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Some results on noncommutative Hardy-Lorentz spaces

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Abstract

Let $\mathcal{A}$ be a maximal subdiagonal algebra of a finite von Neumann algebra $\mathcal{M}$. For $0 < p < \infty$, we define the noncommutative Hardy-Lorentz spaces and establish the Riesz and Szegö factorizations on these spaces. We also present some results of Jordan morphism on these spaces.

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1 Introduction

The concept of maximal subdiagonal algebras $\mathcal{A}$, which appeared earlier in Arveson's paper [1], unifies analytic function spaces and nonselfadjoint operator algebras. In fact, subdiagonal algebras are the noncommutative analogue of weak* Dirichlet algebras. In the case that $\mathcal{M}$ has a finite trace, $H^p(\mathcal{A})$ may be defined to be the closure of $\mathcal{A}$ in the noncommutative L_p space $L_p(\mathcal{M})$. Subsequently, Arveson's pioneering work has been extended to different cases by several authors. For example, Marsalli and West [2] obtained a series of results including a Riesz factorization theorem, the dual relations between $H^p(\mathcal{A})$ and $H^q(\mathcal{A})$, and Labuschagne [3] proved the universal validity of Szegö's theorem for finite subdiagonal algebras. Recently, the noncommutative H_p spaces have been developed by Blecher, Bekjan, Labuschagne, Xu and their coauthors in a series of papers.

The noncommutative Hardy spaces have received a lot of attention since Arveson's pioneering work. Most results on the classical Hardy spaces on the torus have been established in this noncommutative setting. Here we mention only two of them directly related with the objective of this paper. After the fundamental work of Arveson, Labuschagne [3] proved the noncommutative Szegö type theorem for finite subdiagonal algebras, and Blecher and Labuschagne [4] gave several useful variants of this theorem for $L^p(\mathcal{M})$. In [5], Bekjan and Xu presented the more general form of Szegö type factorization theorem: Let $0 < p,q \leq \infty$. Let $x \in L^p(\mathcal{M})$ be an invertible operator such that $x^{-1} \in L^q(\mathcal{M})$. Then there exist a unitary $u \in \mathcal{M}$ and $h \in H^p(\mathcal{A})$ such that $x = uh$ and $h^{-1} \in H^q(\mathcal{A})$. The second result we wish to mention concerns the following direct decomposition: Let $1 < p < \infty$. Then

$$L^p(\mathcal{M}) = H_0^p(\mathcal{A}) \oplus L^p(D) \oplus JH_0^p(\mathcal{A}). \tag{1.1}$$

This result is proved in [6] for $p = 2$, in [2] for the general case as above.

In this article we introduce the noncommutative Hardy-Lorentz spaces $H^{p,\omega}(\mathcal{A})$. If $\omega \equiv 1$, the noncommutative Hardy-Lorentz spaces $H^{p,\omega}(\mathcal{A})$ correspond to the noncommutative Hardy spaces $H^p(\mathcal{A})$. By adapting the ideas and techniques in [2, 5, 7], we establish the Riesz and Szegö factorizations on these spaces. We also present some results of inner-outer type factorization and Jordan morphism according to noncommutative Hardy-Lorentz spaces.

The remainder of this article is organized as follows. After a short introduction to this article, Section 2 consists of some preliminaries and notations, including the noncommutative weighted Lorentz spaces and their elementary properties. Section 3 presents the Riesz and Szegö factorization of noncommutative Hardy-Lorentz spaces. Section 4 contains some results of outer operators according to noncommutative Hardy-Lorentz spaces. The last section is devoted to Jordan morphism on these spaces.

2 Preliminaries

Throughout this paper $\mathcal{M}$ will be a finite von Neumann algebra with a faithful normal tracial state τ. We refer to [8, 9] for the theory of noncommutative integration. Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. We denote by $P(\mathcal{M})$ the complete lattice of all (self-adjoint) projections in $\mathcal{M}$. The closed densely defined linear operator x in $\mathcal{H}$ with domain $D(x)$ is said to be affiliated with $\mathcal{M}$ if and only if $yx \subseteq xy$ for all $y \in \mathcal{M}'$. When x is affiliated with $\mathcal{M}$, x is said to be τ-measurable if for every $\varepsilon > 0$ there exists $e \in P(\mathcal{M})$ such that $e(H) \subseteq D(x)$ and $\tau(e^\perp) < \varepsilon$ (where for any projection e, we let $e^\perp = 1 - e$). The set of all τ-measurable operators will be denoted by $L_0(\mathcal{M})$. The set $L_0(\mathcal{M})$ is an $*$-algebra with sum and product being the respective closure of the algebraic sum and product. The topology of $L_0(\mathcal{M})$ is determined by the convergence in measure. For $0 < p < \infty$, let

$$L^p(\mathcal{M}) = \left\{x \in L_0(\mathcal{M} : \tau\left(|x|^p\right)^{\frac{1}{p}} < \infty\right\}.$$

We define

$$\|x\|_p = \tau\left(|x|^p\right)^{\frac{1}{p}}, \quad x \in L^p(\mathcal{M}).$$

Then $(L^p(\mathcal{M}); \|\cdot\|_p)$ is a Banach (or quasi-Banach for $p < 1$) space. As usual, we put $L^\infty(\mathcal{M};\tau) = \mathcal{M}$ and denote by $\|\cdot\|_\infty$ $(= \|\cdot\|)$ the usual operator norm.

For $x \in L_0(\mathcal{M})$, we define

$$\lambda_t(x) = \tau\left(e_{(t,\infty)}\left(|x|\right)\right) \quad \text{and} \quad \mu_t(x) = \inf\left\{s > 0 : \lambda_s(x) \le t\right\},$$

where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ associated with the interval (t,∞). The function $t \to \lambda_t(x)$ is called the distribution function of x and $t \to \mu_t(x)$ is the generalized singular number of x. We will denote simply by $\lambda(x)$ and $\mu(x)$ the functions $t \to \lambda_t(x)$ and $t \to \mu_t(x)$, respectively. It is easy to check that both are decreasing and continuous from the right on $(0,\infty)$ (*cf.* [10]).

It will be sometimes convenient to write $L^1 = L^1(I)$ for brevity, where $I = [0,1]$. When ω is a nonnegative, integrable function on $[0,1]$ and not identically zero, we say that ω is a weight. For a given weight ω, we write $W(t) = \int_0^t \omega(s)\,ds < \infty$, $0 \le t \le 1$. We also agree that 'decreasing' or 'increasing' will mean 'nonincreasing' or 'nondecreasing', respectively.

Let L_0 be the set of all Lebesgue measurable functions on I. For $f \in L_0$, we define its nonincreasing rearrangement as

$$f^*(t) = \inf\bigl\{s > 0 : d_f(s) = m\bigl\{r : \bigl|f(r)\bigr| > s\bigr\} \le t\bigr\}, \quad t > 0,$$

where m denotes the Lebesgue measure on I. The Lorentz space Λ^p_ω, $0 < p < \infty$, is a subspace of L_0 such that

$$\|f\|_{\Lambda^p_\omega} = \left(\int_0^1 f^*(t)^p \omega(t)\,dt\right)^{\frac{1}{p}} < \infty. \tag{2.1}$$

It is clear that

$$\|f\|_{\Lambda^p_\omega} = \left(\int_0^\infty pt^{p-1} W\bigl(d_f(t)\bigr)\,dt\right)^{\frac{1}{p}}.$$

Let $0 < p < \infty$, we define

$$\Gamma^p_\omega = \left\{f \in L_0 : \|f\|_{\Gamma^p_\omega} = \left(\int_0^1 f^{**}(t)^p \omega(t)\,dt\right)^{\frac{1}{p}} < \infty\right\},$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,ds$. Let ω be a weight in I, we write $W(t) = \int_0^t \omega(s)\,ds \in \Delta_2$ if there exists some constant C such that $W(2t) \le CW(t)$, $0 < t < \frac{1}{2}$. For any $0 < p < \infty$, it is well known that Λ^p_ω is a quasi-Banach space if and only if $W(t) \in \Delta_2$ (*cf.* [11]). Since we deal further with the space Λ^p_ω which is at least a quasi-Banach space, we will assume in what follows that $W(t) \in \Delta_2$.

Let Λ^p_ω be a quasi-Banach space. For $0 < s < \infty$, we define the dilation operator D_s on Λ^p_ω by

$$(D_s f)(t) = f(st)\chi_{[0,1]}(ts), \quad t \in [0,1].$$

Define the lower Boyd index $\alpha_{\Lambda^p_\omega}$ of Λ^p_ω by

$$\alpha_{\Lambda^p_\omega} = \lim_{s\to\infty} \frac{\log s}{\log \|D_{\frac{1}{s}}\|} = \sup_{s>1} \frac{\log s}{\log \|D_{\frac{1}{s}}\|}$$

and the upper Boyd index $\beta_{\Lambda^p_\omega}$ of Λ^p_ω by

$$\beta_{\Lambda^p_\omega} = \lim_{s\to 0} \frac{\log s}{\log \|D_{\frac{1}{s}}\|} = \inf_{0<s<1} \frac{\log s}{\log \|D_{\frac{1}{s}}\|}.$$

Note that

$$\alpha_{\Lambda^p_\omega} = \sup\bigl\{p > 0 : \exists c > 0, \forall 0 < a \le 1, \|D_a f\|_{p,\omega} \le ca^{-\frac{1}{p}}\|f\|_{p,\omega}\bigr\}$$

and

$$\beta_{\Lambda_\omega^p} = \inf\bigl\{p > 0 : \exists c > 0, \forall 0 < a \leq 1, \|D_a f\|_{p,\omega} \leq c a^{-\frac{1}{p}} \|f\|_{p,\omega}\bigr\}.$$

It is clear that $0 \leq \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} \leq \infty$ and $\alpha_{\Lambda_\omega^{rp}} = r\alpha_{\Lambda_\omega^p}$, $\beta_{\Lambda_\omega^{rp}} = r\beta_{\Lambda_\omega^p}$, $r > 0$. If Λ_ω^p is a Banach function space, then $1 \leq \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} \leq \infty$. For further results about the Boyd index of quasi-Banach spaces, the reader is referred to [12, 13].

We consider a couple (E_0, E_1) of topological vector spaces E_0 and E_1, which are both continuously embedded in a topological vector space E. The morphisms $T : (E_0, E_1) \to (E_0, E_1)$ in E are all bounded linear mappings from $E_0 + E_1$ to $E_0 + E_1$ such that $T_{E_0} : E_0 \to E_0$ and $T_{E_1} : E_1 \to E_1$. A (quasi-)Banach space A is said to be an intermediate space between E_0 and E_1 if E is continuously embedded between $E_0 \cap E_1$ and $E_0 + E_1$. The space E is called an interpolation space between E_0 and E_1 if, in addition, $T : (E_0, E_1) \to (E_0, E_1)$ implies $T : E \to E$. Further details may be found in [12, 14].

Let $0 < p < \infty$, we say that Λ_ω^p has order continuous norm if for every net (f_i) in Λ_ω^p such that $f_i \downarrow 0$ we have $\|f_i\|_{p,\omega} \downarrow 0$. It follows from Proposition 2.3.3 and Theorem 2.3.4 of [11] that the norm on Λ_ω^p is order continuous. Then it follows from Theorem 3.2 of [12] that Λ_ω^r is an interpolation space for the couple (L^p, L^q), where $0 < p < \alpha_{\Lambda_\omega^r} \leq \beta_{\Lambda_\omega^r} < q \leq \infty$. Let $1 < p < \infty$, it is well known (Theorem A, [15]) that $\Lambda_\omega^p = \Gamma_\omega^p$ if and only if ω satisfies the condition B_p ($\omega \in B_p$), that is, there exists $B > 0$ such that $\int_t^1 s^{-p}\omega(t)\,dt \leq Bt^{-p}W(t)$, $t \in I$. We refer to [11, 15] for these spaces.

A (quasi-)Banach function space E is called symmetric if for $f \in L_0$ and $g \in E$ with $f^* \leq g^*$, we have $f \in E$ and $\|f\|_E \leq \|g\|_E$. It is called fully symmetric if, in addition, for $f \in L_0$ and $g \in E$ with $\int_0^t f^*(s)\,ds \leq \int_0^t g^*(s)\,ds$, we have $f \in E$ and $\|f\|_E \leq \|g\|_E$. If $1 \leq p < \infty$ and ω is a non-increasing weighted function, it is clear that Λ_ω^p is a fully symmetric Banach function space. Let $0 < p < \infty$, then Γ_ω^p is a fully symmetric quasi-Banach function space.

Let $x \in L_0(\mathcal{M})$ and $0 < p < \infty$. We define

$$\|x\|_{p,\omega} = \|x\|_{\Lambda_\omega^p(\mathcal{M})} = \left(\int_0^1 \mu_t(x)^p \omega(t)\,dt\right)^{\frac{1}{p}}. \tag{2.2}$$

By a simple computation we derive

$$\|x\|_{p,\omega} = \left(\int_0^\infty p t^{p-1} W\bigl(\lambda_t(x)\bigr)\,dt\right)^{\frac{1}{p}}.$$

The noncommutative Lorentz space $\Lambda_\omega^p(\mathcal{M})$ is defined as the space of all τ-measurable operators affiliated with a finite von Neumann algebra $\mathcal{M}$ such that $\|x\|_{p,\omega} < \infty$. If $\omega \equiv 1$, then the noncommutative Lorentz space $\Lambda_\omega^p(\mathcal{M})$ is the usual noncommutative L_p space $L_p(\mathcal{M})$. If $\omega = t^{\frac{p}{q}-1}$, $p, q > 0$, then $\Lambda_\omega^p(\mathcal{M}) = L^{q,p}(\mathcal{M})$. For further results about the noncommutative Lorentz spaces $L^{q,p}(\mathcal{M})$, the reader is referred to [16].

If $0 < p < \infty$, we define

$$\Gamma_\omega^p(\mathcal{M}) = \left\{x \in L_0(\mathcal{M}) : \|x\|_{\Gamma_\omega^p(\mathcal{M})} = \left(\int_0^1 x^{**}(t)^p \omega(t)\,dt\right)^{\frac{1}{p}} < \infty\right\},$$

where $x^{**}(t) = \frac{1}{t}\int_0^t \mu_s(x)\,ds$.

Since Λ_ω^p and Γ_ω^p are quasi-Banach spaces, then the following result follows from Theorem 4 of [17].

Proposition 2.1 *Let $0 < p < \infty$, then $\Lambda_\omega^p(\mathcal{M})$ and $\Gamma_\omega^p(\mathcal{M})$ are quasi-Banach spaces.*

We should introduce the Köthe dual spaces (associate spaces) generalizing the definition that can be found in [11] in the context of classical Lorentz space Λ_ω^p, $0 < p \leq \infty$. We define the Köthe dual space of $\Lambda_\omega^p(\mathcal{M})$ by

$$\Lambda_\omega^p(\mathcal{M})' = \Big\{ f \in L_0(\mathcal{M}); \|x\|_{\Lambda_\omega^p(\mathcal{M})'} = \sup_{\|y\|_{p,\omega} \leq 1} \tau\big(|xy|\big) < \infty \Big\}.$$

If $x \in \Lambda_\omega^p(\mathcal{M})'$, it is clear that

$$\|x\|_{\Lambda_\omega^p(\mathcal{M})'} = \sup_{\|y\|_{p,\omega} \leq 1} \|xy\|_1 = \sup\big\{\big|\tau(xy)\big| : \|y\|_{p,\omega} \leq 1\big\}.$$

Lemma 2.1 *Let $\mathcal{M}$ have no minimal projection for every measurable function f with*

$$\lim_{t\to\infty} d_f(t) = 0,$$

then there exists $x \in L_0(\mathcal{M})$ such that $\mu_t(x) = f^*(t)$.

Proof By Lemma 1.8 of [18], there exist $Q_{k2^{-n}} \in \mathcal{M}_{\text{proj}}$ such that $\tau(Q_{k2^{-n}}) = k2^{-n}$ and $Q_{k_1 2^{-n_1}} \leq Q_{k_2 2^{-n_2}}$, $k_1 2^{-n_1} \leq k_2 2^{-n_2}$, where $n, k, n_i, k_i \in \mathbb{N}$, $i = 1, 2$. Let $Q_\lambda = \sup_{\frac{k}{2^n} \leq \lambda} Q_{k2^{-n}}$, $\lambda \in [0,1]$. Then $\{Q_\lambda\}_{\lambda\in[0,1]}$ is an increasing family of projections in $\mathcal{M}$, and it is clear that $\tau(Q_\lambda) = \lambda$, $Q_0 = 0$ and $Q_1 = 1$. Therefore, it is a spectral family of $\mathcal{M}$. Let $f \in L_0$ with $\lim_{t\to\infty} d_f(t) = 0$. For $x = \int_0^\infty f^*(\lambda)\, dQ_\lambda$, we have

$$\tau\big(e_{(t,\infty)}(x)\big) = \tau\bigg(\int_{\{\lambda\geq 0: f^*(\lambda) > t\}} dQ_\lambda\bigg) = \int_{\{\lambda \geq 0: f^*(\lambda)>t\}} d\lambda = d_{f^*}(t) = d_f(t) \to 0, \quad t \to \infty,$$

which implies $x \in L_0(\mathcal{M})$ and $\mu_t(y) = f^*(t)$, $t > 0$. □

Lemma 2.2 *Let $1 < p < \infty$. Then the injection $\Lambda_\omega^p(\mathcal{M})' \hookrightarrow L_0(\mathcal{M})$ is continuous.*

Proof For $x \in \Lambda_\omega^p(\mathcal{M})'$ with $\|x\|_{\Lambda_\omega^p(\mathcal{M})'} \leq 1$. Since $\mu_t(x) \to 0$, $t \to \infty$, then $|x|$ admits the Schmidt decomposition $|x| = \int_0^\infty \mu_t(x)\, d\widetilde{e}_t$, where $\widetilde{e}_t = e_{\mu_t(x)-0}$, $t > 0$, and $e_{0-0} = 1$ (*cf.* [19]). Given $\delta > 0$, it is clear that $\mu_t(x) \geq \mu_\delta(x)\chi_{[\frac{\delta}{2},\delta)}$. Thus $|x| = \int_0^\infty \mu_t(x)\, d\widetilde{e}_t \geq \mu_\delta(x)q$, where $q = \int_0^\infty \chi_{[\frac{\delta}{2},\delta)}\, d\widetilde{e}_t$. Since $\tau(q) < \infty$, we obtain $q \in \Lambda_\omega^p(\mathcal{M})'$. Therefore, $\|x\|_{\Lambda_\omega^p(\mathcal{M})'} = \||x|\|_{\Lambda_\omega^p(\mathcal{M})'} \geq \mu_\delta(x)\|q\|_{\Lambda_\omega^p(\mathcal{M})'}$ and $\|q\|^{-1}_{\Lambda_\omega^p(\mathcal{M})'} \geq \mu_\delta(x)$, which complete the proof. □

Proposition 2.2 *Let $1 < p < \infty$. Then $\Lambda_\omega^p(\mathcal{M})'$ is a noncommutative Banach function space.*

Proof It is clear that $\|\cdot\|_{\Lambda_\omega^p(\mathcal{M})'}$ is subadditive, homogenous and positive. If $\|x\|_{\Lambda_\omega^p(\mathcal{M})'} = 0$, then $\|xy\|_1 = 0$ for every $y \in \Lambda_\omega^p(\mathcal{M})$. Since $e_A(|x|) \in \Lambda_\omega^p(\mathcal{M})$, we have $xe_A(|x|) = 0$, where A is a subset of $(0,\infty)$, which implies that $x = 0$. The proof of Theorem 8.11 of [20] shows that it is sufficient to prove the noncommutative form of the Riesz-Fischer theorem, *i.e.*, we have an estimate

$$\left\|\sum_{n=1}^{\infty} x_n\right\|_{\Lambda_\omega^p(\mathcal{M})'} \le \sum_{n=1}^{\infty} \|x_n\|_{\Lambda_\omega^p(\mathcal{M})'}, \quad x_n \ge 0, n = 1, 2, 3, \ldots,$$

whenever the right-hand side is finite. Let $\sum_{n=1}^{\infty} \|x_n\|_{\Lambda_\omega^p(\mathcal{M})'} < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges to some x in $L_0(\mathcal{M})$. Indeed, set $z_n = \sum_{k=1}^{n} x_k$, it is clear that $\{z_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\Lambda_\omega^p(\mathcal{M})'$. It follows from Lemma 2.2 that $\{z_n\}_{n=1}^{\infty}$ converges to some x in $L_0(\mathcal{M})$. Since $\sum_{n=1}^{\infty} \|x_n y\|_{\Lambda_\omega^p(\mathcal{M})'} \le \sum_{n=1}^{\infty} \|x_n\|_{\Lambda_\omega^p(\mathcal{M})'} < \infty$, then

$$\left\|\left(\sum_{n=1}^{\infty} x_n\right) y\right\|_1 = \left\|\sum_{n=1}^{\infty} x_n y\right\|_1 \le \sum_{n=1}^{\infty} \|x_n y\|_1 \le \sum_{n=1}^{\infty} \|x_n\|_{\Lambda_\omega^p(\mathcal{M})'} < \infty$$

holds for each $y \in \Lambda_\omega^p(\mathcal{M})$ with $\|y\|_{\Lambda_\omega^p(\mathcal{M})} \le 1$. Thus $\sum_{n=1}^{\infty} x_n \in \Lambda_\omega^p(\mathcal{M})'$ and

$$\left\|\sum_{n=1}^{\infty} x_n\right\|_{\Lambda_\omega^p(\mathcal{M})'} \le \sum_{n=1}^{\infty} \|x_n\|_{\Lambda_\omega^p(\mathcal{M})'} < \infty.$$

□

Proposition 2.3 *Let $\mathcal{M}$ have no minimal projection, then the associate space $\Lambda_\omega^p(\mathcal{M})'$ is a noncommutative Banach function space. For $x \in \Lambda_\omega^p(\mathcal{M})'$,*

$$\|x\|_{\Lambda_\omega^p(\mathcal{M})'} = \sup\left\{\int_0^1 \mu_t(x)\mu_t(y)\,dt : \|y\|_{\Lambda_\omega^p(\mathcal{M})} \le 1\right\}. \tag{2.3}$$

Proof Let $x \in \Lambda_\omega^p(\mathcal{M})'$ and $y \in \Lambda_\omega^p(\mathcal{M})$, then $xy \in L^1(\mathcal{M})$. By Theorem 4.2 of [10], we have $\tau(|xy|) \le \int_0^\infty \mu_t(x)\mu_t(y)\,dt$. Thus

$$\|x\|_{\Lambda_\omega^p(\mathcal{M})'} \le \sup\left\{\int_0^1 \mu_t(x)\mu_t(y)\,dt : \|y\|_{\Lambda_\omega^p(\mathcal{M})} \le 1\right\}.$$

To prove the other inequality, let $y \in \Lambda_\omega^p(\mathcal{M})$ with $\|y\|_{\Lambda_\omega^p(\mathcal{M})} \le 1$. By Proposition 2.2 of [21], we obtain

$$\begin{aligned}
\int_0^1 \mu_t(x)\mu_t(y)\,dt &= \sup\left\{\tau\left(|x|\widetilde{y}\right) : \widetilde{y} \in L_0(\mathcal{M}), \widetilde{y} \sim y\right\} \\
&\le \sup_{\|\widetilde{y}\|_{\Lambda_\omega^p(\mathcal{M})} \le 1} \tau\left(|x|\widetilde{y}\right) = \sup_{\|\widetilde{y}\|_{\Lambda_\omega^p(\mathcal{M})} \le 1} \tau\left(u^* x\widetilde{y}\right) \\
&\le \sup_{\|\widetilde{y}\|_{\Lambda_\omega^p(\mathcal{M})} \le 1} \tau\left(|x\widetilde{y}|\right) = \|x\|_{\Lambda_\omega^p(\mathcal{M})'}.
\end{aligned}$$

Hence, $\|x\|_{\Lambda_\omega^p(\mathcal{M})'} \ge \sup\{\int_0^\infty \mu_t(x)\mu_t(y)\,dt : \|y\|_{\Lambda_\omega^p(\mathcal{M})} \le 1\}$. □

Proposition 2.4 *Let $x \in \Lambda_\omega^p(\mathcal{M})'$, then we have*

$$\|x\|_{\Lambda_\omega^p(\mathcal{M})'} = \left\|\mu_t(x)\right\|_{\Lambda_\omega^p(I)'}.$$

Moreover, $(\Lambda_\omega^p)'(\mathcal{M}) = \Lambda_\omega^p(\mathcal{M})'$.

Proof This result follows immediately from Lemma 2.1 and Proposition 2.3. First, let $\mathcal{M}$ have no minimal projection. For every $f \in \Lambda_\omega^p$, by Lemma 2.1, there exists $y \in \Lambda_\omega^p(\mathcal{M})$ such that $\mu(y) = f^*(t)$, $t > 0$. Thus, by Proposition 2.3, we obtain

$$\begin{aligned}\|x\|_{\Lambda_\omega^p(\mathcal{M})'} &= \sup\left\{\int_0^1 \mu_t(x)\mu_t(y)\,dt : \|y\|_{\Lambda_\omega^p(\mathcal{M})} \le 1\right\}\\ &= \sup\left\{\int_0^1 \mu_t(x)f^*(t)\,dt : \|f\|_{\Lambda_\omega^p} \le 1\right\}\\ &= \left\|\mu_t(x)\right\|_{\Lambda_\omega^p(I)'}.\end{aligned}$$

If $\mathcal{M}$ has minimal projections, we consider the von Neumann algebra tensor product $\mathcal{M}\bar{\otimes}L^\infty([0,1],m)$ denoted by $\overline{\mathcal{M}}$, equipped with the tensor product trace $\tau \otimes m$, then $\overline{\mathcal{M}}$ has no minimal projection. By the trivial fact $\mu(x) = \mu(x \otimes 1)$ and argument of above, we have $\|x\|_{\Lambda_\omega^p(\mathcal{M})'} = \|\mu_t(x)\|_{\Lambda_\omega^p(I)'}$, which implies that

$$\Lambda_\omega^p(\mathcal{M})' = \left\{x \in L_0(\mathcal{M}) : \mu_t(x) \in \left(\Lambda_\omega^p\right)'\right\} = \left(\Lambda_\omega^p\right)'(\mathcal{M}). \qquad \square$$

Definition 2.1 Let $0 < p \le \infty$. If $l \in \Lambda_\omega^p(\mathcal{M})^*$, then l is called normal if $x_\alpha \downarrow 0$ holds in $\Lambda_\omega^p(\mathcal{M})$ implies $l(x_\alpha) \to 0$.

If $l \in \mathcal{M}^*$ then, by Theorem 5.11 of [22], l is normal if and only if l is ultra-weak topology on $\mathcal{M}$. Then a similar discussion of Lemma 5.10 of [23] leads to the following lemma.

Lemma 2.3 *Let $1 < \alpha_{\Lambda_\omega^p} \le \beta_{\Lambda_\omega^p} < \infty$, $1 < p < \infty$ and $l \in \Lambda_\omega^p(\mathcal{M})^*$. If $e \in \mathcal{M}_{\text{proj}}$, we define l_e by setting $l_e(x) = l(ex)$, $x \in \mathcal{M}$, then l_e is a normal linear functional on $\mathcal{M}$.*

Proof Let $e \in \mathcal{M}$ and $x_0 > x_\alpha \downarrow 0$ hold in $\mathcal{M}$, then $ex_\alpha e \downarrow 0$ holds in $L^1(\mathcal{M})$, and so $\mu_t(ex_\alpha e) \downarrow 0$ holds in L^1. Thus,

$$\mu_t(ex_\alpha) = \mu_t(x_\alpha e) = \mu_t\left(|x_\alpha e|^2\right)^{\frac{1}{2}} = \mu_t\left(ex_\alpha^2 e\right)^{\frac{1}{2}} = \|x_0\|^{\frac{1}{2}}\mu_t(ex_\alpha e)^{\frac{1}{2}} \downarrow 0.$$

By Theorem 2.3.4 of [11], we have $\|ex_\alpha\|_{\Lambda_\omega^p(\mathcal{M})} = (\int_0^1 \mu_t(ex_\alpha)^p\omega(t))^{\frac{1}{p}} \downarrow 0$. Therefore, $l_e(x_\alpha) \to 0$. This tells us that l_e is a normal linear functional on $\mathcal{M}$. $\square$

Proposition 2.5 *Let $1 < \alpha_{\Lambda_\omega^p} \le \beta_{\Lambda_\omega^p} < \infty$, $1 < p < \infty$. If $l \in \Lambda_\omega^p(\mathcal{M})^*$, then there exists a unique operator $y \in \Lambda_\omega^p(\mathcal{M})'$ such that $l(x) = \tau(yx)$ for every $x \in \mathcal{M}$ and $\|l\| \ge \|y\|_{\Lambda_\omega^p(\mathcal{M})'}$.*

Proof It follows from Theorem 3.2 of [12] that Λ_ω^p is an interpolation space for the couple (L^1, L^∞). Thus $L^\infty \hookrightarrow \Lambda_\omega^p \hookrightarrow L^1$, and so $\mathcal{M} \hookrightarrow \Lambda_\omega^p(\mathcal{M}) \hookrightarrow L^1(\mathcal{M})$, where '$\hookrightarrow$' denotes a continuous embedding. It follows that there exists some constant C such that

$$C^{-1}\|x\|_1 \le \|x\|_{p,\omega} \le C\|x\|, \quad x \in \mathcal{M}$$

and $W(t) \le C$, $t \in [0,1]$. We apply Lemma 2.3 and some techniques from the proof of Theorem 5.11 in [23]. The result follows in the same way as in [23]. $\square$

Proposition 2.6 *Let $1 < \alpha_{\Lambda_\omega^p} \le \beta_{\Lambda_\omega^p} < \infty$, $1 < p < \infty$. Then*

(i) $\Lambda^p_\omega(\mathcal{M})^* = \Lambda^p_\omega(\mathcal{M})' = (\Lambda^p_\omega)'(\mathcal{M}) = \Gamma^q_{\widetilde{\omega}}(\mathcal{M})$, *where* $\widetilde{\omega}(t) = t^q W(t)^{-q}\omega(t)$, $t > 0$.
(ii) *If* e_i *is an increasing sequence of projections in* $\mathcal{M}$ *converging strongly to* 1, *we have* $\lim_{i\to\infty} \|xe_i - x\|_{p,\omega} = 0$, $\lim_{i\to\infty} \|e_i x - x\|_{p,\omega} = 0$, $\forall x \in \Lambda^p_\omega(\mathcal{M})$.
(iii) $\Lambda^p_\omega(\mathcal{M})^*$ *separates points.*

Proof (i): Combining Proposition 2.4 with Remark 2.4.8 of [11], we obtain $\Lambda^p_\omega(\mathcal{M})' = (\Lambda^p_\omega)'(\mathcal{M}) = \Gamma^q_{\widetilde{\omega}}(\mathcal{M})$. For $l \in \Lambda^p_\omega(\mathcal{M})^*$, by Proposition 2.5, there exists $y \in \Lambda^p_\omega(\mathcal{M})'$ such that

$$l(x) = l_y(x) = \tau(xy), \quad \forall x \in \mathcal{M}.$$

Let $l^s = l - l_y$, we have $l^s \in \Lambda^p_\omega(\mathcal{M})^*$ and $l^s(x) = 0$, $x \in \mathcal{M}$. For $x \in \Lambda^p_\omega(\mathcal{M})$, it is well known that $\tau(e_{(\frac{1}{n},\infty)}(|x|)) < \infty$, $n = 1, 2, 3, \ldots$. Let $x_n = xe_{[0,\frac{1}{n}]}(|x|)$, $n = 1, 2, \ldots$, then $x - x_n \in \mathcal{M}$. This implies that $|l^s(x)| = |l^s(x_n)| \le \|l^s\|\|x_n\|_{p,\omega}$, $n = 1, 2, \ldots$. Note that $|x_n| \le |x|$ and $\mu(x_n) \to 0$, $n \to \infty$. Then, by Theorem 2.3.4 of [11], we have $\|x_n\|_{p,\omega} = \|\mu(x_n)\|_{p,\omega} \to 0$, $n \to \infty$, and so $|l^s(x)| = |l^s(x_n)| = 0$. Therefore, $l^s = 0$, that is, $l = l_y$. (ii): Without loss of generality, we suppose $0 \le x \in \Lambda^p_\omega(\mathcal{M})$. Let $x = \int_0^\infty \lambda\, de_\lambda$ be the spectral decomposition of x. We write $q_i = 1 - e_i$, then q_i converges strongly to 0 and $q_i > q_{i+1}$, $n = 1, 2, \ldots$. Set $x_i = (xq_i x)^{\frac{1}{2}}$, $n = 1, 2, \ldots$. Since $\tau(1) < \infty$, then $x_i^2 \to 0$ in the measure topology. Thus, by Theorem 2.3.4 of [11] and $\mu_t(x_n) \le \mu_t(x)$, we have

$$\begin{aligned}\|x_n\|_{p,\omega} &= \big\|\mu_t(x_n)\big\|_{\Lambda^p_\omega(I)} \\ &= \big\|\mu_t\big(x_n^2\big)^{\frac{1}{2}}\big\|_{\Lambda^p_\omega(I)} \downarrow_n 0.\end{aligned}$$

It follows that $\|xq_n\|_{p,\omega} = \|(xq_n x)^{\frac{1}{2}}\|_{p,\omega} = \|x_n\|_{p,\omega} \to 0$, $n \to \infty$. (iii): Suppose that there exists $0 \ne x_0 \in \Lambda^p_\omega(\mathcal{M})$ such that $l(x_0) = 0$ holds for all $l \in \Lambda^p_\omega(\mathcal{M})^*$. Then, for every $y \in \Lambda^p_\omega(\mathcal{M})'$, we have $l_y(x_0) = \tau(x_0 y) = 0$. By (i), we have $\Lambda^p_\omega(\mathcal{M})' = \Gamma^q_{\widetilde{\omega}}(\mathcal{M})$, then $e_{(a,\infty)}(|x_0|) \in \Lambda^p_\omega(\mathcal{M})'$, $\forall a > 0$. Thus $\tau(x_0 e_{(a,\infty)}(|x_0|)) = l_{e_{(a,\infty)}(|x_0|)}(x_0) = 0$, $\forall a > 0$, which implies that $x_0 = 0$, contradiction. Therefore, $\Lambda^p_\omega(\mathcal{M})^*$ separates points. □

The identity in $\mathcal{M}$ is denoted by 1, and we denote by $\mathcal{D}$ a von Neumann subalgebra of $\mathcal{M}$; moreover, we let $\mathcal{E} : \mathcal{M} \to \mathcal{D}$ be the unique normal faithful conditional expectation such that $\tau \circ \mathcal{E} = \tau$. A finite subdiagonal algebra of $\mathcal{M}$ with respect to $\mathcal{E}$ (or $\mathcal{D}$) is a w^*-closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ satisfying the following conditions:
(i) $\mathcal{A} + J\mathcal{A}$ is w^*-dense in $\mathcal{M}$;
(ii) $\mathcal{E}$ is multiplicative on $\mathcal{A}$, *i.e.*, $\mathcal{E}(ab) = \mathcal{E}(a)\mathcal{E}(b)$ for all $a, b \in \mathcal{A}$;
(iii) $\mathcal{A} \cap J\mathcal{A} = \mathcal{D}$.
$\mathcal{D}$ is then called the diagonal of $\mathcal{A}$, where $J\mathcal{A} = \{x^* : x \in \mathcal{A}\}$. We say that $\mathcal{A}$ is a maximal subdiagonal algebra in $\mathcal{M}$ with respect to $\mathcal{E}$ in case that $\mathcal{A}$ is not properly contained in any other subalgebra of $\mathcal{M}$ which is subdiagonal with respect to $\mathcal{E}$. It is proved by Exel (Theorem 7, [24]) that a finite subdiagonal algebra $\mathcal{A}$ is automatically maximal. This maximality yields the following useful characterization of $\mathcal{A}$:

$$\mathcal{A} = \big\{x \in \mathcal{M} : \tau(xa) = 0, \forall a \in \mathcal{A}_0\big\},$$

where $\mathcal{A}_0 = \mathcal{A} \cap \ker\mathcal{E}$ (see [1]).

If K is a subset of $\Lambda^p_\omega(\mathcal{M})$, $[K]_{p,\omega}$ will denote the closure of K in $\Lambda^p_\omega(\mathcal{M})$ (with respect to the w^*-topology in the case of $p = \infty$). Since $\mathcal{M}$ is a finite von Neumann algebra, then $\mathcal{M} = \Lambda^\infty_\omega(\mathcal{M}) \subseteq \Lambda^p_\omega(\mathcal{M})$, $0 < p \leq \infty$.

Definition 2.2 Let $\mathcal{M}$ be a finite von Neumann algebra, we define noncommutative weighted Hardy spaces by $H^{p,\omega}(\mathcal{A}) = [\mathcal{A}]_{p,\omega}$ and $H^{p,\omega}_0(\mathcal{A}) = [\mathcal{A}_0]_{p,\omega}$.

In what follows, we will keep all previous notations throughout the paper, and C will always denote a constant, which may be different in different places. Unless otherwise stated, it will be assumed throughout that Lorentz spaces Λ^p_ω satisfy the following property: for $f \in L_0$ and $g \in \Lambda^p_\omega$ with $\int_0^t f^*(s)\,ds \leq \int_0^t g^*(s)\,ds$, we have

$$f \in \Lambda^p_\omega \quad \text{and} \quad \|f\|_{p,\omega} \leq C\|g\|_{p,\omega}. \tag{2.4}$$

For two nonnegative (possibly infinite) quantities A and B, by $A \lesssim B$ we mean that there exists a constant $C > 0$ such that $A \leq CB$.

Remark 2.1

(i) If $\omega \equiv 1$, then $H^{p,\omega}(\mathcal{A}) = H^p(\mathcal{A})$ and $H^{p,\omega}_0(\mathcal{A}) = H^p_0(\mathcal{A})$. In [25], Section 3 it is shown that for $1 \leq p \leq \infty$,

$$H^p(\mathcal{A}) = [\mathcal{A}]_p = \{x \in L^p(\mathcal{M}) : \tau(xy) = 0 \text{ for all } y \in \mathcal{A}_0\}$$

and

$$H^p_0(\mathcal{A}) = [\mathcal{A}_0]_p = \{x \in L^p(\mathcal{M}) : \tau(xy) = 0 \text{ for all } y \in \mathcal{A}\}.$$

Subsequently, Bekjan and Xu (Proposition 3.3, [5]) showed that $H^q(\mathcal{A}) = H^p(\mathcal{A}) \cap L^q(\mathcal{M})$ and $H^q_0(\mathcal{A}) = H^p_0(\mathcal{A}) \cap L^q(\mathcal{M})$, where $0 < p < q \leq \infty$.

(ii) Let $0 < p \leq q \leq \infty$. Since $\mathcal{M}$ is a finite von Neumann algebra, then $\Lambda^q_\omega(\mathcal{M}) \subseteq \Lambda^p_\omega(\mathcal{M})$.

(iii) Let $0 < r_1 < \alpha_{\Lambda^p_\omega} \leq \beta_{\Lambda^p_\omega} < r_2 \leq \infty$. It follows from Theorem 3.2 of [12] that Λ^p_ω is an interpolation space for the couple (L^{r_1}, L^{r_2}). Thus $L^{r_2} \hookrightarrow \Lambda^p_\omega \hookrightarrow L^{r_1}$, and so $L^{r_2}(\mathcal{M}) \hookrightarrow \Lambda^p_\omega(\mathcal{M}) \hookrightarrow L^{r_1}(\mathcal{M})$, where '$\hookrightarrow$' denotes a continuous embedding. Moreover, this implies that $H^{r_2}(\mathcal{A}) \hookrightarrow H^{p,\omega}(\mathcal{A}) \hookrightarrow H^{r_1}(\mathcal{A})$.

(iv) Let $1 < p < \infty$ and $\omega \in B_p$. Then Theorem A of [15] implies that Λ^p_ω satisfies property (2.4).

(v) If $1 < \alpha_{\Lambda^p_\omega} \leq \beta_{\Lambda^p_\omega} < \infty$ and Λ^p_ω satisfies property (2.4), it follows from Proposition 1 of [26] that $\mathcal{E}$ is bounded on $\Lambda^p_\omega(\mathcal{M})$.

3 Riesz and Szegö factorization of noncommutative weighted Hardy spaces

Proposition 3.1 *Let $1 < p < \infty$ and $1 < \alpha_{\Lambda^p_\omega} \leq \beta_{\Lambda^p_\omega} < \infty$. Then*

$$H^{p,\omega}(\mathcal{A}) = \{x \in \Lambda^p_\omega(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}_0\},$$
$$H^{p,\omega}_0(\mathcal{A}) = \{x \in \Lambda^p_\omega(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}\}.$$

Moreover, if $1 < \alpha_{\Lambda^p_\omega} \leq \beta_{\Lambda^p_\omega} < \infty$, $1 < p < \infty$ *and* $r < \alpha_{\Lambda^p_\omega}$, *then*

$$H^r(\mathcal{A}) \cap \Lambda^p_\omega(\mathcal{M}) = H^{p,\omega}(\mathcal{A}),$$
$$H^r_0(\mathcal{A}) \cap \Lambda^p_\omega(\mathcal{M}) = H^{p,\omega}_0(\mathcal{A}).$$

Proof Since $1 < \alpha_{\Lambda^p_\omega} \leq \beta_{\Lambda^p_\omega} < \infty$ and $\mathcal{A}_0$ is an ideal of $\mathcal{A}$, then

$$\mathcal{A} \subseteq \{x \in \Lambda^p_\omega(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}_0\}.$$

For $x \in H^{p,\omega}(\mathcal{A})$, there exist $x_n \in \mathcal{A}$ such that $\|x - x_n\|_{p,\omega} \to 0$, $n \to \infty$ and $\tau(x_n y) = 0$, $\forall y \in \mathcal{A}_0$, $n = 1, 2, \ldots$. Hence, $x_n \to x$ in the measure topology. This means that $x_n y \to xy$ in the measure topology holds for every $y \in \mathcal{A}_0$. By Remark 2.1(iii), we obtain $\Lambda^p_\omega(\mathcal{M}) \subseteq L^1(\mathcal{M})$. Using Theorem 3.5 of [10], we obtain

$$\lim_{n\to\infty} \left|\tau(x_n y - xy)\right| \leq \lim_{n\to\infty} \tau\left(|x_n y - xy|\right) \lesssim \lim_{n\to\infty} \|x_n y - xy\|_{p,\omega} = 0,$$

which implies that $\tau(xy) = 0$, and so

$$H^{p,\omega}(\mathcal{A}) \subseteq \{x \in \Lambda^p_\omega(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}_0\}.$$

Conversely, let $x \in \{x \in \Lambda^p_\omega(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}_0\}$ and $x \notin H^{p,\omega}(\mathcal{A})$. By (i) and (iii) of Proposition 2.6, there exists $y \in \Lambda^p_\omega(\mathcal{M})'$ such that $\tau(xy) \neq 0$ and $\tau(ya) = 0$, $\forall a \in H^{p,\omega}(\mathcal{A})$. Since $(f + g)^{**} \leq f^{**} + g^{**}$, $\Gamma^q_{\tilde{\omega}}$ is Banach function spaces over [0,1], where $\frac{1}{q} + \frac{1}{p} = 1$. By Corollary II. 6.7 of [27], we have $\Gamma^q_{\tilde{\omega}} \subseteq L^1$, which implies that $\Gamma^q_{\tilde{\omega}}(\mathcal{M}) \subseteq L^1(\mathcal{M})$. Combining this with Proposition 2.6, we obtain $y \in \Lambda^p_\omega(\mathcal{M})' = \Gamma^q_{\tilde{\omega}}(\mathcal{M}) \subseteq L^1(\mathcal{M})$. Note that $\tau(ya) = 0$, $\forall a \in \mathcal{A} \subseteq H^{p,\omega}(\mathcal{A})$. Then by Proposition 3.9 of [2] and Remark 2.1(i), we have $y \in H^1_0(\mathcal{A})$ and $\mathcal{E}(y) = 0$. Set $1 \leq s < \alpha_{\Lambda^p_\omega}$, then $x \in \{z \in L_s(\mathcal{M}) : \tau(za) = 0, \forall a \in \mathcal{A}_0\} = H^s(\mathcal{A})$. By Corollary 2.2 of [5], we deduce $\tau(xy) = \tau(\mathcal{E}(xy)) = \tau(\mathcal{E}(x)\mathcal{E}(y)) = 0$. This is a contradiction, and so the first equality holds.

A similar argument to the proof of above shows that

$$H^{p,\omega}_0(\mathcal{A}) \subseteq \{x \in \Lambda^p_\omega(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}\}.$$

On the other hand, let $x \in \{x \in \Lambda^p_\omega(\mathcal{M}) : \tau(xy) = 0, \forall y \in \mathcal{A}\}$. Since $\mathcal{D} \subseteq \mathcal{A}$, then $\tau(xy) = \tau(\mathcal{E}(x)y) = 0$, $\forall y \in \mathcal{D}$. It follows that $\mathcal{E}(x) = 0$. Since $x \in \Lambda^p_\omega(\mathcal{M}) \subseteq L^1(\mathcal{A})$, it follows from Corollary 2.2 of [5] that $\tau(xy) = \tau(\mathcal{E}(xy)) = \tau(\mathcal{E}(x)\mathcal{E}(y)) = 0$ holds for all $y \in \mathcal{A}_0$. This implies that $x \in H^{p,\omega}(\mathcal{A})$, and so there exist $x_n \in \mathcal{A}$, $n = 1, 2, \ldots$, such that $\|x - x_n\|_{p,\omega} \to 0$, $n \to \infty$. On the other hand, Remark 2.1(v) shows that $\mathcal{E}$ is bounded on $\Lambda^p_\omega(\mathcal{M})$. It follows from the fact $\mathcal{E}(x) = 0$ that

$$\left\|x_n - \mathcal{E}(x_n) - x\right\|_{p,\omega} \lesssim \|x_n - x\|_{p,\omega} + \left\|\mathcal{E}(x_n) - \mathcal{E}(x)\right\|_{p,\omega} \to 0, \quad n \to \infty.$$

Since $x_n - \mathcal{E}(x_n) \in \mathcal{A}_0$, then $x \in H^{p,\omega}_0(\mathcal{A})$, which implies the second equality holds.

For the third equality, it is clear that $H^r(\mathcal{A}) \cap \Lambda^p_\omega(\mathcal{M}) \supseteq H^{p,\omega}(\mathcal{A})$. Conversely, if $r \geq 1$ and $x \in H^r(\mathcal{A}) \cap \Lambda^p_\omega(\mathcal{M})$, then $x \in \{z \in \Lambda^p_\omega(\mathcal{M}) : \tau(za) = 0, \forall a \in \mathcal{A}_0\}$ since $H^r(\mathcal{A}) = \{z \in L_r(\mathcal{A}) : \tau(za) = 0, \forall a \in \mathcal{A}_0\}$ and $L_r(\mathcal{A}) \supseteq \Lambda^p_\omega(\mathcal{M})$. By the first equality, we have $x \in H^{p,\omega}(\mathcal{A})$.

On the other hand, if $r < 1$ and $x \in H^r(\mathcal{A}) \cap \Lambda_\omega^p(\mathcal{M})$, there exists $1 \le s < \alpha_{\Lambda_\omega^p}$ such that $\Lambda_\omega^s(\mathcal{M}) \subseteq L_s(\mathcal{M})$, and so, by Proposition 3.3 of [5], we have $H^r(\mathcal{A}) \cap \Lambda_\omega^p(\mathcal{M}) = H^r(\mathcal{A}) \cap L_s(\mathcal{M}) \cap \Lambda_\omega^p(\mathcal{M}) = H^s(\mathcal{A}) \cap \Lambda_\omega^p(\mathcal{M}) = H^{p,\omega}(\mathcal{A})$. Similarly, the fourth equality holds. □

Theorem 3.1 *Let $0 < p,q < \infty$ and $0 < \alpha_{\Lambda_\omega^p} \le \beta_{\Lambda_\omega^p} < \infty$. If $x \in \Lambda_\omega^p(\mathcal{M})$ is an invertible operator such that $x^{-1} \in \Lambda_\omega^q(\mathcal{M})$, then there exist a unitary $u \in \mathcal{M}$ and $h \in H^{p,\omega}(\mathcal{A})$ such that $x = uh$, $h^{-1} \in H^{q,\omega}(\mathcal{A})$.*

Proof By the fact $\alpha_{\Lambda_\omega^{rp}} = r\alpha_{\Lambda_\omega^p}$, $\beta_{\Lambda_\omega^{rp}} = r\beta_{\Lambda_\omega^p}$, $r > 0$ and $0 < \alpha_{\Lambda_\omega^p} \le \beta_{\Lambda_\omega^p} < \infty$, we have $0 < \alpha_{\Lambda_\omega^q} \le \beta_{\Lambda_\omega^q} < \infty$. First, let $1 < p,q \le \infty$, $\alpha_{\Lambda_\omega^p} > 1$, $\alpha_{\Lambda_\omega^q} > 1$, and let $x \in \Lambda_\omega^p(\mathcal{M})$ with $x^{-1} \in \Lambda_\omega^q(\mathcal{M})$. Take r_1, r_2 with $1 \le r_1 < \alpha_{\Lambda_\omega^p}$, $1 \le r_2 < \alpha_{\Lambda_\omega^q}$. By Theorem 3.1 of [5], there exist a unitary $u \in \mathcal{M}$ and $h \in H^{r_1}(\mathcal{A})$ such $h = u^*x$ and $h^{-1} \in H^{r_2}(\mathcal{A})$. It is clear that $h = u^*x \in \Lambda_\omega^p(\mathcal{M})$. It follows from Proposition 3.1 that $h \in H^{p,\omega}(\mathcal{A})$. Similarly, $h^{-1} \in H^{q,\omega}(\mathcal{A})$.

On the other hand, if $\min(p,q) \le 1$ or $\min(\alpha_{\Lambda_\omega^p}, \alpha_{\Lambda_\omega^p}) \le 1$, there exists an integer n such that $\min(np, nq) > 1$ and $\min(n\alpha_{\Lambda_\omega^p}, n\alpha_{\Lambda_\omega^q}) = \min(\alpha_{\Lambda_\omega^{np}}, \alpha_{\Lambda_\omega^{nq}}) > 1$. Let $x = v|x|$ be the polar decomposition of x. We write $x_1 = v|x|^{\frac{1}{n}}$, $x_k = |x|^{\frac{1}{n}}$, $2 \le k \le n$, then $x = x_1x_2 \cdots x_n$. Thus, $x_k \in \Lambda_\omega^{np}(\mathcal{M})$ and $x_k^{-1} \in \Lambda_\omega^{nq}(\mathcal{M})$. Therefore, there exist $u_n \in \mathcal{M}$ and $h_n \in H^{np,\omega}(\mathcal{A})$ such that $x_n = u_nh_n$ and $x_n^{-1} \in H^{nq,\omega}(\mathcal{A})$. Repeating this argument, we again get the same factorization for $x_{n-1}u_n$: $x_{n-1}u_n = u_{n-1}h_{n-1}$, and then for $x_{n-2}u_{n-1}$, and so on. In this way, we obtain the factorization $x = uh_1 \cdots h_n$, $u \in \mathcal{M}$, where $u \in \mathcal{M}$ is a unitary and $h_k \in H^{np,\omega}(\mathcal{A})$ such that $h_k^{-1} \in H^{nq,\omega}(\mathcal{A})$. We write $h = h_1 \cdots h_n$, then $x = uh$ is the desired factorization. □

Remark 3.1

(i) Let $x \in \Lambda_\omega^\infty(\mathcal{M}) = \mathcal{M}$ be an invertible operator such that $x^{-1} \in \Lambda_\omega^\infty(\mathcal{M})$. Theorem 3.1 of [5] implies that there exist a unitary $u \in \mathcal{M}$ and $h \in \mathcal{A}$ such that $x = uh$, $h^{-1} \in \mathcal{A}$.

(ii) Let $0 < p < \infty$ and $0 < \alpha_{\Lambda_\omega^p} \le \beta_{\Lambda_\omega^p} < \infty$. If $x \in \Lambda_\omega^p(\mathcal{M})$ is an invertible operator such that $x^{-1} \in \Lambda_\omega^\infty(\mathcal{M}) = \mathcal{M}$, then there exist a unitary $u \in \mathcal{M}$ and $h \in H^{p,\omega}(\mathcal{A})$ such that $x = uh$, $h^{-1} \in \mathcal{A}$.

Indeed, if $1 < p < \infty$, $1 < \alpha_{\Lambda_\omega^p}$, take $r > 0$ with $1 \le r < \alpha_{\Lambda_\omega^p}$. By Theorem 3.1 of [5], there exist a unitary $u \in \mathcal{M}$ and $h \in H^r(\mathcal{A})$ such that $h = u^*x$ and $h^{-1} \in \mathcal{A}$. Since $h = u^*x \in \Lambda_\omega^p(\mathcal{M})$, it follows from Proposition 3.1 that $h \in H^{p,\omega}(\mathcal{A})$. Therefore, by adapting the proof of Theorem 3.1, we complete the proof.

Proposition 3.2 *Let $0 < p < q < \infty$ and $0 < \alpha_{\Lambda_\omega^q} \le \beta_{\Lambda_\omega^q} < \infty$. Then*

$$H^{p,\omega}(\mathcal{A}) \cap \Lambda_\omega^q(\mathcal{M}) = H^{q,\omega}(\mathcal{A}), \qquad H_0^{p,\omega}(\mathcal{A}) \cap \Lambda_\omega^q(\mathcal{M}) = H_0^{q,\omega}(\mathcal{A}).$$

Proof Since $0 < p < q < \infty$, then $\Lambda_\omega^q(\mathcal{M}) \subseteq \Lambda_\omega^p(\mathcal{M})$, which implies that $H^{p,\omega}(\mathcal{A}) \cap \Lambda_\omega^q(\mathcal{M}) \supseteq H^{q,\omega}(\mathcal{A})$. Conversely, let $x \in H^{p,\omega}(\mathcal{A}) \cap \Lambda_\omega^q(\mathcal{M})$. We write $a = (x^*x + 1)^{\frac{1}{2}}$, then $a \in \Lambda_\omega^q(\mathcal{M})$ and $a^{-1} \in \mathcal{M}$. Applying Remark 3.1(ii) to a, there exist $u \in \mathcal{M}$ and $h \in H^{q,\omega}(\mathcal{A})$ such that $a = uh$ and $h^{-1} \in \mathcal{A}$. Since $h^*h = x^*x + 1$, then there exists $v \in \mathcal{M}$ with $\|v\| \le 1$ such that $x = vh$. Since $\alpha_{\Lambda_\omega^{rp}} = r\alpha_{\Lambda_\omega^p}$, $r > 0$ and $0 < \alpha_{\Lambda_\omega^q}$, then there exists $0 < \alpha_{\Lambda_\omega^p}$. Let $0 < r < \alpha_{\Lambda_\omega^p}$. By Remark 2.1(iii), we deduce $x \in H^{p,\omega}(\mathcal{A}) \subseteq H^r(\mathcal{A})$. Thus $v = xh^{-1} \in H^{p,\omega}(\mathcal{A}) \subseteq H^r(\mathcal{A})$. It follows from Proposition 3.3 of [5] that $v \in \mathcal{A}$. Therefore, $x \in \mathcal{A} \cdot H^{q,\omega}(\mathcal{A}) = H^{q,\omega}(\mathcal{A})$. Similarly, the second equality holds. □

Theorem 3.2 *Let* $0 < p < \infty$, $0 < q \leq \infty$ *and* $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$. *Then, for every* $x \in H^{p,\omega}(\mathcal{A})$ *and* $\varepsilon > 0$, *there exist* $y \in H^{q,\omega}(\mathcal{A})$ *and* $z \in H^{r,\omega}(\mathcal{A})$ *such that* $x = yz$, *where* $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Proof The case where $\max\{q, r\} = \infty$ is trivial. Thus we assume $\max\{q, r\} < \infty$. Let $x \in H^{p,\omega}(\mathcal{A})$ and $\varepsilon > 0$. We write $a = (x^*x + \varepsilon)^{\frac{1}{2}}$, then $x \in \Lambda_\omega^p(\mathcal{M})$ and $a^{-1} \in \mathcal{M}$. Let $v \in \mathcal{M}$ be a contraction operator such that $x = va$. Now applying Remark 3.1(ii) to $a^{\frac{p}{r}}$, we have $a^{\frac{p}{r}} = uz$, $z^{-1} \in \mathcal{M}$, where u is a unitary in $\mathcal{M}$ and $z \in H^{r,\omega}(\mathcal{A})$. Let $y = va^{\frac{p}{q}}u \in \Lambda_\omega^q(\mathcal{M})$, then $x = yz$ and so $y = xz^{-1}$. Since $x \in H^{p,\omega}(\mathcal{A})$ and $z^{-1} \in \mathcal{A}$, we have $y \in H^{p,\omega}(\mathcal{M})$. By Proposition 3.2 and the fact $p < q$, we obtain $y \in H^{q,\omega}(\mathcal{A})$. □

Remark 3.2 Let $0 < p, q \leq \infty$, $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. For every $x \in H^{p,\omega}(\mathcal{A})$ and $\varepsilon > 0$, let y, z be as in Theorem 3.2, then $\|y\|_{q,\omega}\|z\|_{r,\omega} \leq C(1 + \varepsilon)\|x\|_{p,\omega}$ for some constant C (independent of x). Indeed, the case $p = \infty$ follows from Theorem 3.4 of [5] since $\Lambda_\omega^\infty(\mathcal{M}) = \mathcal{M}$. The other case follows from Theorem 3.2.

4 Outer operators according to $H^{p,\omega}(\mathcal{A})$

Let x be a τ-measurable operator with $\lim_{t\to\infty} \mu_t(x) = 0$. The Fuglede-Kadison determinant $\Delta(x)$ is defined by

$$\Delta(x) = \exp\bigl(\tau\bigl(\log|x|\bigr)\bigr) = \exp\biggl(\int_0^\infty \log t \, d\nu_{|x|}(t)\biggr),$$

where $d\nu_{|x|}$ denotes the probability measure on $\mathbb{R}^+$ which is obtained by composing the spectral measure of $|x|$ with the trace τ. We refer the reader to [1, 8] for more information on determinant.

Proposition 4.1 *Let* $0 < p < q \leq \infty$, $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$ *and* $h \in H_\omega^q(\mathcal{A})$, *then*

(i) $[h\mathcal{A}]_{p,\omega} = H^{p,\omega}(\mathcal{A})$ *if and only if* $[h\mathcal{A}]_{q,\omega} = H^{q,\omega}(\mathcal{A})$.
(ii) $[\mathcal{A}h]_{p,\omega} = H^{p,\omega}(\mathcal{A})$ *if and only if* $[\mathcal{A}h]_{q,\omega} = H^{q,\omega}(\mathcal{A})$.
(iii) $[\mathcal{A}h\mathcal{A}]_{p,\omega} = H^{p,\omega}(\mathcal{A})$ *if and only if* $[\mathcal{A}h\mathcal{A}]_{q,\omega} = H^{q,\omega}(\mathcal{A})$.

Proof We shall prove only the third equivalence. The other cases are similar. Since $p < q$, then $H^{q,\omega}(\mathcal{A})$ is dense in $H^{p,\omega}(\mathcal{A})$. It follows from $[\mathcal{A}h\mathcal{A}]_{\Lambda_\omega^q(\mathcal{M})} = H^{q,\omega}(\mathcal{A})$ that $[\mathcal{A}h\mathcal{A}]_{\Lambda_\omega^p(\mathcal{M})} = H^{p,\omega}(\mathcal{A})$. Conversely, since $\alpha_{\Lambda_\omega^p} > 0$, there is $r > 0$ such that $\Lambda_\omega^p(\mathcal{M}) \subseteq L_r(\mathcal{M})$, and so $[\mathcal{A}h\mathcal{A}]_r = H^r(\mathcal{A})$. Using Proposition 4.1 of [5], we have $[\mathcal{A}h\mathcal{A}]_\infty = H^\infty(\mathcal{A})$, which means that $[\mathcal{A}h\mathcal{A}]_{q,\omega} = H^{q,\omega}(\mathcal{A})$. □

Definition 4.1 Let $0 < p < \infty$ and $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$. An operator $h \in H^{p,\omega}(\mathcal{A})$ is called left outer, right outer or bilaterally outer according to $[h\mathcal{A}]_{p,\omega} = H^{p,\omega}(\mathcal{A})$, $[\mathcal{A}h]_{p,\omega} = H^{p,\omega}(\mathcal{A})$ or $[\mathcal{A}h\mathcal{A}]_{p,\omega} = H^{p,\omega}(\mathcal{A})$.

Remark 4.1

(i) Proposition 4.1 justifies the relative independence of the index p in Definition 4.1.
(ii) Since $\Lambda_\omega^\infty(\mathcal{M}) = \mathcal{M}$, Proposition 4.1 of [5] and Proposition 4.1 imply that Definition 4.1 coincides with the definition in the sense of [5].

Proposition 4.2 *Let* $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$ *and* $h \in H^{p,\omega}(\mathcal{A})$.

(i) *If h is left or right outer, then $\Delta(h) = \Delta(\mathcal{E}(h))$. Conversely, if $\Delta(h) = \Delta(\mathcal{E}(h))$ and $\Delta(h) > 0$, then h is left and right outer (so bilaterally outer too).*
(ii) *If $\mathcal{A}$ is antisymmetric and h is bilaterally outer, then $\Delta(h) = \Delta(\mathcal{E}(h))$.*

Proof Since $\alpha_{\Lambda_\omega^p} > 0$, then there is $r > 0$ such that $H^{p,\omega}(\mathcal{A}) \subseteq H^r(\mathcal{A})$. Applying Theorem 4.4 of [5] to $h \in H^{p,\omega}(\mathcal{A}) \subseteq H^r(\mathcal{A})$, we obtain that (i) and (ii) hold. □

In the classical function algebra setting, one assumes that $\mathcal{D} = \mathcal{A} \cap J\mathcal{A}$ is one-dimensional, which forces $\mathcal{E} = \tau(\cdot)1$. If in our setting this is the case, then we say that $\mathcal{A}$ is antisymmetric. It is worth remarking that the antisymmetric maximal subdiagonal subalgebras of commutative von Neumann algebras are precisely the weak* Dirichlet algebras. It is clear that $\mathcal{A}$ is antisymmetric if and only if $\dim \mathcal{D} = 1$ (equivalently, $\mathcal{D} = \mathbb{C}1$).

The following corollary is a consequence of Proposition 4.2.

Corollary 4.1 *Let $h \in H^{p,\omega}(\mathcal{A})$, $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$.*
(i) *If $\Delta(h) > 0$, then h is left outer if and only if h is right outer.*
(ii) *Assume that $\mathcal{A}$ is antisymmetric, then the following properties are equivalent:*
  (a) *h is left outer;*
  (b) *h is right outer;*
  (c) *h is bilaterally outer;*
  (d) *$\Delta(\mathcal{E}(h)) = \Delta(h) > 0$.*

We will say that h is outer if it is at the same time left and right outer. If $h \in H_\omega^p(\mathcal{A})$ with $\Delta(h) > 0$, then h is outer if and only if $\Delta(h) = \Delta(\mathcal{E}(h))$. Also in the case that $\mathcal{A}$ is antisymmetric, h with $\Delta(h) > 0$ is outer if and only if it is left, right or bilaterally outer.

The following corollary is an immediate consequence of Corollary 4.7 of [5] and the proof of Proposition 4.1.

Corollary 4.2 *Let $0 < p, q \leq \infty$ and $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$. If $h \in H^{p,\omega}(\mathcal{A})$ with $h^{-1} \in H^{q,\omega}(\mathcal{A})$, then h is outer.*

Theorem 4.1 *Let $0 < p < \infty$ and $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$. If $x \in \Lambda_\omega^p(\mathcal{M})$ with $\Delta(x) > 0$, then there exist a unitary $u \in \mathcal{M}$ and an outer $h \in H^{p,\omega}(\mathcal{A})$ such that $x = uh$.*

Proof First, let $\alpha_{\Lambda_\omega^p} > 1$ and $p > 1$, there is $\alpha_{\Lambda_\omega^p} > r > 1$ such that $\mathcal{M} \subseteq \Lambda_\omega^p(\mathcal{M}) \subseteq L_r(\mathcal{M})$ and $\mathcal{A} \subseteq H^{p,\omega}(\mathcal{A}) \subseteq H^r(\mathcal{A})$. Applying Theorem 4.8 of [5] to $x \in \Lambda_\omega^p(\mathcal{M}) \subseteq L_r(\mathcal{M})$, there exist a unitary $u \in \mathcal{M}$ and an outer $h \in H^r(\mathcal{A})$ such that $x = uh$. It follows from Proposition 3.1 that $h = u^*x \in \Lambda_\omega^p(\mathcal{M}) \cap H^r(\mathcal{A}) = H^{p,\omega}(\mathcal{M})$. By adapting the proof of Theorem 3.1, we complete the proof. □

Corollary 4.3 *Let $0 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$ and $x \in \Lambda_\omega^p(\mathcal{M})$ with $0 < p < \infty$ such that $\Delta(x) > 0$, then there exist a unitary $u \in \mathcal{A}$ (inner) and an outer $h \in H^{p,\omega}(\mathcal{A})$ such that $x = uh$.*

Proof Let $\alpha_{\Lambda_\omega^p} > 1$ and $p > 1$, there exists $\alpha_{\Lambda_\omega^p} > r > 1$ such that $\mathcal{M} \subseteq \Lambda_\omega^p(\mathcal{M}) \subseteq L_r(\mathcal{M})$ and $\mathcal{A} \subseteq H^{p,\omega}(\mathcal{A}) \subseteq H^r(\mathcal{A})$. Applying Corollary 4.9 of [5] to $x \in \Lambda_\omega^p(\mathcal{M}) \subseteq L_r(\mathcal{M})$, there exist a unitary $u \in \mathcal{A}$ and an outer $h \in H^r(\mathcal{A})$ such that $x = uh$. Thus, Proposition 3.1 implies that $h = u^*x \in \Lambda_\omega^p(\mathcal{M}) \cap H^r(\mathcal{A}) = H^{p,\omega}(\mathcal{M})$. By adapting the end of the proof of Theorem 3.1, we get the desired factorization of x. For simplicity we consider only the case $\alpha_{\Lambda_\omega^p} > \frac{1}{2}$ and

$p > \frac{1}{2}$. Let $x = v|x|$ be the polar decomposition of x. Using a similar discussion of above to $|x|^{\frac{1}{2}}$, there exist a unitary $u_1 \in \mathcal{A}$ and an outer $h_1 \in H^{2p,\omega}(\mathcal{A})$ such that $|x|^{\frac{1}{2}} = u_1 h_1$ since $\Delta(|x|^{\frac{1}{2}}) > 0$. Similarly, we have $v|x|^{\frac{1}{2}} u_1 = u_2 h_2$, where $u_2 \in \mathcal{A}$ and an outer $h_2 \in H^{2p,\omega}(\mathcal{A})$. Therefore, $u = u_2$ and $h = h_2 h_1$ yield the desired factorization of x. □

Corollary 4.4 *Let $0 < p < \infty$, $0 < \alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} < \infty$ and $h \in H^{p,\omega}(\mathcal{A})$ with $\Delta(h) > 0$, then h is outer if and only if for any $x \in H^{p,\omega}(\mathcal{A})$ with $|x| = |h|$, we have $\Delta(\mathcal{E}(x)) \le \Delta(\mathcal{E}(h))$.*

Proof Let h be outer and $x \in H^{p,\omega}(\mathcal{A})$ with $|x| = |h|$. Taking $0 < r < \alpha_{\Lambda^p_\omega}$, we obtain that $x \in H^r(\mathcal{A})$. From Corollary 4.11 of [5], we get $\Delta(\mathcal{E}(x)) \le \Delta(\mathcal{E}(h))$. Conversely, since $0 < \alpha_{\Lambda^p_\omega}$, there is $0 < r < \alpha_{\Lambda^p_\omega}$ such that $H^{p,\omega}(\mathcal{A}) \subseteq H^r(\mathcal{A})$. From Corollary 4.11 of [5] and Remark 4.1(i), we obtain that h is outer. □

5 Jordan morphism

Let x be an operator, we write $\operatorname{Re} x = \frac{x+x^*}{2}$ and $\operatorname{Im} x = \frac{x-x^*}{2i}$. If $u \in \operatorname{Re}\mathcal{A}$, then $u = \operatorname{Re} x$ for some $x \in \mathcal{A}$. Let $a = x - \frac{1}{2}\mathcal{E}(x - x^*)$, it is clear that $a \in \mathcal{A}$, $u = \operatorname{Re} a$ and $\mathcal{E}(\operatorname{Im} a) = 0$. Therefore, there exists $\widetilde{u} = \operatorname{Im} a \in \operatorname{Re}\mathcal{A}$ such that $a = u + i\widetilde{u} \in \mathcal{A}$ and $\mathcal{E}(\widetilde{u}) = \mathcal{E}(\operatorname{Im} a) = 0$. By a similar discussion of [2], we have $\widetilde{u} \in \operatorname{Re}\mathcal{A}$ is unique. Thus, we can define $\widetilde{u} = \operatorname{Im} a$, where $a \in \mathcal{M}$ is the unique element of $\mathcal{M}$ with $u = \operatorname{Re} a$ and $\mathcal{E}(\operatorname{Im} a) = 0$. It is obvious that $\sim: u \mapsto \widetilde{u}$ is real linear. We called $\widetilde{u}$ the conjugate of u. We define the Herglotz transform $H : \Lambda^p_\omega(\mathcal{M}) \to \Lambda^p_\omega(\mathcal{M})$ by $H(u) = u + i\widetilde{u}$. It is clear that H is real linear.

Theorem 5.1 *Let Λ^p_ω be a fully symmetric quasi-Banach function space and $1 < \alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} < \infty$. The real linear maps*

$$\sim: \operatorname{Re}\mathcal{A} \to \operatorname{Re}\mathcal{A}, \qquad H : \operatorname{Re}\mathcal{A} \to \mathcal{A}$$

extend to real linear maps

$$\sim: \Lambda^p_\omega(\mathcal{M})^{sa} \to \Lambda^p_\omega(\mathcal{M})^{sa}, \qquad H : \Lambda^p_\omega(\mathcal{M})^{sa} \to H^{p,\omega}(\mathcal{A}).$$

If $x \in \Lambda^p_\omega(\mathcal{M})^{sa}$, then $H(x) = x + i\widetilde{x} \in H^{p,\omega}(\mathcal{A})$ and $\mathcal{E}(\widetilde{x}) = 0$. Both $\sim$ and H are bounded.

Proof Let $r_1, r_2 > 0$ with $1 < r_1 < \alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} < r_2 < \infty$. By Theorem 5.4 of [2], we have that $\sim$ and H extend to bounded real linear maps

$$\sim: L^{r_i}(\mathcal{M})^{sa} \to L^{r_i}(\mathcal{M})^{sa}, \qquad H : L^{r_i}(\mathcal{M})^{sa} \to H^{r_i}(\mathcal{A})$$

and $\mathcal{E}(\widetilde{x}) = 0$, $H(x) = x + i\widetilde{x} \in H^{r_i}(\mathcal{A})$, $i = 1, 2$. Now consider the standard complexification $\eqsim : L^{r_i}(\mathcal{M}) \to L^{r_i}(\mathcal{M})$, $i = 1, 2$, that is, $\widetilde{\widetilde{x}} = \widetilde{\operatorname{Re} x} + i\widetilde{\operatorname{Im} x}$. From the method in [2], we know that $x, y \prec x + iy$, $x, y \in L^{r_i}(\mathcal{M})^{sa}$, $i = 1, 2$. Thus,

$$\bigl\| \eqsim(x + iy) \bigr\|_{r_i} = \|\widetilde{x} + i\widetilde{y}\|_{r_i} \lesssim \|x + iy\|_{r_i},$$

and so $\eqsim$ is bounded on $L^{r_i}(\mathcal{M})$, $i = 1, 2$. Since $1 < \alpha_{\Lambda^p_\omega} \le \beta_{\Lambda^p_\omega} < \infty$, it follows from Theorem 3.7 of [12] that $\Lambda^p_\omega(\mathcal{M})$ is an interpolation space for the couple $(L^{r_1}(\mathcal{M}), L^{r_2}(\mathcal{M}))$, which implies that $\eqsim$ is bounded on $\Lambda^p_\omega(\mathcal{M})$. Thus, both $\sim$ and H are bounded. By the discussion of above we know that $H(x) \in H^{r_1}(\mathcal{M})$, $\forall x \in L^{r_1}(\mathcal{M})^{sa}$. Since $\Lambda^p_\omega(\mathcal{M}) \subseteq L^{r_1}(\mathcal{M})$

and H is bounded, we have $H(x) \in H^{r_1}(\mathcal{M})$, $\mathcal{E}(\widetilde{x}) = 0$ and $H(x) \in \Lambda_\omega^p(\mathcal{M})$ hold for all $x \in \Lambda_\omega^p(\mathcal{M})^{sa}$. Combining this with Proposition 3.1, we obtain $H(x) \in H^{p,\omega}(\mathcal{A})$. $\square$

Using the same method as that in Theorem 6.2 of [2], we obtain the following result.

Theorem 5.2 *Let Λ_ω^p be a fully symmetric quasi-Banach function space and $1 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$, then*

$$\Lambda_\omega^p(\mathcal{M}) = H_0^{p,\omega}(\mathcal{A}) \oplus \Lambda_\omega^p(\mathcal{D}) \oplus JH_0^{p,\omega}(\mathcal{A}).$$

The relevant projections are $x \mapsto \frac{1}{2}[x + i\overline{\widetilde{x}} - \mathcal{E}(x)]$; $x \mapsto \mathcal{E}(x)$; $x \mapsto \frac{1}{2}[x - i\overline{\widetilde{x}} - \mathcal{E}(x)]$, where $\overline{\sim} : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_\omega^p(\mathcal{M})$ is the standard complexification of $\sim$, that is, $\overline{\widetilde{x}} = \widetilde{\mathrm{Re}\, x} + i\widetilde{\mathrm{Im}\, x}$.

Let B_1, B_2 be two Banach algebras, a linear map $\varphi : B_1 \to B_2$ is a homomorphism if $\varphi(ab) = \varphi(a)\varphi(b)$, $a, b \in B_1$. We say that a linear map $\varphi : B_1 \to B_2$ is an anti-morphism if $\varphi(ab) = \varphi(b)\varphi(a)$, $a, b \in B_1$. Given a unital Banach algebra B with identity 1, under the term irreducible representation of B, we understand a continuous homomorphism $\pi : B \to \mathcal{B}(X)$, where $\mathcal{B}(X)$ is the set of all bounded linear operators on some Banach space X such that $\pi(B)$ is an irreducible subalgebra of $\mathcal{B}(X)$ in the sense of admitting only trivial invariant subspaces. A Jordan morphism between two von Neumann algebras $\mathcal{M}_1$ and $\mathcal{M}_2$ is a linear mapping $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ which preserves the Jordan product, *i.e.*, $\varphi(ab + ba) = \varphi(a)\varphi(b) + \varphi(b)\varphi(a)$ for all $a, b \in \mathcal{M}_1$. For further information, we refer the reader to [7].

Lemma 5.1 *Let $1 \leq p < \infty$ and let $\varphi : \mathcal{A} \to \mathcal{A}$ be a Jordan morphism such that $\pi \circ \varphi$ is either a morphism or an anti-morphism for each irreducible representation π of $\mathcal{M}$. If φ extends to a bounded map $\varphi : H^{p,\omega}(\mathcal{A}) \to H^{p,\omega}(\mathcal{A})$ with norm $\|\varphi\|_{p,\omega}$, then for any integer $0 < k \leq p$, φ extends to a map $\varphi : H^{\frac{p}{k},\omega}(\mathcal{A}) \to H^{\frac{p}{k},\omega}(\mathcal{A})$ with norm not exceeding $2C^{2k-1}(\|\varphi\|_{p,\omega})^k$, where C is a constant.*

Proof Let $\varepsilon > 0$ and $x \in \mathcal{A}$ be given. Using Remark 3.2, we obtain $h_1, \ldots, h_k \in \mathcal{A}$ so that $x = h_1 h_2 \cdots h_k$ with

$$\prod_{n=1}^{k} \|h_n\|_{p,\omega} \leq \bigl(C(1+\varepsilon)\bigr)^{k-1} \|x\|_{\frac{p}{k},\omega}.$$

By Lemma 4.5 of [7], we have

$$\left|\varphi\left(\prod_{n=1}^{k} h_n\right)\right| \leq \left|\prod_{n=1}^{k} \varphi(h_n)\right| + \prod_{n=1}^{k} \bigl|\varphi(h_{k+1-n})\bigr|.$$

Thus, by the fact that Λ_ω^p is a fully symmetric quasi-Banach function space, we deduce

$$\begin{aligned}\|\varphi(x)\|_{\frac{p}{k},\omega} &= \left\|\varphi\left(\prod_{n=1}^{k} h_n\right)\right\|_{\frac{p}{k},\omega} \leq \left\|\left|\prod_{n=1}^{k} \varphi(h_n)\right| + \prod_{n=1}^{k} \bigl|\varphi(h_{k+1-n})\bigr|\right\|_{\frac{p}{k},\omega} \\ &\leq C\left(\left\|\prod_{n=1}^{k} \varphi(h_n)\right\|_{\frac{p}{k},\omega} + \left\|\prod_{n=1}^{k} \bigl|\varphi(h_{k+1-n})\bigr|\right\|_{\frac{p}{k},\omega}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq 2C^k \prod_{n=1}^{k} \|\varphi(h_n)\|_{p,\omega} \leq 2C^k \big(\|\varphi\|_{p,\omega}\big)^k \prod_{n=1}^{k} \|h_n\|_{p,\omega} \\
&\leq 2C^k \big(\|\varphi\|_{p,\omega}\big)^k \big(C(1+\varepsilon)\big)^{k-1} \|x\|_{\frac{p}{k},\omega}.
\end{aligned}
$$

This implies that the required estimate holds. □

Lemma 5.2 *Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a continuous identity-preserving Jordan morphism extending continuously to a map $\varphi : H^{p,\omega}(\mathcal{A}) \to H^{p,\omega}(\mathcal{A})$, $1 \leq p < \infty$ with norm $\|\varphi\|_{p,\omega}$. For any $k \in \mathbb{N}$ and any $x \in \mathcal{A}$ such that $\varphi(x)$ is normal, we have $\|\varphi(x)\|_{pk,\omega} \leq (\|\varphi\|_{p,\omega})^{\frac{1}{k}} \|x\|_{pk,\omega}$. Let $1 < p < \infty$ and $1 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$. If, in addition, φ is contractive on $\mathcal{A}$ and Λ_ω^p is a fully symmetric quasi-Banach function space, it extends to a unique bounded hermitian map $\widetilde{\varphi}$ on $\Lambda_\omega^p(\mathcal{M})$ which is positive on $\mathcal{A} + J\mathcal{A} \subseteq \Lambda_\omega^p(\mathcal{M})$. If $\widetilde{\varphi}$ is positive on all of $\Lambda_\omega^p(\mathcal{M})$, we get*

$$
\|\varphi(x)\|_{p2^k,\omega} \leq C2^{\frac{1}{2}} \big(\|\varphi\|_{p,\omega}\big)^{\frac{1}{2^k}} \|x\|_{p2^k,\omega}, \quad k \in \mathbb{N}, x \in \mathcal{A}, \tag{5.1}
$$

where C is a constant. If, moreover, $\widetilde{\varphi}$ satisfies the condition

$$
\widetilde{\varphi}\big(|x^*|^{2^k} + |x|^{2^k}\big) \geq \big|\widetilde{\varphi}(x^*)\big|^{2^k} + \big|\widetilde{\varphi}(x)\big|^{2^k}, \quad k \in \mathbb{N}, x \in \mathcal{M}, \tag{5.2}
$$

we obtain the estimate

$$
\|\varphi(x)\|_{p2^k,\omega} \leq C\big(2\|\varphi\|_{p,\omega}\big)^{\frac{1}{2^k}} \|x\|_{p2^k,\omega}, \quad x \in \mathcal{A}, k \in \mathbb{N}, \tag{5.3}
$$

where C is a constant.

Let B be a linear complement of $\mathcal{D}$ in $\mathcal{A}$. It is well known that $J\mathcal{A} = \mathcal{D} \oplus JB$ and hence that $\mathcal{A} + J\mathcal{A} = B \oplus \mathcal{D} \oplus JB$. Let $\varphi : \mathcal{A} \to \mathcal{M}$ be a contractive identity-preserving Jordan morphism, we may extend φ to a map $\widetilde{\varphi}$ on $\mathcal{A} + J\mathcal{A}$ by setting $\varphi = \widetilde{\varphi}$ on $\mathcal{A}$ and defining the action on JB by $\widetilde{\varphi}(x) = \varphi(x^*)^*$, $x \in JB$. Since φ is order-preserving on $\mathcal{D}$, it follows that $\widetilde{\varphi}$ preserves adjoints. Indeed, $\widetilde{\phi}$ is order-preserving, positive and contractive on $\mathcal{A} + J\mathcal{A}$. For further information, we refer the reader to [7].

Proof Our proof follows Labuschagne's argument of (Lemma 4.8, [7]). Given some $x \in \mathcal{A}$ such that $\varphi(x)$ is normal, then

$$
\begin{aligned}
\|\varphi(x)\|_{pk,\omega} &= \big(\big\||\varphi(x)|^k\big\|_{p,\omega}\big)^{\frac{1}{k}} = \big(\big\||\varphi(x)^k|\big\|_{p,\omega}\big)^{\frac{1}{k}} \\
&= \big(\big\|\varphi(x^k)\big\|_{p,\omega}\big)^{\frac{1}{k}} \leq \big(\|\varphi\|_{p,\omega}\big\|x^k\big\|_{p,\omega}\big)^{\frac{1}{k}} \\
&\leq \big(\|\varphi\|_{p,\omega}\big)^{\frac{1}{k}} \|x\|_{pk,\omega}, \quad k \in \mathbb{N}.
\end{aligned}
$$

Let $1 < p < \infty$, $1 < \alpha_{\Lambda_\omega^p} \leq \beta_{\Lambda_\omega^p} < \infty$, and let φ be an identity-preserving contractive Jordan morphism on $\mathcal{A}$. From the preceding discussion (above the proof of this lemma), φ uniquely extends to a positive map on $\mathcal{A} + J\mathcal{A}$. Moreover, φ then acts positively on the von Neumann algebra $\mathcal{D}$. By Theorem 5.2, we have $H^{p,\omega}(\mathcal{A}) = H_0^{p,\omega}(\mathcal{A}) \oplus \Lambda_\omega^p(\mathcal{D})$. Since $\mathcal{D}$ is dense in $\Lambda_\omega^p(\mathcal{D})$, then the continuous action of φ on $H^{p,\omega}(\mathcal{A})$ (and hence also on $L^p(\mathcal{M})$)

ensures that the unique extension of φ to $H^{p,\omega}(\mathcal{A})$ acts positively on $\Lambda^p_\omega(\mathcal{D})$. The extension of φ to a hermitian map $\widetilde{\varphi}$ on $\Lambda^p_\omega(\mathcal{M})$ can now be defined by

$$\widetilde{\varphi}(x) = \varphi\left(x^*\right)^*, \quad x \in JH_0^{p,\omega}(\mathcal{A}) \quad \text{and} \quad \widetilde{\varphi}(x) = \varphi(x), \quad x \in H^{p,\omega}(\mathcal{A}). \tag{5.4}$$

Since this construction canonically contains that construction in the discussion above this lemma, $\widetilde{\varphi}$ constructed as (5.4) will restrict to $\mathcal{A} + J\mathcal{A}$, which yields precisely the map in the preceding discussion (above the proof of this lemma). From Theorem 5.2, the $\|\cdot\|_{p,\omega}$ boundedness of $\widetilde{\varphi}$ follows from the $\|\cdot\|_{p,\omega}$ boundedness of φ and of the conjugate linear map $x \to x^*$. Let $1 \leq q < \infty$, we write $|x|_q = (\frac{1}{2}(|x|^q + |x^*|^q))^{\frac{1}{q}}$. It is well known that $|x|^q \leq 2|x|_q^q$, and so $|x| \leq 2^{\frac{1}{q}}|x|_q$. If $\widetilde{\varphi}$ acts positively on all of $\Lambda^p_\omega(\mathcal{M})$, it will surely map $\|\cdot\|_\infty$ boundedly $\mathcal{M}$ into $\mathcal{M}$. By Lemma 7.3 of [28], we get

$$\widetilde{\varphi}(x)^*\widetilde{\varphi}(x) + \widetilde{\varphi}(x)\widetilde{\varphi}(x)^* \leq \widetilde{\varphi}\left(x^*x + xx^*\right), \quad x \in \mathcal{M}.$$

Since $\widetilde{\varphi}(y)^2 \leq \widetilde{\varphi}(y^2)$ for each $y \in \mathcal{M}^+$, we have

$$\left\|\widetilde{\varphi}(y)\right\|_{p2^k,\omega} = \left\|\widetilde{\varphi}(y)^2\right\|_{p2^{k-1},\omega}^{\frac{1}{2}} \leq \left\|\widetilde{\varphi}\left(y^2\right)\right\|_{p2^{k-1},\omega}^{\frac{1}{2}} \leq \left\|\widetilde{\varphi}\left(y^{2^k}\right)\right\|_{p,\omega}^{\frac{1}{2^k}}, \quad k \in \mathbb{N}, y \in \mathcal{M}^+. \tag{5.5}$$

Now note that

$$\left|\widetilde{\varphi}(x)\right|_2^2 = \frac{1}{2}\left(\left|\widetilde{\varphi}(x)\right|^2 + \left|\widetilde{\varphi}\left(x^*\right)\right|^2\right) \leq \widetilde{\varphi}\left(|x|^2 + \left|x^*\right|^2\right) = \widetilde{\varphi}\left(|x|_2^2\right), \quad x \in \mathcal{M},$$

which implies that

$$\left\|\left|\widetilde{\varphi}(x)\right|_2\right\|_{p2^k,\omega} = \left\|\left|\widetilde{\varphi}(x)\right|_2^2\right\|_{p2^{k-1},\omega}^{\frac{1}{2}} \leq \left\|\widetilde{\varphi}\left(|x|_2^2\right)\right\|_{p2^{k-1},\omega}^{\frac{1}{2}}.$$

Combining this with (5.5), we get

$$\left\|\left|\widetilde{\varphi}(x)\right|_2\right\|_{p2^k,\omega} \leq \left(\left(\left\|\widetilde{\varphi}\left(\left(|x|_2^2\right)^{2^{k-1}}\right)\right\|_{p,\omega}\right)^{\frac{1}{2^{k-1}}}\right)^{\frac{1}{2}} = \left(\left\|\widetilde{\varphi}\left(|x|_2^{2^k}\right)\right\|_{p,\omega}\right)^{\frac{1}{2^k}}, \quad x \in \mathcal{M}, k \in \mathbb{N}.$$

Consequently, given $x \in \mathcal{A}$, the positivity of $\widetilde{\varphi}$ and the fact that $|\varphi(x)| \leq 2^{\frac{1}{2}}|\varphi(x)|_2$ imply that

$$\begin{aligned}
\left\|\varphi(x)\right\|_{p2^k,\omega} &= \left\|\left|\varphi(x)\right|\right\|_{p2^k,\omega} \leq 2^{\frac{1}{2}}\left\|\left|\varphi(x)\right|_2\right\|_{p2^k,\omega} \\
&\leq 2^{\frac{1}{2}}\left\|\left|\varphi(x)\right|_2^{2^k}\right\|_{p,\omega}^{\frac{1}{2^k}} \leq 2^{\frac{1}{2}}\left\|\varphi\left(|x|_2^{2^k}\right)\right\|_{p,\omega}^{\frac{1}{2^k}} \\
&\leq 2^{\frac{1}{2}}\|\varphi\|_{p,\omega}^{\frac{1}{2^k}}\left\||x|_2\right\|_{p2^k,\omega} \leq C2^{\frac{1}{2}}\|\varphi\|_{p,\omega}^{\frac{1}{2^k}}\|x\|_{p2^k,\omega}.
\end{aligned}$$

The final inequality is a consequence of Lemma 2.1 of [29]. Observe that $|x| \leq 2^{\frac{1}{2^k}}|x|_{2^k}$, a similar proof using $|x|_{2^k}$ instead of $|x|_2$ will suffice in the case that $\widetilde{\varphi}$ satisfies condition (5.2). □

Lemma 5.3 *Let $1 < p \leq \infty$ be given and let (q_n) be a sequence of reals in $[1,p)$ increasing to p. Then $x \in \Lambda^p_\omega(\mathcal{M})$ if and only if $x \in \bigcap_{n=1}^\infty \Lambda^{q_n}_\omega \subseteq \Lambda^p_\omega(\mathcal{M})$, and $\sup_n \|x\|_{q_n,\omega} < \infty$. In this case $\|x\|_{p,\omega} = \lim_{n\to\infty} \|x\|_{q_n,\omega}$.*

Proof Let $x \in \Lambda_\omega^p(\mathcal{M}) \subseteq \Lambda_\omega^{q_n}(\mathcal{M})$, then

$$\begin{aligned}\|x\|_{q_n,\omega}^{q_n} &= \int_0^1 \mu_t(x)^{q_n}\omega(t)\,dt \le \|1\|_{\frac{p}{p-q_n},\omega}\big\|\mu_t(x)^{q_n}\big\|_{\frac{p}{q_n},\omega} \\ &= \|x\|_{p,\omega}^{q_n} W(1)^{q_n(\frac{1}{q_n}-\frac{1}{p})}.\end{aligned}$$

Thus, $\limsup_{n\to\infty}\|x\|_{q_n,\omega} \le \|x\|_{p,\omega}$. By the Levi lemma and the dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}\|x\|_{q_n,\omega}^{q_n} &= \lim_{n\to\infty}\int_{\{t\in[0,1]:\mu_t(x)>1\}}\mu_t(x)^{q_n}\,dt + \lim_{n\to\infty}\int_{\{t\in[0,1]:\mu_t(x)\le 1\}}\mu_t(x)^{q_n}\,dt \\ &= \int_{\{t\in[0,1]:\mu_t(x)>1\}}\mu_t(x)^{p}\,dt + \int_{\{t\in[0,1]:\mu_t(x)\le 1\}}\mu_t(x)^{p}\,dt = \|x\|_{p,\omega}^p.\end{aligned}$$ □

Lemma 5.4 *Let $1 \le p < \infty$ be given and let (q_n) be a sequence of reals in (p,∞) decreasing to p. Given $x \in \bigcup_{n=1}^\infty \Lambda_\omega^{q_n} \subseteq \Lambda_\omega^p(\mathcal{M})$, we have $\|x\|_{p,\omega} = \lim_{n\to\infty}\|x\|_{q_n,\omega}$.*

Proof Let $x \in \Lambda_\omega^{q_n} \subseteq \Lambda_\omega^{q_{n+1}} \subseteq \Lambda_\omega^p(\mathcal{M})$. Let $E_1 = \{t \in [0,1] : \mu_t(x) \ge 1\}$ and $E_2 = \{t \in [0,1] : \mu_t(x) < 1\}$, then $\mu_t(x)^p \le \mu_t(x)^{q_n}$, $t \in E_1$ and $\mu_t(x)^p \ge \mu_t(x)^{q_n}$, $t \in E_2$. This implies that

$$\lim_{n\to\infty}\int_{E_2}\mu_t(x)^{q_n}\omega(t)\,dt = \int_{E_2}\mu_t(x)^p\omega(t)\,dt$$

and

$$\lim_{n\to\infty}\int_{E_1}\big(\mu_t(x)^{q_n} - \mu_t(x)^p\big)\omega(t)\,dt = \int_{E_1}0\omega(t)\,dt = 0,$$

which yield the desired result. □

Using the same method as that in Theorem 4.12 of [7], we obtain the following result.

Theorem 5.3 *Let Λ_ω^p be a fully symmetric quasi-Banach function space, and let $\varphi : \mathcal{A} \to \mathcal{A}$ be a contractive identity-preserving Jordan morphism such that:*

(a) *$\pi \circ \varphi$ is either a homomorphism or an anti-morphism for each irreducible representation of $\mathcal{M} \supseteq \mathcal{A}$; and*

(b) *for some $1 < p < \infty$, φ extends to a bounded map $\varphi : H^{p,\omega}(\mathcal{A}) \to H^{p,\omega}(\mathcal{A})$.*

If the canonical hermitian extension $\widetilde{\varphi}$ to all of $\Lambda_\omega^p(\mathcal{M})$ (see Lemma 5.2) is even positive, then, for any other $1 \le q < \infty$, if $1 < \alpha_{\Lambda_\omega^q} \le \beta_{\Lambda_\omega^q} < \infty$, φ will induce a unique closed operator $\overline{\varphi} : D(\overline{\varphi}) \subseteq H^{q,\omega}(\mathcal{A}) \to H^{q,\omega}(\mathcal{A})$ with the following properties:

(i) *$D(\overline{\varphi}) \supseteq \bigcup_{r>q} H^{r,\omega}(\mathcal{A})$;*

(ii) *for any $r > q$, $\overline{\varphi}$ restricts to a bounded map $\overline{\varphi} : H^{r,\omega}(\mathcal{A}) \to H^{q,\omega}(\mathcal{A})$.*

(iii) *for any $q > s \ge 1$, $\overline{\varphi}$ extends uniquely to a bounded map $\overline{\varphi} : H^{q,\omega}(\mathcal{A}) \to H^{s,\omega}(\mathcal{A})$.*

If $\widetilde{\varphi} : \Lambda_\omega^p(\mathcal{M}) \to \Lambda_\omega^p(\mathcal{M})$ satisfies the slightly stronger condition (5.2) in Lemma 5.2, then $\overline{\varphi} : H^{q,\omega}(\mathcal{A}) \to H^{q,\omega}(\mathcal{A})$ itself is a bounded everywhere defined operator.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by the corresponding author YZH. YZH and JJS prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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