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The mean consistency of wavelet density estimators

Zijuan Geng and Jinru Wang*

*Correspondence:
wangjinru@bjut.edu.cn
Department of Applied Mathematics, Beijing University of Technology, Beijing, 100124, P.R. China

Abstract

The wavelet estimations have made great progress when an unknown density function belongs to a certain Besov space. However, in many practical applications, one does not know whether the density function is smooth or not. It makes sense to consider the mean L_p-consistency of the wavelet estimators for $f \in L_p$ ($1 \leq p \leq \infty$). In this paper, the authors will construct wavelet estimators and analyze their $L_p(\mathbb{R})$ performance. They prove that, under mild conditions on the family of wavelets, the estimators are shown to be L_p ($1 \leq p \leq \infty$)-consistent for both noiseless and additive noise models.

Keywords: density estimation; wavelet; L_p norm; consistency; random noise

1 Introduction

Wavelet analysis plays important roles in both pure and applied mathematics such as signal processing, image compressing, and numerical solutions. One of the important applications is to estimate an unknown density function based on random samples [1–3]. Optimal convergence rate and consistency are two basic asymptotic criteria of the quality for an estimator. Some perfect achievements have been made for the wavelet estimation in L_p norm by Donoho *et al.* [4] *etc.*, when an unknown density function belongs to Besov spaces. However, in many practical applications, we do not know whether the density function is smooth or not [5]. Therefore, it is natural to consider the mean consistency of the wavelet estimators, which means $E\|f - \hat{f}_n\|_p$ ($1 \leq p \leq \infty$) converges to zero as the sample size n tends to infinity.

In 2005, Chacón and Rodríguez-Casal [6] discussed the mean L_1-consistency of the wavelet estimator based on random samples without any noise. However, in practice, the observed samples are contaminated by random noises. Devroye [7] proved the mean consistency of the kernel estimator in L_1 norm. Liu and Taylor [8] investigated L_∞-consistency of the kernel estimator. Ramírez and Vidakovic [9] proposed linear and nonlinear wavelet estimators and showed that they are L_2-consistent.

This paper studies the mean L_p-consistency of the wavelet estimator. In Section 2, we briefly describe the preliminaries on wavelet scaling functions and orthogonal projection kernels. In Section 3, for the classical model, the mean L_p-consistency is given, which generalizes Chacón's theorem [6]. The last section deals with the L_p-consistency for the additive noise model.

2 Wavelet scaling function and orthonormal projection kernel

In this section, we shall recall some useful and well-known concepts and lemmas. As usual, $L_p(\mathbb{R})$, $p \geq 1$, denotes the classical Lebesgue space on the real line $\mathbb{R}$.

Definition 2.1 (see [10]) A multi-resolution analysis (MRA) of $L_2(\mathbb{R})$ is a set of increasing, closed linear subspaces $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$, called scaling spaces, satisfying:

(i) $\bigcap_{-\infty}^{\infty} V_j = \{0\}$, $\overline{\bigcup_{-\infty}^{\infty} V_j} = L_2(\mathbb{R})$;
(ii) $f(\cdot) \in V_0$ if and only if $f(2^j \cdot) \in V_j$ for all $j \in \mathbb{Z}$;
(iii) $f(\cdot) \in V_0$ if and only if $f(\cdot - k) \in V_0$ for all $k \in \mathbb{Z}$;
(iv) there exists a function $\varphi(\cdot) \in V_0$ such that $\{\varphi(\cdot - k)\}$ is an orthonormal basis in V_0. The function $\varphi(\cdot)$ is called the scaling function.

It is easy to show that $\{\varphi_{jk}(x), k \in \mathbb{Z}\}$ forms an orthonormal basis in V_j, where $\varphi_{jk}(x) = 2^{j/2}\varphi(2^j x - k)(x)$, $j, k \in \mathbb{Z}$.

Condition S There exists a bounded nonincreasing function $\Phi(\cdot)$ such that $\int \Phi(|u|)\,du < \infty$, and $|\varphi(u)| \leq \Phi(|u|)$ (a.e.).

Condition S is not very restrictive. For example, the Meyer scaling functions satisfy that condition; compactly supported and bounded scaling functions do as well. Furthermore, Condition S implies that $\varphi \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ and $\operatorname{ess\,sup} \sum_k |\varphi(x-k)| < \infty$. We denote $\theta_\varphi(x) = \sum_k |\varphi(x-k)|$.

The following lemmas are taken from [2], which will be used later on.

Lemma 2.2 *If the scaling function φ satisfies $\operatorname{ess\,sup} \sum_{k\in\mathbb{Z}} |\varphi(x-k)| < \infty$, then for any sequence $\{\lambda_k\}_{k\in\mathbb{Z}} \in l_p$, one has $C_1\|\lambda\|_{l_p} 2^{(\frac{j}{2}-\frac{j}{p})} \leq \|\sum_k \lambda_k \varphi_{j,k}\|_p \leq C_2\|\lambda\|_{l_p} 2^{(\frac{j}{2}-\frac{j}{p})}$, where $C_1 = (\|\theta_\varphi\|_\infty^{\frac{1}{p}} \|\varphi\|_1^{\frac{1}{q}})^{-1}$, $C_2 = (\|\theta_\varphi\|_\infty^{\frac{1}{q}} \|\varphi\|_1^{\frac{1}{p}})^{-1}$, $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.*

Definition 2.3 (see [2]) If the scaling function φ satisfies $\operatorname{ess\,sup} \sum_k |\varphi(x-k)| < \infty$, the kernel function

$$K(x,y) = \sum_k \varphi(x-k)\varphi(y-k)$$

is called orthonormal projection kernel associated with φ.

For $f \in L_p(\mathbb{R})$ $(1 \leq p \leq \infty)$, if $\operatorname{ess\,sup} \sum_k |\varphi(x-k)| < \infty$, it is not hard to show that

$$\int K_j(x,y)f(y)\,dy = K_j f = \sum_k \alpha_{jk}\varphi_{jk}(x),$$

where $K_j(x,y) = 2^j K(2^j x, 2^j y)$, $\alpha_{jk} = \int \varphi_{jk}(x) f(x)\,dx$.

Lemma 2.4 *If the scaling function φ satisfies Condition* S, *then*

(i) $\int K(x,y)\,dy = 1$ *(a.e.)*;
(ii) $|K(x,y)| \leq C_1\Phi(\frac{|x-y|}{C_2})$ *(a.e.), where C_1, C_2 are positive constants depending on Φ.*

Let $F(x) = C_1\Phi(\frac{|x|}{C_2})$, then $F \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ and $|K(x,y)| \leq F(x-y)$ (a.e.).

Lemma 2.5 *If the scaling function φ satisfies Condition* S, *then for $f \in L_p(\mathbb{R})$, $1 \le p < \infty$, one has*

$$\lim_{j\to\infty} \|K_j f - f\|_p = 0.$$

The above result is also true if $f \in L_\infty(\mathbb{R})$ is uniformly continuous.

Lemma 2.6 (Rosenthal's inequality) *Let $X_1, \ldots, X_n$ be independent random variables such that $E(X_i) = 0$ and $|X_i| < M$, then there exists a constant $C(p) > 0$ such that*

$$\text{(i)}\quad E\left(\left|\sum_{i=1}^n X_i\right|^p\right) \le C(p)\left(M^{p-2}\sum_{i=1}^n E(X_i^2) + \left(\sum_{i=1}^n E(X_i^2)\right)^{\frac{p}{2}}\right), \quad p > 2,$$

$$\text{(ii)}\quad E\left(\left|\sum_{i=1}^n X_i\right|^p\right) \le C(p)\left(\sum_{i=1}^n E(X_i^2)\right)^{\frac{p}{2}}, \quad 0 < p \le 2.$$

3 Mean consistency for L_p norm

In this section, based on the random sample without noise, we shall construct the wavelet estimator and give its L_p-consistency.

Let $X_1, X_2, \ldots, X_n$ are independent identically distributed (i.i.d.) random samples without noise, φ be compactly supported scaling function, the wavelet estimator be defined as follows:

$$\hat{f}_n(x) = \sum_k \hat{\alpha}_{jk}\varphi_{jk}(x), \qquad \hat{\alpha}_{jk} = \frac{1}{n}\sum_{i=1}^n \varphi_{jk}(X_i). \tag{1}$$

Obviously, one gets $E\hat{\alpha}_{jk} = \frac{1}{n}\sum_{i=1}^n \int \varphi_{jk}(x)f(x)\,dx = \alpha_{jk}$. On the other hand, one obtains

$$\begin{aligned}\hat{f}_n(x) &= \sum_k \hat{\alpha}_{jk}\varphi_{jk}(x)\\ &= \sum_k\left(\frac{1}{n}\sum_{i=1}^n \varphi_{jk}(X_i)\right)\varphi_{jk}(x)\\ &= \frac{1}{n}\sum_{i=1}^n\sum_k \varphi_{jk}(X_i)\varphi_{jk}(x)\\ &= \frac{1}{n}\sum_{i=1}^n K_j(x, X_i).\end{aligned} \tag{2}$$

Theorem 3.1 *Let a scaling function $\varphi(x)$ be compactly supported and bounded, $\hat{f}_n(x)$ be the wavelet estimator defined in* (1). *If we take $2^j \sim n^{\frac{1}{2}}$, then for any $f \in L_p(\mathbb{R})$, $1 \le p < \infty$, one has*

$$\lim_{n\to\infty} E\|f - \hat{f}_n\|_p = 0. \tag{3}$$

Note The notation $A \lesssim B$ indicates that $A \le cB$ with a positive constant c, which is independent of A and B. If $A \lesssim B$ and $B \lesssim A$, we write $A \sim B$.

Proof Due to $(E\|f-\hat{f}_n\|_p)^p \le E\|f-\hat{f}_n\|_p^p$, one only needs to consider $E\|f-\hat{f}_n\|_p^p$.

Firstly, thanks to the triangular inequality and convexity inequality, one can decompose $E\|f-\hat{f}_n\|_p^p$ into a bias term and a stochastic term, respectively. That is,

$$\begin{aligned} E\|f-\hat{f}_n\|_p^p &= E\|f-E\hat{f}_n+E\hat{f}_n-\hat{f}_n\|_p^p \\ &\le E\big(\|f-E\hat{f}_n\|_p+\|E\hat{f}_n-\hat{f}_n\|_p\big)^p \\ &\le 2^{p-1}\big(\|f-E\hat{f}_n\|_p^p+E\|\hat{f}_n-E\hat{f}_n\|_p^p\big). \end{aligned}$$

(i) For the bias term $\|f-E\hat{f}_n\|_p^p$, one has

$$\begin{aligned} E\hat{f}_n(x) &= E\frac{1}{n}\sum_{i=1}^n K_j(x,X_i) = EK_j(x,X_1) \\ &= \int K_j(x,y)f(y)\,dy = K_jf(x). \end{aligned}$$

Since $\varphi(x)$ satisfies Condition S, taking $2^j \sim n^{\frac{1}{2}}$, due to Lemma 2.4 and Lemma 2.5, one gets

$$\lim_{n\to\infty}\|f-E\hat{f}_n\|_p = \lim_{n\to\infty}\|f-K_jf\|_p = 0.$$

(ii) For the stochastic term $E\|\hat{f}_n-E\hat{f}_n\|_p^p$, one can estimate it as follows:

$$\begin{aligned} E\|\hat{f}_n-E\hat{f}_n\|_p^p &= E\int|\hat{f}_n-E\hat{f}_n|^p\,dx \\ &= \int E|\hat{f}_n-E\hat{f}_n|^p \\ &= \int E\left|\frac{1}{n}\sum_{i=1}^n K_j(x,X_i)-E\frac{1}{n}\sum_{i=1}^n K_j(x,X_i)\right|^p dx \\ &= \frac{1}{n^p}\int E\left|\sum_{i=1}^n\big(K_j(x,X_i)-EK_j(x,X_i)\big)\right|^p dx \\ &= \frac{1}{n^p}\int E\left|\sum_{i=1}^n Y_i\right|^p dx. \end{aligned}$$

Denote $Y_i = K_j(x,X_i)-EK_j(x,X_i)$, then $\{Y_i\}$ are i.i.d. samples, and $EY_i = 0$. One obtains

$$\begin{aligned} |Y_i| &= \big|K_j(x,X_i)-EK_j(x,X_i)\big| \le \big|K_j(x,X_i)\big| + \big|EK_j(x,X_i)\big| \\ &\le \left|2^j\sum_k\varphi\big(2^jx-k\big)\varphi\big(2^jX_i-k\big)\right| + \int\left|2^j\sum_k\varphi\big(2^jx-k\big)\varphi\big(2^jy-k\big)\right|f(y)\,dy \\ &\le 2^j\|\varphi\|_\infty\|\theta_\varphi\|_\infty + 2^j\|\varphi\|_\infty\|\theta_\varphi\|_\infty\int f(x)\,dx \\ &\lesssim 2^{j+1}. \end{aligned}$$

(i) For $2 \le p < \infty$, Rosenthal's inequality, Lemma 2.6, tells us that

$$E\left(\left|\sum_{i=1}^{n} Y_i\right|^p\right) \le C(p)\left(\left(2^{j+1}\right)^{p-2}\sum_{i=1}^{n} E\left(Y_i^2\right) + \left(\sum_{i=1}^{n} E\left(Y_i^2\right)\right)^{p/2}\right)$$
$$\lesssim \left(2^{j+1}\right)^{p-2}\sum_{i=1}^{n} E\left(Y_i^2\right) + \left(\sum_{i=1}^{n} E\left(Y_i^2\right)\right)^{p/2}$$

and

$$\begin{aligned}\left(\sum_{i=1}^{n} E\left(Y_i^2\right)\right)^{p/2} &= \left(nE(Y_1)\right)^{p/2} \\ &= n^{p/2}\left(E\left(K_j(x,X_1) - EK_j(x,X_1)\right)^2\right)^{p/2} \\ &\le n^{p/2}\left(EK_j^2(x,X_1)\right)^{p/2} \\ &= n^{p/2}\left(\int K_j^2(x,y)f(y)\,dy\right)^{p/2} \\ &\le n^{p/2}\left(\int 2^{2j}F^2\left(2^j x - 2^j y\right)f(y)\,dy\right)^{p/2} \\ &= n^{p/2}2^{jp/2}\left(\int 2^j F^2\left(2^j x - 2^j y\right)f(y)\,dy\right)^{p/2},\end{aligned}$$

then

$$\begin{aligned}\int\left(\sum_{i=1}^{n} E\left(Y_i^2\right)\right)^{p/2} dx &= n^{p/2}2^{jp/2}\int\left(\int 2^j F^2\left(2^j x - 2^j y\right)f(y)\,dy\right)^{p/2} dx \\ &= n^{p/2}2^{jp/2}\int\left(\int F^2(t)f\left(x - t/2^j\right)dt\right)^{p/2} dx \\ &= n^{p/2}2^{jp/2}\|F\|_2^p\int\left(\int \frac{F^2(t)}{\|F\|_2^2}f\left(x - t/2^j\right)dt\right)^{p/2} dx \\ &\lesssim n^{p/2}2^{jp/2}\int\int \frac{F^2(t)}{\|F\|_2^2}f^{p/2}\left(x - t/2^j\right)dt\,dx \\ &\lesssim n^{p/2}2^{jp/2}\int\int F^2(t)f^{p/2}\left(x - t/2^j\right)dx\,dt \\ &\lesssim n^{p/2}2^{jp/2}.\end{aligned}$$

Therefore, one gets

$$\begin{aligned}E\|\hat{f}_n - E\hat{f}_n\|_p^p &\lesssim \frac{1}{n^p}\left(\left(2^{j+1}\right)^{p-2}n2^j + n^{p/2}2^{jp/2}\right) \\ &= \frac{2^{(j+1)(p-2)}2^j n}{n^p} + \frac{n^{p/2}2^{jp/2}}{n^p} \\ &= \left(\frac{2^j}{n}\right)^{p-1} + \left(\frac{2^j}{n}\right)^{p/2}. \qquad (4)\end{aligned}$$

Taking $2^j \sim n^{\frac{1}{2}}$, one obtains the following desired result:

$$\lim_{n\to\infty} E\|\hat{f}_n - E\hat{f}_n\|_p^p = 0. \tag{5}$$

(ii) For $1 \le p < 2$, let $A = \{x \mid |\hat{f}_n - E\hat{f}_n| < 1\}$, $B = \{x \mid |\hat{f}_n - E\hat{f}_n| \ge 1\}$, then one has

$$\begin{aligned} E\|\hat{f}_n - E\hat{f}_n\|_p^p &= E\int |\hat{f}_n - E\hat{f}_n|^p\,dx \\ &= E\int_A |\hat{f}_n - E\hat{f}_n|^p\,dx + E\int_B |\hat{f}_n - E\hat{f}_n|^p\,dx \\ &\le E\int_A |\hat{f}_n - E\hat{f}_n|\,dx + E\int_B |\hat{f}_n - E\hat{f}_n|^2\,dx \\ &\le E\int |\hat{f}_n - E\hat{f}_n|\,dx + E\int |\hat{f}_n - E\hat{f}_n|^2\,dx \\ &= E\|\hat{f}_n - E\hat{f}_n\|_1 + E\|\hat{f}_n - E\hat{f}_n\|_2^2. \end{aligned} \tag{6}$$

Obviously, one knows that $f \in L_1(\mathbb{R})$ which guarantees that $\lim_{n\to\infty} E\|\hat{f}_n - E\hat{f}_n\|_2^2 = 0$.

Moreover, $E\|\hat{f}_n - E\hat{f}_n\|_1 = \int \frac{1}{n}E|\sum_{i=1}^n Y_i|\,dx$, according to Rosenthal's inequality, Lemma 2.6, one has

$$\begin{aligned} \frac{1}{n}E\left|\sum_{i=1}^n Y_i\right| &\le \frac{1}{n}\left(\sum_{i=1}^n E(Y_i)^2\right)^{\frac{1}{2}} \\ &= \frac{1}{n^{1/2}}\left(E(Y_1)^2\right)^{\frac{1}{2}} \\ &\le \left(\frac{2^j}{n}\right)^{1/2}\left(\int 2^j F^2\left(2^j x - 2^j y\right) f(y)\,dy\right)^{1/2} \\ &= \left(\frac{2^j}{n}\right)^{1/2} G_j * f(x) \\ &= A(x), \end{aligned} \tag{7}$$

where $G(x) = F^2(x)$. On the other hand,

$$\begin{aligned} \frac{1}{n}E\left|\sum_{i=1}^n Y_i\right| &\le \frac{1}{n}\sum_{i=1}^n E\left|K_j(x,X_i) - EK_j(x,X_i)\right| \\ &\lesssim E\left|K_j(x,X_1)\right| \\ &= \int \left|K_j(x,y)\right| f(y)\,dy \\ &\le \int 2^j F\left(2^j x - 2^j y\right) f(y)\,dy \\ &= \int F(t) f\left(x - \frac{t}{2^j}\right) dt \\ &\le f(x)\int F(t)\,dt + \int F(t)\left|f\left(x - \frac{t}{2^j}\right) - f(x)\right| dt \\ &= B(x) + C(x), \end{aligned} \tag{8}$$

where $B(x) = f(x)\int F(t)\,dt$, $C(x) = \int F(t)|f(x - \frac{t}{2^j}) - f(x)|\,dt$. Then we get

$$E\|\hat{f}_n - E\hat{f}_n\|_1 \le \int \min\{A(x), B(x) + C(x)\}\,dx \le \int \min\{A(x), B(x)\}\,dx + \int C(x)\,dx.$$

One knows that

$$\int C(x)\,dx = \int\int F(t)\left|f\left(x - \frac{t}{2^j}\right) - f(x)\right|\,dt\,dx = \int F(t)\int\left|f\left(x - \frac{t}{2^j}\right) - f(x)\right|\,dx\,dt,$$

since

$$F(t)\int\left|f\left(x - \frac{t}{2^j}\right) - f(x)\right|\,dx \le 2F(t)\|f\|_1 \quad \text{and} \quad \lim_{j\to\infty}\int\left|f\left(x - \frac{t}{2^j}\right) - f(x)\right|\,dx = 0.$$

Then one gets $\lim_{n\to\infty}\int C(x)\,dx = 0$. Next, for $\int B(x)\,dx = \int f(x)\int F(t)\,dt\,dx = \|F\|_1\|f\|_1$. One has $B(x) \in L_1(\mathbb{R})$. By the Lebesgue dominated convergence theorem, one gets

$$\lim_{n\to\infty}\int \min\{A(x), B(x)\}\,dx = \int \lim_{n\to\infty}\min\{A(x), B(x)\}\,dx \le \int \lim_{n\to\infty} A(x)\,dx.$$

It remains only to show that $\lim_{n\to\infty} A(x) = 0$. Since the function $G(x) = F^2(x)$ is radially decreasing,

$$\lim_{j\to\infty} G_j * f(x) = \|F\|_2^2 f(x) \quad \text{(a.e.)}$$

and $\|F\|_2^2 f(x)$ is finite for almost all x, we have $\lim_{n\to\infty} A(x) = \lim_{n\to\infty}(\frac{2^j}{n}G_j * f)^{1/2} = 0$. Finally, we get

$$\lim_{n\to\infty} E\|\hat{f}_n - E\hat{f}_n\|_1 = 0. \qquad \square$$

Remark Theorem 3.1 can be considered as a natural extension of Theorem 1 in [6].

Next we shall consider L_∞-consistency.

Theorem 3.2 *Let a scaling function $\varphi(x)$ satisfy* $\operatorname{supp}\varphi \subset [-A, A]$ *and bounded, $\hat{f}_n(x)$ be the wavelet estimator defined in* (1). *If $f \in L_\infty(\mathbb{R})$ is uniformly continuous and $f(x) \lesssim \frac{1}{(1+|x|)^{2+\delta}}$ for any $\delta > 0$, taking $2^j \sim n^{\frac{1}{4}}$, then one gets*

$$\lim_{n\to\infty} E\|f - \hat{f}_n\|_\infty = 0. \tag{9}$$

Proof The proof is similar to Theorem 3.1. We have

$$E\|f - \hat{f}_n\|_\infty \le \|f - E\hat{f}_n\|_\infty + E\|\hat{f}_n - E\hat{f}_n\|_\infty.$$

Since φ satisfies Condition S and f is uniformly continuous, by Lemma 2.4 and Lemma 2.5, one gets

$$\lim_{n\to\infty}\|f - E\hat{f}_n\|_p = \lim_{n\to\infty}\|f - K_j f\|_\infty = 0. \tag{10}$$

For the stochastic term, it can be proved that

$$\begin{aligned}|\hat{f}_n - E\hat{f}_n| &= \left|\sum_k \hat{\alpha}_{jk}\varphi_{jk}(x) - \sum_k \alpha_{jk}\varphi_{jk}(x)\right| \leq \sum_k |\hat{\alpha}_{jk} - \alpha_{jk}|\left|\varphi_{jk}(x)\right| \\ &\leq 2^{j/2}\sum_k |\hat{\alpha}_{jk} - \alpha_{jk}|\|\varphi\|_\infty \quad \text{(a.e.)},\end{aligned}$$

then one has $\|\hat{f}_n - E\hat{f}_n\|_\infty \lesssim 2^{j/2}\sum_k |\hat{\alpha}_{jk} - \alpha_{jk}|$. So one obtains

$$E\|\hat{f}_n - E\hat{f}_n\|_\infty \lesssim 2^{j/2}\sum_k E|\hat{\alpha}_{jk} - \alpha_{jk}|. \tag{11}$$

According to Rosenthal's inequality, Lemma 2.6, one has

$$\begin{aligned}E|\hat{\alpha}_{jk} - \alpha_{jk}| &= E\left|\frac{1}{n}\sum_{i=1}^n \bigl(\varphi_{jk}(X_i) - E\varphi_{jk}(X_i)\bigr)\right| \\ &\leq \frac{1}{n}\left|\sum_{i=1}^n E\bigl(\varphi_{jk}(X_i) - E\varphi_{jk}(X_i)\bigr)^2\right|^{1/2} \\ &\leq \frac{1}{n^{1/2}}\left|E\varphi_{jk}^2(X_1)\right|^{1/2} \\ &= \frac{1}{n^{1/2}}\left(\int 2^j\varphi^2\bigl(2^jx - k\bigr)f(x)\,dx\right)^{1/2} \\ &= \frac{1}{n^{1/2}}\left(\int_{|t-k|\leq A}\varphi^2(t-k)f\left(\frac{t}{2^j}\right)dt\right)^{1/2}.\end{aligned}$$

Moreover, one has $E\|\hat{f}_n - E\hat{f}_n\|_\infty \lesssim (\frac{2^j}{n})^{1/2}\sum_k(\int_{|t-k|\leq A}f(\frac{t}{2^j})\,dt)^{1/2}$ and

$$\begin{aligned}&\sum_k\left(\int_{|t-k|\leq A}f\left(\frac{t}{2^j}\right)dt\right)^{1/2} \\ &\quad\leq \sum_{|k|\leq A+1}\left(\int_{|t-k|\leq A}f\left(\frac{t}{2^j}\right)dt\right)^{1/2} + \sum_{|k|\geq A+1}\left(\int_{|t-k|\leq A}f\left(\frac{t}{2^j}\right)dt\right)^{1/2} \\ &\quad\lesssim \sum_{|k|\leq A+1}\left(\int f\left(\frac{t}{2^j}\right)dt\right)^{1/2} + \sum_{|k|\geq A+1}\left(\int_{|t-k|\leq A}\frac{1}{(1+|t/2^j|)^{2+\delta}}\,dt\right)^{1/2} \\ &\quad\lesssim 2^{j/2} + \sum_{k\geq A+1}\left(\int_{|t-k|\leq A}\frac{1}{(1+|(k-A)/2^j|)^{2+\delta}}\,dt\right)^{1/2} \\ &\quad= 2^{j/2} + \sum_{k\geq A+1}\left(\frac{2A}{(1+|(k-A)/2^j|)^{2+\delta}}\right)^{1/2} \\ &\quad\lesssim 2^{j/2} + \sum_{k\geq 1}\frac{1}{(1+|k/2^j|)^{1+\delta/2}} \\ &\quad\lesssim 2^{j/2} + \int\frac{1}{(1+|t/2^j|)^{1+\delta/2}}\,dt \\ &\quad\lesssim 2^j.\end{aligned} \tag{12}$$

Therefore, one gets $E\|\hat{f}_n - E\hat{f}_n\|_\infty \lesssim (\frac{2^{3j}}{n})^{1/2}$. Taking $2^j \sim n^{\frac{1}{4}}$, one obtains

$$\lim_{n\to\infty} E\|f - \hat{f}_n\|_\infty = 0. \qquad \square$$

4 Additive noise model

In practical situations, direct data is not always available. One of the classical models is described as follows:

$$Y_i = X_i + \epsilon_i,$$

where X_i stands for the random samples with unknown density f_X and ϵ_i denotes the i.i.d. random noise with density g. To estimate the density f_X is a deconvolution problem.

In 2002, Fan and Koo [11] studied the wavelet estimation for random samples with smooth and super smooth noise over a Besov ball. In 2014, Li and Liu [12] considered the wavelet estimation for random samples with moderately ill-posed noise. In this section, we consider the mean L_p-consistency for $f_X \in L_p(\mathbb{R})$ with additive noise.

The Fourier transform of $f \in L_1(\mathbb{R})$ is defined as follows:

$$\tilde{f}(t) = \int f(x)e^{-itx}\,dx.$$

It is well known that $\tilde{f}_Y(t) = \tilde{f}_X(t)\tilde{g}(t)$. For $\tilde{g}(t) \neq 0$ ($\forall t \in \mathbb{R}$), the wavelet estimator is given by

$$\hat{f}_{X,n}(x) = \sum_k \hat{\alpha}_{jk}\varphi_{jk}(x), \tag{13}$$

where

$$\hat{\alpha}_{jk} = \frac{1}{n}\sum_{i=1}^{n}(H_j\varphi)_{jk}(Y_i); \qquad (H_j\varphi)(y) = \frac{1}{2\pi}\int e^{ity}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^j t)}\,dt, \tag{14}$$

and φ is the Meyer scaling function.

Lemma 4.1 *If $f_X \in L_2(\mathbb{R})$, then $\hat{\alpha}_{jk}$ defined in* (14) *is the unbiased estimation of α_{jk}.*

Proof The Plancherel formula tells us that

$$\alpha_{jk} = \int f_X(x)\varphi_{jk}(x)\,dx = \frac{1}{2\pi}\int \tilde{f}_X(t)\overline{\tilde{\varphi}_{jk}(t)}\,dt = \frac{1}{2\pi}\int \frac{\tilde{f}_Y(t)}{\tilde{g}(t)}\tilde{\varphi}_{jk}(-t)\,dt.$$

On the other hand, one gets

$$\begin{aligned} E\tilde{\alpha}_{jk} &= E\left(\frac{1}{n}\sum_{i=1}^{n}(H_j\varphi)_{jk}(Y_i)\right) = E(H_j\varphi)_{jk}(Y_1) \\ &= \int\left(\frac{1}{2\pi}\int 2^{\frac{j}{2}}e^{it(2^j y-k)}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^j t)}\,dt\right)f_Y(y)\,dy \\ &= \frac{1}{2\pi}\int\int 2^{\frac{j}{2}}e^{it(2^j y-k)}f_Y(y)\,dy\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^j t)}\,dt \end{aligned}$$

$$= \frac{1}{2\pi}\int 2^{\frac{j}{2}} e^{-itk}\tilde{f}_Y(-2^j t)\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^j t)}\,dt$$
$$= \frac{1}{2\pi}\int \frac{\tilde{f}_Y(t)}{\tilde{g}(t)}\tilde{\varphi}_{jk}(-t)\,dt.$$

Therefore, $E\tilde{\alpha}_{jk} = \alpha_{jk}$. □

The next two theorems deal with the different cases for $p \geq 2$ and $1 \leq p < 2$, respectively.

Theorem 4.1 *Let $\varphi(x)$ be the Meyer scaling function, $\tilde{g}(t) \gtrsim (1+|t^2|)^{-\frac{\beta}{2}}$ $(\beta \geq 0)$, and $\hat{f}_{X,n}(x)$ is the wavelet estimator defined in* (13). *If $f_X \in L_p(\mathbb{R})$, $2 \leq p < \infty$, taking $2^j \sim n^{\frac{1-\epsilon}{1+2\beta}}$ $(\epsilon > 0)$, then one gets*

$$\lim_{n\to\infty} E\|f_X - \hat{f}_{X,n}\|_p = 0. \tag{15}$$

Proof Similarly, one needs to consider a bias term and a stochastic term.

(i) For the bias term, one observes that

$$E\hat{f}_{X,n}(x) = E\Bigl(\sum_k \hat{\alpha}_{jk}\varphi_{jk}(x)\Bigr)$$
$$= E\Bigl(\sum_k \frac{1}{n}\sum_{i=1}^{n}(H_j\varphi)_{jk}(Y_i)\varphi_{jk}(x)\Bigr)$$
$$= E\Bigl(\sum_k (H_j\varphi)_{jk}(Y_1)\varphi_{jk}(x)\Bigr).$$

Note that $\int\sum_k |(H_j\varphi)_{jk}(y)||\varphi_{jk}(x)|f_Y(y)\,dy \leq 2^j\|(H_j\varphi)\|_\infty\|\theta_\varphi\|_\infty\|f_Y\|_1 < \infty$, then

$$E\hat{f}_{X,n}(x) = E\Bigl(\sum_k (H_j\varphi)_{jk}(Y_1)\varphi_{jk}(x)\Bigr)$$
$$= \sum_k E(H_j\varphi)_{jk}(Y_1)\varphi_{jk}(x)$$
$$= \sum_k \alpha_{jk}\varphi_{jk}(x)$$
$$= K_j f_X(x).$$

From Lemma 2.5, one gets

$$\lim_{n\to\infty}\|f_X - E\hat{f}_{X,n}\|_p = \lim_{n\to\infty}\|f_X - K_j f_X\|_p = 0. \tag{16}$$

(ii) For the stochastic term. Due to Lemma 2.2, it can be found that

$$\|\hat{f}_{X,n} - E\hat{f}_{X,n}\|_p^p = \Bigl\|\sum_k \hat{\alpha}_{jk}\varphi_{jk}(x) - \sum_k \alpha_{jk}\varphi_{jk}(x)\Bigr\|_p^p = \Bigl\|\sum_k (\hat{\alpha}_{jk}-\alpha_{jk})\varphi_{jk}(x)\Bigr\|_p^p$$
$$\lesssim 2^{j(\frac{p}{2}-1)}\sum_k |\hat{\alpha}_{jk}-\alpha_{jk}|_p^p,$$

so one gets

$$
\begin{aligned}
E\|\hat{f}_{X,n} - E\hat{f}_{X,n}\|_p^p &\lesssim 2^{j(\frac{p}{2}-1)} E \sum_k |\hat{\alpha}_{jk} - \alpha_{jk}|_p^p \\
&= 2^{j(\frac{p}{2}-1)} \sum_k E|\hat{\alpha}_{jk} - \alpha_{jk}|_p^p. \qquad (17)
\end{aligned}
$$

Firstly, we estimate $\mathrm{E}|\hat{\alpha}_{jk} - \alpha_{jk}|_p^p$. We have

$$
\begin{aligned}
|\hat{\alpha}_{jk} - \alpha_{jk}| &= \left| \frac{1}{n} \sum_{i=1}^n (H_j\varphi)_{jk}(Y_i) - \frac{1}{n} \sum_{i=1}^n E(H_j\varphi)_{jk}(Y_i) \right| \\
&= \frac{1}{n} \left| \sum_{i=1}^n Z_{ik} \right|,
\end{aligned}
$$

where $Z_{ik} = (H_j\varphi)_{jk}(Y_i) - E(H_j\varphi)_{jk}(Y_i)$ and $EZ_{ik} = 0$. Then

$$
\begin{aligned}
|Z_{ik}| &= \left| (H_j\varphi)_{jk}(Y_i) - E(H_j\varphi)_{jk}(Y_i) \right| \\
&\leq \left| (H_j\varphi)_{jk}(Y_i) \right| + E\left| (H_j\varphi)_{jk}(Y_i) \right| \\
&= \left| \frac{1}{2\pi} \int 2^{\frac{j}{2}} e^{it(2^j Y_i - k)} \frac{\tilde{\varphi}(t)}{\tilde{g}(-2^j t)}\, dt \right| + \int \left| \frac{1}{2\pi} \int 2^{\frac{j}{2}} e^{it(2^j y - k)} \frac{\tilde{\varphi}(t)}{\tilde{g}(-2^j t)}\, dt \right| f_Y(y)\, dy \\
&\lesssim 2^{j(\frac{1}{2}+\beta)}.
\end{aligned}
$$

Thanks to Rosenthal's inequality, Lemma 2.6, one has

$$
\begin{aligned}
E|\hat{\alpha}_{jk} - \alpha_{jk}|_p^p &= \frac{1}{n^p} E\left| \sum_{i=1}^n Z_{ik} \right|^p \\
&\lesssim \frac{1}{n^p} \left(\left(2^{j(\frac{1}{2}+\beta)}\right)^{p-2} \sum_{i=1}^n E|Z_{ik}|^2 + \left(\sum_{i=1}^n E|Z_{ik}|^2 \right)^{\frac{p}{2}} \right) \\
&= \frac{2^{j(\frac{1}{2}+\beta)(p-2)}}{n^{p-1}} E|Z_{1k}|^2 + \frac{1}{n^{\frac{p}{2}}} \left(E|Z_{1k}|^2 \right)^{\frac{p}{2}}. \qquad (18)
\end{aligned}
$$

One only needs to consider $\sum_k (E|Z_{1k}|^2)^{\frac{p}{2}}$. Define $A = \int |(H_j\varphi)(y)|^2\, dy = 2\pi \int_R |\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^j t)}|^2\, dt \lesssim \int |(1+|2^j t|)^{\beta/2} \tilde{\varphi}(t)|^2\, dt \lesssim 2^{2j\beta}$, and

$$
\begin{aligned}
\left(E|Z_{1k}|^2\right)^{\frac{p}{2}} &= \left(E\left| (H_j\varphi)_{jk}(Y_1) - E(H_j\varphi)_{jk}(Y_1) \right|^2 \right)^{\frac{p}{2}} \\
&\leq \left(E\left| (H_j\varphi)_{jk}(Y_1) \right|^2 \right)^{\frac{p}{2}} \\
&= \left(\int \left| (H_j\varphi)_{jk}(y) \right|^2 f_Y(y)\, dy \right)^{\frac{p}{2}} \\
&= A^{\frac{p}{2}} \left(\int \frac{|(H_j\varphi)_{jk}(y)|^2}{A} f_Y(y)\, dy \right)^{\frac{p}{2}} \\
&\leq A^{\frac{p}{2}-1} \int \left| (H_j\varphi)_{jk}(y) \right|^2 f_Y(y)^{\frac{p}{2}}\, dy.
\end{aligned}
$$

Moreover,

$$\begin{aligned}\sum_k \big|(H_j\varphi)_{jk}(y)\big|^2 &= \sum_k \biggl(\frac{2^{\frac{j}{2}}}{2\pi}\biggl|\int e^{it(2^jy-k)}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}\,dt\biggr|\biggr)^2 \\ &\lesssim 2^j \sum_k \biggl(\biggl|\int_{-\frac{4\pi}{3}}^{\frac{4\pi}{3}} e^{it(2^jy-k)}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}\,dt\biggr|\biggr)^2 \\ &\leq 2^j \sum_k \biggl(\biggl|\int_0^{\frac{4\pi}{3}} e^{it2^jy}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}e^{-itk}\,dt\biggr| + \biggl|\int_{-\frac{4\pi}{3}}^0 e^{it2^jy}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}e^{-itk}\,dt\biggr|\biggr)^2 \\ &\leq 2^j \biggl(\sum_k \biggl|\int_0^{\frac{4\pi}{3}} e^{it2^jy}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}e^{-itk}\,dt\biggr|^2 \\ &\quad + \sum_k \biggl|\int_{-\frac{4\pi}{3}}^0 e^{it2^jy}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}e^{-itk}\,dt\biggr|^2\biggr).\end{aligned}$$

Note that $e^{it2^jy}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}I_{[0,2\pi]} \in L_2[0,2\pi]$, $\{e^{-itk}, k \in Z\}$ is an orthonormal basis of $L_2[0,2\pi]$, and by the Parseval formulas, one gets

$$\sum_k \biggl|\int_0^{\frac{4\pi}{3}} e^{it2^jy}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}e^{-itk}\,dt\biggr|^2 = \int_0^{\frac{4\pi}{3}} \biggl|e^{it2^jy}\frac{\tilde{\varphi}(t)}{\tilde{g}(-2^jt)}\biggr|^2\,dt = 2^{2j\beta}.$$

Similarly, $\sum_k |\int_{-\frac{4\pi}{3}}^0 e^{it2^jy}\frac{\hat{\varphi}(t)}{\hat{g}(-2^jt)}e^{-itk}\,dt|^2 = 2^{2j\beta}$. Then $\sum_k |(H_j\varphi)_{jk}(y)|^2 \lesssim 2^{j(2\beta+1)}$.

For the density function $f_Y \in L_1(\mathbb{R}) \cap L_p(\mathbb{R})$, $2 \leq p < \infty$, one has $f_Y \in L_{p/2}(\mathbb{R})$. Moreover, $\sum_k (E|Z_{1k}|^2)^{\frac{p}{2}} \lesssim A^{\frac{p}{2}-1}2^{2j\beta} = 2^{j(\beta p+1)}$. Therefore,

$$\begin{aligned}\sum_k E|\hat{\alpha}_{jk} - \alpha_{jk}|_p^p &= \frac{2^{j(\frac{1}{2}+\beta)(p-2)}}{n^{p-1}}\sum_k E|Z_{1k}|^2 + \frac{1}{n^{\frac{p}{2}}}\sum_k \bigl(E|Z_{1k}|^2\bigr)^{\frac{p}{2}} \\ &\lesssim \frac{2^{j(\frac{1}{2}+\beta)(p-2)}2^{j(2\beta+1)}}{n^{p-1}} + \frac{2^{j(\beta p+1)}}{n^{\frac{p}{2}}} = \frac{2^{j(\beta p+1)}}{n^{\frac{p}{2}}}\biggl(\biggl(\frac{2^j}{n}\biggr)^{\frac{p}{2}-1} + 1\biggr).\end{aligned}$$

Then we get $E\|\hat{f}_{X,n} - E\hat{f}_{X,n}\|_p^p \lesssim 2^{j(\frac{p}{2}-1)}\frac{2^{j(\beta p+1)}}{n^{\frac{p}{2}}}((\frac{2^j}{n})^{\frac{p}{2}-1}+1) \lesssim (\frac{2^{j(2\beta+1)}}{n})^{\frac{p}{2}}$. Taking $2^j \sim n^{\frac{1-\epsilon}{1+2\beta}}$ ($\epsilon > 0$), one obtains $\lim_{n\to\infty} E\|\hat{f}_{X,n} - E\hat{f}_{X,n}\|_p^p = 0$. □

Theorem 4.2 *Let $\varphi(x)$ be the Meyer scaling function, $\tilde{g}(t)| \gtrsim (1+|t|^2)^{-\frac{\beta}{2}}$ ($\beta \geq 0$), $\hat{f}_{X,n}(x)$ is the estimator defined in* (13). *If $f_X \in L_2(\mathbb{R}) \cap L_p(\mathbb{R})$ ($1 \leq p < 2$) and $\operatorname{supp} f_X \subset [-B,B]$, taking $2^j \sim n^{\frac{1-\epsilon}{2+2\beta}}$, then one has*

$$\lim_{n\to\infty} E\|f_X - \hat{f}_{X,n}I_{[-B,B]}\|_p = 0. \tag{19}$$

Proof For the bias term, we get $E\hat{f}_{X,n}I_{[-B,B]}(x) = K_jf_XI_{[-B,B]}(x)$, then

$$\begin{aligned}\|f_X - E\hat{f}_{X,n}I_{[-B,B]}\|_p^p &= \|f_X - K_jf_XI_{[-B,B]}\|_p^p \\ &= \int_{\mathbb{R}} \bigl|f_X(x) - K_jf_XI_{[-B,B]}(x)\bigr|^p\,dx\end{aligned}$$

$$\begin{aligned} &= \int_{-B}^{B} \left| f_X(x) - K_j f_X(x) \right|^p dx \\ &\le \left\| f_X(x) - K_j f_X(x) \right\|_p^p. \end{aligned}$$

So one gets $\lim_{n\to\infty} \| f_X - E\hat{f}_{X,n} I_{[-B,B]} \|_p \le \lim_{n\to\infty} \| f - K_j f \|_p = 0$.

Next we only consider the stochastic term. For any $1 \le p < 2$,

$$\begin{aligned} E\|\hat{f}_{X,n} I_{[-B,B]} - E\hat{f}_{X,n} I_{[-B,B]}\|_p^p \le{}& E\|\hat{f}_{X,n} I_{[-B,B]} - E\hat{f}_{X,n} I_{[-B,B]}\|_1 \\ &+ E\|\hat{f}_{X,n} I_{[-B,B]} - E\hat{f}_{X,n} I_{[-B,B]}\|_2^2. \end{aligned} \tag{20}$$

Because $\lim_{n\to\infty} E\|\hat{f}_{X,n} I_{[-B,B]} - E\hat{f}_{X,n} I_{[-B,B]}\|_2^2 \le \lim_{n\to\infty} E\|\hat{f}_{X,n} - E\hat{f}_{X,n}\|_2^2 = 0$, we only need to consider $E\|\hat{f}_{X,n} I_{[-B,B]} - E\hat{f}_{X,n} I_{[-B,B]}\|_1$. Clearly,

$$\left| \sum_k \varphi(x-k)(H_j\varphi)(y-k) \right| \le \sum_k \left|\varphi(x-k)\right| \left|(H_j\varphi)(y-k)\right| \le \|\theta_\varphi\|_\infty \|H_j\varphi\|_\infty \lesssim 2^{j\beta};$$

define

$$D(x,y) = \sum_k \varphi(x-k)(H_j\varphi)(y-k),$$

then

$$\begin{aligned} \hat{f}_{X,n}(x) &= \sum_k \hat{\alpha}_{jk} \varphi_{jk}(x) \\ &= \sum_k \left(\frac{1}{n} \sum_{i=1}^n (H_j\varphi)_{jk}(Y_i) \right) \varphi_{jk}(x) \\ &= \frac{1}{n} \sum_{i=1}^n \sum_k (H_j\varphi)_{jk}(Y_i) \varphi_{jk}(x) \\ &= \frac{1}{n} \sum_{i=1}^n D_j(x, Y_i). \end{aligned}$$

We know that

$$E\|\hat{f}_{X,n} I_{[-B,B]} - E\hat{f}_{X,n} I_{[-B,B]}\|_1 = \int_{-B}^{B} E|\hat{f}_{X,n} - E\hat{f}_{X,n}|\, dx,$$

now we estimate $E|\hat{f}_{X,n} - E\hat{f}_{X,n}|$. Using Rosenthal's inequality, Lemma 2.6, one gets

$$\begin{aligned} E|\hat{f}_{X,n} - E\hat{f}_{X,n}| &\le \frac{1}{n} \left(\sum_{i=1}^n E\left|D_j(x,Y_i) - ED_j(x,Y_i)\right|^2 \right)^{1/2} \\ &\le \frac{1}{n^{1/2}} \left(E\left|D_j(x,Y_1)\right|^2 \right)^{1/2} \\ &\le \frac{2^j}{n^{1/2}} \left(\int \left(\sum_k \left|(H_j\varphi)(2^j y - k)\right| \left|\varphi(2^j x - k)\right| \right)^2 f_Y(y)\, dy \right)^{1/2} \\ &\lesssim \frac{2^j}{n^{1/2}} 2^{j\beta} \|\theta_\varphi\|_\infty \|f\|_1^{1/2}. \end{aligned} \tag{21}$$

Then $E\|\hat{f}_{X,n}I_{[-B,B]} - E\hat{f}_{X,n}I_{[-B,B]}\|_1 \lesssim \frac{2^{j(1+\beta)}}{n^{1/2}}$, taking $2^j \sim n^{\frac{1-\epsilon}{2+2\beta}}$, one gets

$$\lim_{n\to\infty} E\|\hat{f}_{X,n}I_{[-B,B]} - E\hat{f}_{X,n}I_{[-B,B]}\|_1 = 0. \qquad \square$$

Remark If g is the Dirac function δ, then the conclusions with additive noise reduce to the classical model results without noise.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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