lambda}\}$*;
(iii) *f_1 is upper 0-level closed on $K_1(\bar{x},\bar{\lambda}) \times K_2(\bar{x},\bar{\lambda}) \times \{\bar{\lambda}\}$*;
(iv) *f_2 is strongly upper 0-level closed at $K_1(\bar{x},\bar{\lambda}) \times K_2(\bar{x},\bar{\lambda}) \times \{\lambda\}$*;
(v) *g is lower pseudocontinuous in $(x,\bar{\lambda})$.*

Then (OPLEC) *is well-posed at $\bar{\lambda}$. Furthermore, if $S_f(\lambda)$ is a singleton, for all $\lambda \in \Lambda$, and* (OPLEC) *possesses a unique solution, then this problem is uniquely well-posed at $\bar{\lambda}$.*

Proof Let the function $F =: (F_1, F_2) : Y \times Y \to \mathbb{R}^2$ be given by

$$F\big((x,\lambda_1),(y,\lambda_2)\big) = \big(g(y,\lambda_2) - g(x,\lambda_1), 0\big);$$

that is, for all $(x,\lambda_1),(y,\lambda_2) \in Y := X \times \Lambda$,

$$F_1\big((x,\lambda_1),(y,\lambda_2)\big) = g(y,\lambda_2) - g(x,\lambda_1) \quad \text{and} \quad F_2\big((x,\lambda_1),(y,\lambda_2)\big) = 0.$$

Hence, we have $F_2(\cdot,\mathbf{y})$ is strongly upper 0-level closed on $X \times \Lambda$, for all $\mathbf{y} \in X \times \Lambda$. To apply Theorem 3.2, we need to check only that $F_1(\cdot,\mathbf{y})$ is upper 0-level closed on $X \times \Lambda$ for all $\mathbf{y} \in X \times \Lambda$. For each fixed point $\mathbf{y} := (y,\lambda) \in X \times \Lambda$, let $\{\mathbf{x}_n\} := \{(x_n,\lambda_n)\}$ be any sequence in $X \times \Lambda$ converging to $\mathbf{x} := (x,\bar{\lambda})$ and $F_1(\mathbf{x}_n,\mathbf{y}) \geq 0$; that is, we obtain

$$g(y,\lambda) - g(x_n,\lambda_n) \geq 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

We will show that $F_1(\mathbf{x},\mathbf{y}) \geq 0$; that is, we have to prove that $g(y,\lambda) \geq g(x,\bar{\lambda})$. Suppose, on the contrary, that $g(y,\lambda) < g(x,\bar{\lambda})$. $\mathbf{y} \in X \times \Lambda$. The lower pseudocontinuity of g at $(x,\bar{\lambda})$

implies that

$$g(y,\lambda) < \liminf_{n\to\infty} g(x_n,\lambda_n).$$

Thus there are $t \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$g(y,\lambda) - g(x_n,\lambda_n) \leq g(y,\lambda) - t < 0,$$

which gives a contradiction with (4.1). Applying Theorem 3.2, we have (LVQEPLEC) generated by the function F is well-posed at $\bar{\lambda}$. Consequently, (OPLEC) is well-posed at $\bar{\lambda}$. The assertion on unique well-posedness is easy to demonstrate. This completes the proof. □

For $\xi, \varepsilon > 0$, the approximate solution set of (OPLEC) is defined by

$$M(\xi,\varepsilon) = \left\{ \begin{array}{l} (x,\lambda) \in K_1(x,\lambda) \times \Lambda \text{ s.t.} \\ g(x,\lambda) \leq \inf_{\hat{\lambda}\in\Lambda, y\in S_f(\hat{\lambda})} g(y,\hat{\lambda}) + \varepsilon, \\ f(x,y,\lambda) + \xi e \geq_l 0, \forall y \in K_2(x,\lambda) \end{array} \right\},$$

where $e = (0,1) \in \mathbb{R}^2$.

Theorem 4.2 *Let X and Λ be two metric spaces. Then the following assertions hold*:

(i) *If* (OPLEC) *is uniquely well-posed, then* $\operatorname{diam} M(\xi,\varepsilon) \downarrow 0$ *as* $(\xi,\varepsilon) \downarrow (0,0)$.

(ii) *Conversely, assume that* $\operatorname{diam} M(\xi,\varepsilon) \downarrow 0$ *as* $(\xi,\varepsilon) \downarrow (0,0)$, *and that the following conditions hold*:

- (a) *X and Λ are complete*;
- (b) *K_1 is closed and K_2 is lsc at* $(\bar{x},\bar{\lambda})$;
- (c) *Z_f is lsc on* $Z_{1,f}(\bar{\lambda}) \times \bar{\lambda}$;
- (d) *f_1 is upper 0-level closed at* $K_1(\bar{x},\bar{\lambda}) \times K_2(\bar{x},\bar{\lambda}) \times \{\bar{\lambda}\}$;
- (e) *f_2 is strongly upper 0-level closed on* $K_1(\bar{x},\bar{\lambda}) \times K_2(\bar{x},\bar{\lambda}) \times \{\bar{\lambda}\}$;
- (f) *g be lower pseudocontinuous at* $(\bar{x},\bar{\lambda})$.

Then (OPLEC) *is uniquely well-posed at* $\bar{\lambda}$.

Proof (1) Suppose that (OPLEC) is uniquely well-posed. Assume, on the contrary, that there are a sequence $(\xi_n,\epsilon_n) \downarrow 0$, $n_0 \in \mathbb{N}$, and $r > 0$ such that

$$\operatorname{diam} M(\xi_n,\epsilon_n) > r, \quad \forall n \geq n_0.$$

Then, for each $n \geq n_0$, there exist (x_n^1,λ_n^1) and (x_n^2,λ_n^2) in $M(\xi_n,\epsilon_n)$ such that

$$d\big((x_n^1,\lambda_n^1),(x_n^2,\lambda_n^2)\big) > \frac{r}{2}.$$

Since $\{(x_n^1,\lambda_n^1)\}$ and $\{(x_n^2,\lambda_n^2)\}$ are approximating sequences for (OPLEC), they have to converge to the same unique solution and hence we arrive at a contradiction.

(2) Let $\{\mathbf{x}_n\} := \{(x_n,\lambda_n)\}$ be any approximating sequence for (OPLEC). Then there is a sequence $(\xi_n,\epsilon_n) \downarrow (0,0)$, as $n \to \infty$, such that, for all $n \in \mathbb{N}$,

$$g(x_n,\lambda_n) \leq g(y,\hat{\lambda}) + \varepsilon_n \quad \text{for all } (y,\hat{\lambda}) \in S_f(\hat{\lambda}) \times \Lambda,$$

$$f_1(x_n, y, \lambda_n) \geq 0, \quad \forall y \in K_2(x_n, \lambda_n)$$

and

$$f_2(x_n, z, \lambda_n) + \xi_n \geq 0, \quad \forall z \in Z_f(x_n, \lambda_n).$$

Consequently, we can obtain, for all $n \in \mathbb{N}$,

$$g(x_n, \lambda_n) \leq \inf_{\hat{\lambda} \in \Lambda, y \in S_f(\hat{\lambda})} g(y, \hat{\lambda}) + \epsilon_n.$$

This means that $\mathbf{x}_n := (x_n, \lambda_n) \in M(\xi_n, \epsilon_n)$, and hence $\{\mathbf{x}_n\}$ is a Cauchy sequence $X \times \Lambda$. The completeness of $X \times \Lambda$ implies that $\{\mathbf{x}_n\}$ converges to a point $\bar{\mathbf{x}} := (\bar{x}, \bar{\lambda})$. Since K_1 is closed at $(\bar{x}, \bar{\lambda})$ and $x_n \in K_1(x_n, \lambda_n)$, one has $\bar{x} \in K_1(\bar{x}, \bar{\lambda})$. Using the same argument as for Theorem 3.2, one sees that $\bar{\mathbf{x}}$ solves (OPLEC). Next, we will show that (OPLEC) has a unique solution. If (OPLEC) has two distinct solutions $(\bar{x}_1, \bar{\lambda}_1)$ and $(\bar{x}_2, \bar{\lambda}_2)$, they must belong to $M(\xi, \epsilon)$ for all $\xi, \epsilon > 0$. This yields the contradiction that

$$0 < d\big((\bar{x}_1, \bar{\lambda}_1), (\bar{x}_2, \bar{\lambda}_2)\big) \leq \operatorname{diam} M(\xi, \epsilon).$$

This completes the proof. □

For the well-posedness of (OPLEC) in terms of measures of noncompactness we have the following result. Let us consider only the case of the Hausdorff measure η; we get the corresponding results for the case μ and ι.

Theorem 4.3

(i) *If* (OPLEC) *is well-posed at* $\bar{\lambda}$, *then* $\eta(M(\xi, \varepsilon)) \downarrow 0$ *as* $(\xi, \varepsilon) \downarrow (0, 0)$.

(ii) *Conversely, suppose that* $\eta(M(\xi, \varepsilon)) \downarrow 0$ *as* $(\xi, \varepsilon) \downarrow (0, 0)$, *and the following conditions hold*:

- (a) X *and* Λ *are complete*;
- (b) K_1 *is closed and* K_2 *is lsc on* $X \times \Lambda$;
- (c) Z_f *is lsc on* $Z_{1,f}(\bar{\lambda}) \times \bar{\lambda}$;
- (d) f_1 *is upper b-upper level closed in* $K_1(X, \Lambda) \times K_2(X, \Lambda) \times \Lambda$, *for all* $b < 0$;
- (e) f_2 *is strongly upper b-upper level closed in* $K_1(X, \Lambda) \times K_2(X, \Lambda) \times \Lambda$, *for all* $b < 0$;
- (f) g *is lsc in* $X \times \Lambda$.

Then (OPLEC) *is well-posed at* $\bar{\lambda}$.

Proof (1) Suppose that (OPLEC) is well-posed at $\bar{\lambda}$. For all $\xi, \varepsilon > 0$, the solution set $\mathbf{S}_g(\bar{\lambda})$ of (OPLEC) satisfies obviously the containment $\mathbf{S}_g(\bar{\lambda}) \subseteq M(\xi, \varepsilon)$. Consequently, we have

$$H\big(M(\xi, \varepsilon), \mathbf{S}_g(\bar{\lambda})\big) = H^*\big(M(\xi, \varepsilon), \mathbf{S}_g(\bar{\lambda})\big).$$

Any sequence $\{\mathbf{x}_n\}$ in $\mathbf{S}_g(\bar{\lambda})$ is an approximating sequence of (OPLEC) and has a subsequence convergent to some point of $\mathbf{S}_g(\bar{\lambda})$. So, $\mathbf{S}_g(\bar{\lambda})$ is compact. Thus, there exist $y_1, y_2, \ldots, y_n \in Y := X \times \Lambda$ such that

$$\mathbf{S}_g(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} B(y_k, \varepsilon). \tag{4.2}$$

Next, we claim that

$$M(\xi,\varepsilon)\subseteq\bigcup_{k=1}^{n}B\bigl(y_k,\varepsilon+H\bigl(M(\xi,\varepsilon),\mathbf{S}_g(\bar{\lambda})\bigr)\bigr).\tag{4.3}$$

Let $y\in M(\xi,\varepsilon)$ and suppose that $y\notin\bigcup_{k=1}^{n}B(y_k,\varepsilon+H(M(\xi,\varepsilon),\mathbf{S}_g(\bar{\lambda})))$. This implies that

$$y\notin B\bigl(y_k,\varepsilon+H\bigl(M(\xi,\varepsilon),\mathbf{S}_g(\bar{\lambda})\bigr)\bigr),\quad\forall k=1,2,\ldots,n,$$

which gives

$$d(y,y_k)\geq\epsilon+H\bigl(M(\xi,\varepsilon),\mathbf{S}_g(\bar{\lambda})\bigr)\geq\epsilon+\inf_{b\in\mathbf{S}_g(\bar{\lambda})}d(y,b),\quad\forall k=1,2,\ldots,n.$$

Since $\mathbf{S}_g(\bar{\lambda})$ is compact, there is $b\in\mathbf{S}_g(\bar{\lambda})$ such that

$$d(b,y_k)\geq-d(y,b)+d(y,y_k)\geq\epsilon,\quad\forall k=1,2,\ldots,n,$$

which leads to a contradiction with (4.2), and hence (4.3) holds. Consequently, we have

$$\eta\bigl(M(\xi,\varepsilon)\bigr)\leq H\bigl(M(\xi,\varepsilon),\mathbf{S}_g(\bar{\lambda})\bigr)+\eta\bigl(\mathbf{S}_g(\bar{\lambda})\bigr)=H\bigl(M(\xi,\varepsilon),\mathbf{S}_g(\bar{\lambda})\bigr).$$

Hence, we obtain $H(M(\xi,\varepsilon),\mathbf{S}_g(\bar{\lambda}))\downarrow 0$ as $(\xi,\varepsilon)\downarrow(0,0)$. Indeed, suppose that there exist a real number $\rho>0$, a sequence $(\xi_n,\varepsilon_n)\downarrow(0,0)$ and $\mathbf{x}_n\in M(\xi_n,\varepsilon_n)$ such that

$$d\bigl(\mathbf{x}_n,\mathbf{S}_g(\bar{\lambda})\bigr)\geq\rho\quad\text{for all } n\in\mathbb{N}.$$

Being an approximating sequence for (OPLEC), $\{\mathbf{x}_n\}$ has a subsequence convergent to some point of $\mathbf{S}_g(\bar{\lambda})$, by which one arrives at a contradiction with $\rho>0$. We conclude that $\eta(M(\xi,\varepsilon))\downarrow 0$ as $(\xi,\varepsilon)\downarrow(0,0)$.

(2) Assume that $\eta(M(\xi,\varepsilon))\downarrow 0$ as $(\xi,\varepsilon)\downarrow(0,0)$. We first check that $M(\xi,\varepsilon)$ is closed for all $\xi,\varepsilon>0$. Let $\mathbf{m}_n:=(m_n,\lambda'_n)\in M(\xi,\varepsilon)$ with $\mathbf{m}_n\to\mathbf{m}:=(m,\lambda')$. Hence,

$$\begin{aligned}&g\bigl(m_n,\lambda'_n\bigr)\leq\inf_{\hat{\lambda}\in\Lambda,y\in S_f(\hat{\lambda})}g(y,\hat{\lambda})+\varepsilon,\\&f_1\bigl(m_n,z,\lambda'_n\bigr)\geq 0,\qquad f_2\bigl(m_n,y,\lambda'_n\bigr)+\xi\geq 0,\quad\forall z\in K_2\bigl(m_n,\lambda'_n\bigr),y\in Z_f\bigl(m_n,\lambda'_n\bigr).\end{aligned}\tag{4.4}$$

Since K_1 is closed at (m,λ'), $m\in K_1(m,\lambda')$. By the semicontinuity of g at (m,λ'), we have

$$g\bigl(m,\lambda'\bigr)\leq\liminf_{n\to\infty}g\bigl(m_n,\lambda'_n\bigr)\leq\inf_{\hat{\lambda}\in\Lambda,y\in S_f(\hat{\lambda})}g(m,\hat{\lambda})+\varepsilon.$$

Furthermore, we claim that

$$f_1\bigl(m,z,\lambda'\bigr)\geq 0,\qquad f_2\bigl(m,y,\lambda'\bigr)+\xi\geq 0\quad\text{for all } z\in K_2\bigl(m,\lambda'\bigr),y\in Z_f\bigl(m,\lambda'\bigr).$$

Indeed, if there exist $z\in K_2(m,\lambda')$ and $y\in Z_f(m,\lambda')$ such that

$$f_1\bigl(m,z,\lambda'\bigr)<0,\qquad f_2\bigl(m,y,\lambda'\bigr)+\xi<0,$$

then there is $z_n \in K_2(m_n, \lambda'_n)$, $y_n \in Z_f(m_n, \lambda'_n)$ such that

$$z_n \to z \quad \text{and} \quad y_n \to y$$

as K_2 is lsc at (m, λ'). By (c) and (d), there is $n_0 \in \mathbb{N}$ such that

$$f_1\bigl(m_n, z_n, \lambda'_n\bigr) < 0, \qquad f\bigl(m_n, y_n, \lambda'_n\bigr) < -\xi$$

for all $n \geq n_0$, which is a contradiction. Hence, $M(\xi, \varepsilon)$ is closed. Note further that $\mathbf{S}_g(\bar{\lambda}) = \bigcap_{\xi>0,\varepsilon>0} M(\xi, \varepsilon)$ and $\eta(M(\xi, \varepsilon)) \downarrow 0$ as $(\xi, \varepsilon) \downarrow (0, 0)$. Therefore, by the earlier-mentioned properties of η, $\mathbf{S}_g(\bar{\lambda})$ is compact and $H(M(\xi, \varepsilon), \mathbf{S}_g(\bar{\lambda})) \downarrow 0$ as $(\xi, \varepsilon) \downarrow (0, 0)$.

Finally, we prove that (OPLEC) is well-posed at $\bar{\lambda}$. Let $\{\mathbf{x}_n\} := \{(x_n, \lambda_n)\}$ be an approximating sequence, *i.e.*, there exists $(\xi_n, \varepsilon_n) \downarrow (0, 0)$ such that

$$g(x_n, \lambda_n) \leq \inf_{\hat{\lambda}\in\Lambda, y\in S_f(\hat{\lambda})} g(y, \hat{\lambda}) + \varepsilon_n,$$

$$f_1(x_n, z, \lambda_n) \geq 0, \qquad f_2(x_n, y, \lambda_n) + \xi_n \geq 0, \quad \forall y \in K_2(x_n, \lambda_n).$$

Consequently, $(x_n, \lambda_n) \in M(\xi_n, \varepsilon_n)$. So,

$$d\bigl(x_n, \mathbf{S}_g(\bar{\lambda})\bigr) \leq H\bigl(M(\xi_n, \varepsilon_n), \mathbf{S}_g(\bar{\lambda})\bigr) \downarrow 0.$$

By the compactness of $\mathbf{S}_g(\bar{\lambda})$, there is a subsequence of $\{\mathbf{x}_n\}$ convergent to some point of $\mathbf{S}_g(\bar{\lambda})$. Thus, (OPLEC) is well-posed. □

5 Conclusions

In this paper, we obtain the well-posedness for lexicographic vector equilibrium problems and optimization problems with lexicographic equilibrium constraints in metric spaces. Sufficient conditions for a family of such problems to be (uniquely) well-posed at the reference point are established. Numerous examples are provided to explain that all the assumptions we impose are very relaxed and cannot be dropped. The results presented in this paper extend and improve some known results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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